EXACT FACTORIZATIONS OF G-CROSSED BRAIDED FUSION CATEGORIES

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Abstract. For a finite group G, a G-crossed braided fusion category is defined as a G-graded fusion category equipped with a G-action and a G-braiding. In this work, we investigate G-crossed braiding structures within exact factorizations of fusion categories, which are analogous to the Zappa–Szép product in group theory. For a fusion category $\mathcal{B}$ faithfully graded by its universal grading group $U(\mathcal{B})$, we establish that if $\mathcal{B} = \mathcal{A} \bullet \mathcal{C}$ is an exact factorization, then the subcategories $\mathcal{A}$ and $\mathcal{C}$ are $U(\mathcal{A})$- and $U(\mathcal{C})$-crossed braided, respectively. We extend these results to G-crossed commutative fusion rings, where we analyze the $U(\mathsf{R})$-action in exact factorizations of fusion rings. Additionally, we introduce the notion of the generalized semidirect product of fusion categories and rings and show its relationship to the bicrossed product, an equivalent formulation of exact factorization. We further establish that an exact factorization $\mathcal{B} = \mathcal{A} \bullet \mathcal{C}$ is braided if and only if $\mathcal{B} \cong \mathcal{A} \boxtimes \mathcal{C}$, and we provide a complete characterization of conditions under which bicrossed products of categories and rings are commutative or braided. Finally, for a general group G, we present criteria for exact factorization and examine the implications of fusion subcategories of G-crossed braided fusion categories.

1. Introduction

A *fusion category* [11, §4.1] is a specific type of monoidal category, equipped with a tensor product that satisfies associativity and unitality axioms. More precisely, fusion categories are finite semisimple tensor categories where the set of simple objects is finite. The fusion rules, which describe how objects combine under the tensor product, are central to the study of anyonic systems and topologically ordered phases, particularly in two spatial dimensions [18]. These algebraic structures encapsulate the fusion properties of anyons, quasiparticles with nontrivial braiding statistics [15]. The fusion rules play a critical role in understanding the non-abelian statistics that emerge in certain topological quantum field theories, which are essential for fault-tolerant quantum computation [17].

G-crossed braided categories [11, §8.24] have become significant mathematical structures for describing symmetry-enriched invariants in low-dimensional quantum field theories. Specifically, G-crossed braided categories emerge from global symmetries in $(1+1)$-dimensional chiral conformal field theory [20], in $(2+1)$-dimensional topological phases of matter [3], and as invariants of three-dimensional homotopy quantum field theories [30]. Moreover, G-crossed braided categories generalize the notion of fusion categories by incorporating a G-grading and G-action that encodes global symmetries, making them powerful tools for analyzing symmetry-enriched topological (SET) phases [7].

Symmetry-enriched topological phases are phases of matter where global symmetries enrich the topological order by introducing defects or domain walls that carry nontrivial braiding statistics [8]. These G-crossed braided categories describe not only the anyonic content but also the interplay between topological order and symmetry. This interplay is crucial for understanding symmetry-protected topological (SPT) phases [26] and symmetry-enriched topological orders, which have applications in quantum information theory, particularly in the design of topological quantum codes [6]. Furthermore, in the context of tensor networks, fusion categories and their G-crossed counterparts are employed in the construction of matrix product states (MPS) and projected entangled pair states (PEPS), which are used to efficiently represent quantum states in many-body quantum systems [27]. These tensor networks provide a powerful framework for numerically studying the ground states of strongly correlated systems, where topological order and symmetry enrichment play a central role [28].

In the study of topological quantum field theories (TQFTs) and related quantum systems, it is crucial to create new examples of objects from existing ones or to comprehend how complex objects can be broken down into simpler components. For example, fusion categories are fully dualizable in the 3-category of tensor categories, which leads to a "ground-up" construction of the Turaev-Viro fully extended 3-dimensional TQFT [16]. Following the idea of exact factorization for groups, we can extend to analogous notions for Hopf algebras [24] and subsequently fusion categories [13]; given two fusion subcategories $\mathcal{A}, \mathcal{C} \subseteq \mathcal{B}$ of $\mathcal{B}$, we say that $\mathcal{B} = \mathcal{A} \cdot \mathcal{C}$ is an exact factorization if $\mathcal{A} \cap \mathcal{C} = \mathrm{Vec}$ and $\mathrm{FPdim}(\mathcal{B}) = \mathrm{FPdim}(\mathcal{A})\mathrm{FPdim}(\mathcal{C})$. Recently, in [21], the notion of a bicrossed product, $\bowtie$, has been formulated and shown to be equivalent to the exact factorization in the context of fusion rings, which are fusion categories without the associativity coherence data.

This paper focuses on exact factorizations of G-crossed braided fusion categories. We begin with the case where G is the universal grading group of a fusion category $\mathcal{B}$, denoted $\mathrm{U}(\mathcal{B})$, as it encapsulates the most general grading structure for $\mathcal{B}$, and any other G-grading arises from a surjective group homomorphism $\pi : \mathrm{U}(\mathcal{B}) \to G$. Several natural questions arise in this context.

Question 1. If $\mathcal{B} = \mathcal{A} \bullet \mathcal{C}$ is an exact factorization and $\mathcal{B}$ is a $U(\mathcal{B})$-crossed braided fusion category, what can be inferred about $\mathcal{A}$ and $\mathcal{C}$?

Our approach is to show that the $U(\mathcal{B})$-crossed action restricts to $\mathcal{A}$ and $\mathcal{C}$ by considering the bicrossed product and lifting our problem to the level of fusion rings, thereby turning isomorphisms in the categories into equalities in the corresponding ring. We prove in Theorem 3.4 that if $\mathcal{B} = \mathcal{A} \bullet \mathcal{C}$ is an exact factorization of fusion categories and $\mathcal{B}$ is $U(\mathcal{B})$-crossed braided, then $\mathcal{A}$ is $U(\mathcal{A})$-crossed braided and $\mathcal{C}$ is $U(\mathcal{C})$-crossed braided. Furthermore, we prove some related propositions concerning the universal grading.

Question 2. Is there a notion of a G-crossed braided fusion ring? What are the implications of Question 1 in the context of fusion rings?

We first define a G-crossed ring in Definition 4.1. In Theorem 4.4 , we show that if $\mathsf{R} = \mathsf{A} \bullet \mathsf{C}$ is an exact factorization of the fusion ring $(\mathsf{R}, \mathsf{B}(\mathsf{R}))$ into a product of fusion subrings $(\mathsf{A}, \mathsf{B}(\mathsf{A}))$ and $(\mathsf{C}, \mathsf{B}(\mathsf{C}))$ and R is $U(\mathsf{R})$-crossed braided, then A is $U(\mathsf{A})$-crossed braided and C is $U(\mathsf{C})$-crossed braided. Furthermore, motivated by the multiplicative structure within the bicrossed product, we establish results concerning the $U(\mathsf{R})$-action in the bicrossed product $\mathsf{R} = \mathsf{A} \bowtie \mathsf{C}$. These results yield insight into the restriction of the $U(\mathsf{R})$-action and also provide an alternative proof of the preceding statement.

Question 3. If $\mathcal{B} = \mathcal{A} \bullet \mathcal{C}$ is an exact factorization and $\mathcal{B}$ is G-crossed braided, what can be inferred about $\mathcal{A}$ and $\mathcal{C}$?

By setting $H = \{g \in G, \mathcal{B}_g \cap \mathcal{A} \neq 0\}$ and $K = \{g \in G, \mathcal{B}_g \cap \mathcal{C} \neq 0\}$, we prove in Proposition 5.3 that $G = HK$ is a factorization, and in Corollary 5.5 that an exact factorization of the trivial components of the gradings, $\mathcal{B}_e = \mathcal{A}_e \bullet \mathcal{C}_e$, implies that $G = HK$ is an exact factorization, i.e, $\mathcal{A}$ is H-crossed braided and $\mathcal{C}$ is K-crossed braided.

Question 4. What are the implications of a G-crossed braided structure when considering other factorization structures, such as the semidirect product, Deligne tensor product, and fiber product, instead of an exact factorization? Can we build new relationships between these structures and exact factorization?

In Definition 6.3, we introduce a new factorization structure, the generalized semidirect product of two categories. For the standard semidirect product, known as the crossed product, we prove in Proposition 6.1 that $\mathcal{C} \rtimes G$ is G-crossed braided if and only if $\mathcal{C}$ is braided and ρ_g is isomorphic to $\mathrm{id}_{\mathcal{C}}$ for all $g \in G$. For the Deligne product, we prove in Proposition 7.1 that if $\mathcal{B} = \mathcal{A} \boxtimes \mathcal{C}$, where $\mathcal{A}$ is H-crossed braided and $\mathcal{C}$ is K-crossed braided, then $\mathcal{B}$ is $H \times K$-crossed braided. For the fiber product, we prove in Proposition 7.3 that if $\mathcal{A}$ and $\mathcal{C}$ are G-crossed braided fusion categories, then the fiber product $\mathcal{A} \boxtimes_G \mathcal{C}$ is G-crossed braided.

Question 5. What are the implications of exact factorization in general braided fusion categories and their analog in rings, commutative rings?

In Proposition 8.1, we prove that $\mathsf{A} \bowtie \mathsf{C}$ is a commutative ring if and only if A and C are commutative rings, and the actions $\triangleleft$ and $\triangleright$ are trivial. In Proposition 8.2, we prove that if $\mathcal{B} = \mathcal{A} \bullet \mathcal{C}$ is an exact factorization of fusion categories, $\mathcal{B}$ is braided if and only if $\mathcal{A}$ and $\mathcal{C}$ are braided and $\mathcal{B} \cong \mathcal{A} \boxtimes \mathcal{C}$. In Proposition 8.3, we provide an if and only if criterion regarding when the bicrossed product $\mathcal{A} \bowtie \mathcal{C}$ is braided.

This paper is organized as follows. A brief introduction to G-crossed braiding, fusion categories, fusion rings, and exact factorizations is given in Section 2. In Section 3, we prove our main result regarding exact factorization in fusion categories when taking the universal grading. In Section 4, we define a G-crossed fusion ring and look at exact factorization with the universal grading. In Section 5, we prove our main results and build several intermediary propositions in the general G-crossed braided case. In Section 6, we define the generalized semidirect product. In Section 7, we study our problem in related factorization structures. In Section 8, we study the general braiding.

2. Preliminaries

We work over an algebraically closed field $\Bbbk$ of characteristic zero.

2.1. **Monoidal and fusion categories.** In this subsection we recall some basic definitions and fix notation, see [11] for more details.

Monoidal categories [11, §2.1] are categories where the primary elements consist of a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, a natural isomorphism $a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$ for all objects $X, Y, Z \in \mathcal{C}$, an identity object $1 \in \mathcal{C}$, and natural isomorphisms $l_X : 1 \otimes X \xrightarrow{\sim} X$ and $r_X : X \otimes 1 \xrightarrow{\sim} X$ for all objects $X \in \mathcal{C}$. These elements satisfy associativity and unitality axioms, given by $(a_{W,X,Y} \otimes \mathrm{id}_Z) \circ a_{W \otimes X,Y,Z} = a_{W,X,Y \otimes Z} \circ a_{W,X \otimes Y,Z} \circ (\mathrm{id}_W \otimes a_{X,Y,Z})$ and $r_X \circ \mathrm{id}_Y = (\mathrm{id}_X \otimes l_Y) \circ a_{X,1,Y}$. For conciseness, we denote $(\mathcal{C}, \otimes, 1)$ as $\mathcal{C}$.

Example 2.1. [11, Example 2.3.3] The category $\mathbb{k}$-Vec of all $\mathbb{k}$-vector spaces is a monoidal category, where $\otimes = \otimes_{\mathbb{k}}$, $1 = \mathbb{k}$, and the morphisms a, ι, l, r are the obvious ones. The same is true about the category of finite dimensional vector spaces over $\mathbb{k}$, denoted by $\mathbb{k}$-vec.

A *monoidal functor* [11, §2.4] between monoidal categories $(\mathcal{C}, \otimes, 1, a, l, r)$ and $(\mathcal{C}_1, \otimes_1, 1_1, a_1, l_1, r_1)$ is a functor $F : \mathcal{C} \to \mathcal{C}_1$ equipped with a natural isomorphism $F_{X,Y} : F(X) \otimes_1 F(Y) \xrightarrow{\sim} F(X \otimes Y)$ for all objects $X, Y \in \mathcal{C}$, and an isomorphism $F_1 : 1_1 \xrightarrow{\sim} F(1)$, such that the following conditions hold for all objects $X, Y, Z \in \mathcal{C}$: $F(X \otimes Y) \circ F(Z) = (F(X) \otimes_1 \mathrm{id}_{F(Y)}) \circ a_1(F(X), F(Y), F(Z)) \circ F(X \otimes Y \otimes Z)$, $F(l_X) \circ l_1 = F(1_X) \circ (F_0 \otimes_1 \mathrm{id}_{F(X)}) \circ a_1^{-1}(F(1), F(X), F(X))$, and $F(r_X) \circ r_1 = F(X \otimes 1) \circ (\mathrm{id}_{F(X)} \otimes_1 F_1) \circ a_1^{-1}(F(X), F(1), F(X))$.

Let $\mathcal{C}$ be a $\mathbb{k}$-linear, abelian, rigid (equipped with left and right duals), monoidal category. $\mathcal{C}$ is a *multitensor category* over $\mathbb{k}$ if the bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is bilinear with respect to morphisms. We consider $\mathcal{C}$ to be *indecomposable* if it is not equivalent to a direct sum of nonzero multitensor categories. If $\mathrm{End}_{\mathcal{C}}(1) \cong \mathbb{k}$, then we call $\mathcal{C}$ a *tensor category* [11, §4.1].

A *multifusion* category [12] is defined as a finite semisimple multitensor category. A *fusion category* [11, §4.1] is a multifusion category which is a finite semisimple tensor category. Given a fusion category $\mathcal{C}$, we denote by $\mathrm{Irr}(\mathcal{C})$ the set of isomorphism classes of simple objects of $\mathcal{C}$. We can decompose the product of two objects $C, C' \in \mathcal{C}$ as

$$C \otimes C' \simeq \bigoplus_{C'' \in \mathrm{Irr}(\mathcal{C})} N_{C''}^{C,C'} C'' \quad ,$$

where $N_{C''}^{C,C'} = \dim_k \mathrm{Hom}_{\mathcal{C}}(C'', C \otimes C')$.

Example 2.2. Let G be a group. The category $\mathrm{Rep}_{\mathbb{k}}(G)$ of all representations of G over $\mathbb{k}$ is a monoidal category, with $\otimes$ being the tensor product of representations: if for a representation V one denotes by ρ_V the corresponding map $G \to \mathrm{GL}(V)$, then

$$\rho_{V \otimes W}(g) := \rho_V(g) \otimes \rho_W(g).$$

The unit object in this category is the trivial representation $1 = \mathbb{k}$. Note that in general, $\mathrm{Rep}(G)$ is a fusion category if $\mathrm{char}(\mathbb{k}) = 0$, or $\mathrm{char}(\mathbb{k})$ is coprime to $|G|$. More generally, let H be a semisimple Hopf algebra over $\mathbb{k}$. The category $\mathrm{Rep}(H)$ of finite-dimensional representations of H has the structure of a fusion category [1, Definition 1.8].

Example 2.3. The category Vec_G consisting of finite-dimensional vector spaces graded by a finite group G is a fusion category. The simple objects in this category are $\{\mathbb{k}_g\}_{g \in G}$, where each $\mathbb{k}_g$

is a one-dimensional vector space graded by $g \in G$. The tensor product in Vec_G is defined as $\mathbb{k}_g \otimes \mathbb{k}_h = \mathbb{k}_{gh}$, and the associativity morphisms are identities.

Example 2.4. [12, §2] Let G be a group, A an abelian group with trivial G-action, and ω a 3-cocycle of G with values in A. This means $\omega : G \times G \times G \to A$ satisfies the equation:

$$\omega(g_1g_2, g_3, g_4)\omega(g_1, g_2, g_3g_4) = \omega(g_1, g_2, g_3)\omega(g_1, g_2g_3, g_4)\omega(g_2, g_3, g_4)$$

for all $g_1, g_2, g_3, g_4 \in G$. The category Vec_G^ω consists of G-graded finite-dimensional $\mathbb{k}$-vector spaces, where the associativity is defined by the 3-cocycle ω. If G is a finite group, then Vec_G^ω is a fusion category.

The natural tensor product operation on finite abelian categories is known as the *Deligne tensor product*. The Deligne tensor product $\mathcal{C} \boxtimes \mathcal{D}$ [11, §4.6] refers to an abelian $\mathbb{k}$-linear category that serves as a universal construction for the functor mapping every $\mathbb{k}$-linear abelian category $\mathcal{A}$ to the category of right exact bilinear bifunctors $\mathcal{C} \times \mathcal{D} \to \mathcal{A}$. Specifically, there exists a bifunctor $\boxtimes : \mathcal{C} \times \mathcal{D} \to \mathcal{C} \boxtimes \mathcal{D}$, which is right exact in both variables, such that for any right exact bilinear bifunctor $F : \mathcal{C} \times \mathcal{D} \to \mathcal{A}$, there exists a unique right exact functor $\overline{F} : \mathcal{C} \boxtimes \mathcal{D} \to \mathcal{A}$ satisfying $\overline{F} \circ \boxtimes = F$.

In a fusion category $\mathcal{C}$, for an object X, its *Frobenius-Perron dimension* $\mathrm{FPdim}(X)$ is defined in [11, Proposition 3.3.4]. The Frobenius-Perron dimension $\mathrm{FPdim}(\mathcal{C})$ of $\mathcal{C}$ is defined by:

$$\mathrm{FPdim}(\mathcal{C}) = \sum_{X \in \mathrm{Irr}(\mathcal{C})} (\mathrm{FPdim}(X))^2.$$

A monoidal category $\mathcal{C} = (\mathcal{C}, \otimes, 1, a, l, r)$ is *braided* [11, §8.1] if it is equipped with a natural isomorphism $c_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X$ for $X, Y \in \mathcal{C}$ (braiding), such that the following hexagon axioms hold for each $X, Y, Z \in \mathcal{C}$: $c_{X \otimes Y, Z} = a_{Z,X,Y} \circ (c_{X,Z} \otimes \mathrm{id}_Y) \circ a^{-1}_{X,Z,Y} \circ (\mathrm{id}_X \otimes c_{Y,Z}) \circ a_{X,Y,Z}$ and $c_{X,Y \otimes Z} = a^{-1}_{Y,Z,X} \circ (\mathrm{id}_Y \otimes c_{X,Z}) \circ a_{Y,X,Z} \circ c_{X,Y} \circ \mathrm{id}_Z \circ a^{-1}_{X,Y,Z}$.

Example 2.5. The category sVec of super-vector spaces is based on the fusion category $\mathrm{Vec}_{\mathbb{Z}/2\mathbb{Z}}$ with a braiding given by

$$c_{V,W} := \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} : \begin{pmatrix} W_0 \otimes V_0 & W_0 \otimes V_1 \\ W_1 \otimes V_0 & W_1 \otimes V_1 \end{pmatrix} \to \begin{pmatrix} V_0 \otimes W_0 & V_1 \otimes W_0 \\ V_0 \otimes W_1 & V_1 \otimes W_1 \end{pmatrix}.$$

In this context, $c_{V,W}(w \otimes v) = (-1)^{\mathrm{gr}(w) \cdot \mathrm{gr}(v)} v \otimes w$, where $\mathrm{gr}(w)$ and $\mathrm{gr}(v)$ denote the parity of w and v in the super-vector space.

2.2. **Fusion rings.** A *fusion ring* [11, §3.1] is a pair $(R, B(R))$, where R is a ring with a fixed $\mathbb{Z}$-basis $B(R) = \{b_0, \dots, b_n\}$, such that:

(i) The structure coefficients of the multiplication are non-negative integers, that is, $b_i b_j = \sum_{k=0}^n N_{i,j}^k b_k$ for some $N_{i,j}^k \in \mathbb{N}_0$,
(ii) $b_0 = 1$ is the unit of the ring,

(iii) there is an involution $* : \{0, 1, \ldots, n\} \to \{0, 1, \ldots, n\}$ such that the structure coefficients satisfy

$$N_{i,j}^0 = \begin{cases} 1 & \text{if } j = i^*, \\ 0 & \text{otherwise,} \end{cases}$$

(iv) the involution $*$ in I_n induces an involution $* : R \to R$, given by $x = \sum_{i \in I_n} a_i x_i \mapsto x^* = \sum_{i \in I_n} a_i x_i^*$, such that $*$ is an anti-automorphism of rings.

Let $(R, B(R))$ be a fusion ring with basis $B(R) = \{b_0, \ldots, b_n\}$. For $i, j, k \in I_n$, we denote by $N_{b_k}^{b_i, b_j} := N_{i,j}^k$.

The *Grothendieck ring* [11, §4.5] of a fusion category $\mathcal{C}$, denoted $K_0(\mathcal{C})$, is a fusion ring with a basis given by the equivalence classes of simple objects. This ring is the free $\mathbb{Z}$-module generated by $\mathrm{Irr}(\mathcal{C})$, with multiplication defined by the tensor product:

$$C \otimes C' = \sum_{C'' \in \mathrm{Irr}(\mathcal{C})} N_{C,C'}^{C''} C'',$$

for $C, C' \in \mathrm{Irr}(\mathcal{C})$. The involution is determined by the duality of the category.

2.3. **Graded fusion categories.** Let $\mathcal{C}$ be a fusion category and G a finite group. A *G-grading* [11, §4.1] on $\mathcal{C}$ is a decomposition of $\mathcal{C}$ into a direct sum of full abelian subcategories

$$\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g,$$

such that $\mathcal{C}_g^* = \mathcal{C}_{g^{-1}}$ and the tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ maps $\mathcal{C}_g \times \mathcal{C}_h$ to $\mathcal{C}_{gh}$. A G-grading on $\mathcal{C}$ is equivalently characterized by a function $\lambda : \mathrm{Irr}(\mathcal{C}) \to G$ such that $\lambda(X^*) = \lambda(X)^{-1}$ and $\lambda(Z) = \lambda(X)\lambda(Y)$ for all $X, Y, Z \in \mathrm{Irr}(\mathcal{C})$. When $\mathcal{C}_g \neq 0$ for all $g \in G$, the grading is said to be *faithful*.

Example 2.6. A G-graded vector space over a field $\Bbbk$, for a finite group G, is naturally a graded fusion category. It consists of a direct sum of vector spaces $V = \bigoplus_{g \in G} V_g$, where each component V_g is finite-dimensional and over $\Bbbk$. This category is semisimple, meaning every object can be decomposed into simple objects, and it admits a tensor product defined by $(V \otimes W)_h = \bigoplus_{g_1 g_2 = h} V_{g_1} \otimes W_{g_2}$, respecting the G-grading. The unit object is $\Bbbk_G = \bigoplus_{g \in G} \Bbbk_g$ with $\Bbbk_1 = \Bbbk$ and $\Bbbk_g = 0$ for $g \neq 1$. The grading is faithful if each $\mathcal{C}_g$ is nonzero, ensuring $\mathrm{FPdim}(\mathcal{C}) = |G| \cdot \mathrm{FPdim}(\mathcal{C}_1)$.

The *adjoint subcategory* $\mathcal{C}_{\mathrm{ad}}$ [11] is the tensor subcategory generated by the elements $X \otimes X^*$ for $X \in \mathcal{C}$. For any fusion category $\mathcal{C}$, there exists a universal grading by a group $U(\mathcal{C})$ with several key properties: it is faithful; the trivial component forms the full fusion subcategory spanned by objects of the form $X \otimes X^*$; every full fusion subcategory $\mathcal{D} \subset \mathcal{C}$ that contains the adjoint category $\mathcal{C}_{\mathrm{ad}}$ is of the form $\mathcal{D} \cong \bigoplus_{h \in H} \mathcal{C}_h$ for some subgroup $H \subset U(\mathcal{C})$; and the group of monoidal automorphisms of the identity functor is canonically isomorphic to $\mathrm{Hom}(U(\mathcal{C}), \Bbbk^\times)$. We denote $U(\mathcal{C})$ as the *universal grading group* of $\mathcal{C}$.

Proposition 2.7 ([14] Corollary 3.7)**.** *Any other faithful grading of $\mathcal{C}$ by a group G is determined by a surjective group homomorphism $\pi : U(\mathcal{C}) \to G$.*

Corollary 2.8. [23, §2] *Suppose $\mathcal{D}$ is a fusion subcategory of $\mathcal{C}$. Then $\mathcal{D}$ is faithfully graded by the subgroup $U_{\mathcal{D}}(\mathcal{C}) = \{g \in U(\mathcal{C}) \mid \mathcal{D} \cap \mathcal{C}_g \neq 0\} \subseteq U(\mathcal{C})$. By the universal property of $U(\mathcal{D})$, there is a surjective group homomorphism $\phi_{\mathcal{D}} : U(\mathcal{D}) \to U_{\mathcal{D}}(\mathcal{C})$.*

Example 2.9. Consider Vec_G, the category of finite-dimensional vector spaces graded by a finite group G. In this case, the universal grading group $U(\mathrm{Vec}_G) = G$.

Note that these notions on fusion categories can be readily generalized to fusion rings. Let $(R, B(R))$ be a fusion ring and G a group. We say that R is graded by G if R can be decomposed as a graded ring [11, §3.6], that is, $R = \bigoplus_{g\in G} R_g$, where R_g are $\mathbb{Z}$-submodules such that $R_g R_{g'} \subseteq R_{gg'}$, and there is a partition $B(R) = \bigsqcup_{g\in G} B(R)_g$. Additionally, we require $R_g^* = R_{g^{-1}}$ and that R_g is the $\mathbb{Z}$-submodule generated by $B(R)_g$. This G-grading is called faithful if $R_g \neq 0$ for all $g \in G$. A G-grading on R induces a map $\deg : B(R) \to G$, given by $b \mapsto |b|$, such that if xy has z in its $B(R)$-decomposition, then $|z| = |x||y|$. Every fusion ring R is faithfully graded by the universal grading group $U(R)$ [14, Theorem 3.5].

2.4. **G-crossed braided fusion categories.** Let G be a finite group and $\mathcal{C}$ be a fusion category. We say that there is a *categorical left action* by tensor autoequivalences of G on $\mathcal{C}$ when, for every $g \in G$, there exists a tensor functor $L_g : \mathcal{C} \to \mathcal{C}$ (resp. $R_g : \mathcal{C} \to \mathcal{C}$) whose action on an object X and a morphism f is given by $L_g(X) = g \triangleright X$, $R_g(X) = X \triangleleft g$, $L_g(f) = g \triangleright f$, and $R_g(f) = f \triangleleft g$. We denote $g \triangleright Y$ as $g(Y)$ for simplicity. Additionally, there are natural isomorphisms $L^2_{g,h} : g \triangleleft (h \triangleleft -) \to gh \triangleleft -$ (resp. $R^2_{g,h} : (- \triangleleft g) \triangleleft h \to - \triangleleft gh$) and $L_0 : \mathrm{id}_{\mathcal{C}} \to e \triangleleft -$ (resp. $R_0 : \mathrm{id}_{\mathcal{C}} \to - \triangleleft e$), such that the diagrams given in [23, §2.7] commute.

A *G-crossed braided fusion category* [11, §8.24] is a fusion category $\mathcal{C}$ equipped with an action of G on $\mathcal{C}$, a (not necessarily faithful) grading $\mathcal{C} = \bigoplus_{g\in G} \mathcal{C}_g$, and isomorphisms

$$c_{X,Y} : X \otimes Y \xrightarrow{\sim} g(Y) \otimes X \text{ for } g \in G,\ X \in \mathcal{C}_g, \text{ and } Y \in \mathcal{C}. \tag{1}$$

called the G-braiding isomorphisms. These structures must satisfy the following conditions:

(1) $g(\mathcal{C}_h) \subset \mathcal{C}_{ghg^{-1}}$ for all $g, h \in G$.
(2) The isomorphisms $c_{X,Y}$ are functorial in X and Y.
(3) The isomorphisms $c_{X,Y}$ are compatible with the G-action, i.e., $g(c_{X,Y}) = c_{g(X),g(Y)}$ for all $g \in G$.
(4) The following diagrams commute for all $g, h \in G$, $X \in \mathcal{C}_g$, and $Y \in \mathcal{C}_h$:

$$\begin{array}{ccc} X \otimes Y \otimes Z & \xrightarrow{c_{X,Y\otimes Z}} & g(Y \otimes Z) \otimes X \\ \downarrow c_{X,Y} \otimes \mathrm{id}_Z & & \downarrow \mathrm{id}_{g(Y)} \otimes c_{X,Z} \\ g(Y) \otimes X \otimes Z & \xrightarrow{g(Y)\otimes c_{X,Z}} & g(Y) \otimes g(Z) \otimes X \end{array}$$

$$\begin{array}{ccc} X \otimes Y \otimes Z & \xrightarrow{c_{X\otimes Y,Z}} & gh(Z) \otimes X \otimes Y \\ & \searrow \mathrm{id}_X \otimes c_{Y,Z} & \downarrow \mathrm{id}_{gh(Z)} \otimes \mathrm{id}_X \otimes \mathrm{id}_Y \\ & & X \otimes h(Z) \otimes Y \end{array}$$

2.5. **Exact factorization of fusion categories.** In this subsection, we will introduce the definition of exact factorization of groups and fusion categories and some of their properties. See [13] for more details.

An *exact factorization* of a group Σ is defined in [5] as a pair (F, G) of subgroups such that $\Sigma = FG$ and $F \cap G = \{e\}$, where e is the identity element of Σ. This implies that the restriction of the multiplication map $\cdot : F \times G \to \Sigma$ is a bijection. In notation, we denote this situation as $\Sigma = F \cdot G$. An exact factorization of a group is also known as a *Zappa–Szép product*.

Exact factorizations can be described in terms of the notion of a *matched pair*. A matched pair of groups is a collection $(F, G, \blacktriangleleft, \blacktriangleright)$ where F and G are groups, $\blacktriangleleft$ and $\blacktriangleright$ are left and right actions of G on F and F on G, respectively, such that:

$$(gt) \blacktriangleleft f = (g \blacktriangleleft (t \blacktriangleright f))(t \blacktriangleleft f), g \blacktriangleright (fl) = (g \blacktriangleright f)((g \blacktriangleleft f) \blacktriangleright l),$$

for all $g, t \in G$ and $f, l \in F$. Then $F \bowtie G := F \times G$ with the multiplication $(f, g)(l, t) = (f(g \blacktriangleright l), (g \blacktriangleleft l)t)$, where $g, t \in G$ and $f, l \in F$, is a group which is an exact factorization. Moreover, any exact factorization of F and G is of this form.

Let $\mathcal{B}$ be a fusion category with fusion subcategories $\mathcal{A}$ and $\mathcal{C}$. We say that $\mathcal{B}$ is an *exact factorization of fusion categories* $\mathcal{A}$ and $\mathcal{C}$ [13, Theorem 3.8] if any of the following equivalent conditions are met: $\mathcal{B}$ is the full abelian subcategory spanned by direct summands of $X \otimes Y$, where $X \in \mathcal{A}$ and $Y \in \mathcal{C}$, and $\mathcal{A} \cap \mathcal{C} = \mathrm{Vec}$; or $\mathcal{A} \cap \mathcal{C} = \mathrm{Vec}$ and $\mathrm{FPdim}(\mathcal{B}) = \mathrm{FPdim}(\mathcal{A}) \cdot \mathrm{FPdim}(\mathcal{C})$; or every simple object of $\mathcal{B}$ can be uniquely expressed as $A \otimes C$ with $A \in \mathrm{Irr}(\mathcal{A})$ and $C \in \mathrm{Irr}(\mathcal{C})$. In this case, we denote $\mathcal{B}$ as $\mathcal{A} \bullet \mathcal{C}$. For a fusion ring $(\mathsf{R}, B(\mathsf{R}))$, and fusion subrings $(\mathsf{A}, B(\mathsf{A}))$, $(\mathsf{C}, B(\mathsf{C}))$, we say $\mathsf{R} = \mathsf{A} \bullet \mathsf{C}$ is an *exact factorization of fusion rings* [21, Definition 2.7] if every element $b \in B(\mathsf{R})$ can be uniquely expressed as $b = ac$, where $a \in B(\mathsf{A})$ and $c \in B(\mathsf{C})$.

Proposition 2.10 ([21] Proposition 3.24)**.** *Let $\mathcal{B} = \mathcal{A} \bullet \mathcal{C}$ be an exact factorization of fusion categories. Then, the universal grading groups satisfy $U(\mathcal{B}) \cong U(\mathcal{A}) \bullet U(\mathcal{C})$.*

Remark 2.11. We identify $a \in U(\mathcal{A})$ with $(a, e) \in U(\mathcal{A}) \times U(\mathcal{C}) = U(\mathcal{B})$.

Proposition 2.12 ([21] Proposition 3.22)**.** *If $\mathsf{R} = \mathsf{A} \bullet \mathsf{C}$ is an exact factorization of fusion rings, then there is an exact factorization of groups $U(\mathsf{R}) \cong U(\mathsf{A}) \bullet U(\mathsf{C})$.*

2.6. **Matched pairs and Bicrossed product.** Exact factorizations of fusion rings can be described in terms of the notion of matched pair. Note that analogous notions have been developed for fusion categories, see [21, Section 4]. A *matched pair* of fusion rings A and C with faithful gradings

$$\mathsf{A} = \sum_{h \in H} \mathsf{A}_h, \qquad \mathsf{B}(\mathsf{A}) = \bigsqcup_{h \in H} \mathsf{B}(\mathsf{A})_h, \qquad \mathsf{C} = \sum_{k \in K} \mathsf{C}_k, \qquad \mathsf{B}(\mathsf{C}) = \bigsqcup_{k \in K} \mathsf{B}(\mathsf{C})_k,$$

for groups H and K, is a collection $(\mathsf{A}, \mathsf{C}, H, K, \blacktriangleright, \blacktriangleleft, \triangleright, \triangleleft)$ where

- $(H, K, \blacktriangleright, \blacktriangleleft)$ is a matched pair of groups between H and K,

- $\triangleright\colon K \times \mathsf{A} \to \mathsf{A}$ and $\triangleleft\colon \mathsf{C} \times H \to \mathsf{C}$ are $\mathbb{Z}$-linear left and right actions, respectively, such that $k \triangleright \mathsf{B}(\mathsf{A})_h = \mathsf{B}(\mathsf{A})_{k \blacktriangleright h}$, $\mathsf{B}(\mathsf{C})_k \triangleleft h = \mathsf{B}(\mathsf{C})_{k \blacktriangleleft h}$, and

$$\begin{aligned} k \triangleright (aa') &= (k \triangleright a)((k \blacktriangleleft |a|) \triangleright a'), && a, a' \in \mathsf{B}(\mathsf{A}), \\ (cc') \triangleleft h &= (c \triangleleft (|c'| \blacktriangleright h))(c' \triangleleft h), && c, c' \in \mathsf{B}(\mathsf{C}), \\ k \triangleright 1 &= 1 = 1 \triangleleft h && k \in K, h \in H, \end{aligned}$$

see [21, Definition 3.10]. Moreover, any exact factorization of A and C is like this.

Given a matched pair of fusion rings $(\mathsf{A}, \mathsf{C}, \mathsf{H}, \mathsf{K}, \blacktriangleright, \blacktriangleleft, \triangleright, \triangleleft)$, we define the *bicrossed product* of A and C, denoted as $\mathsf{A} \bowtie \mathsf{C}$, as the following $\mathbb{Z}$-ring:

(i) $\mathsf{A} \bowtie \mathsf{C} = \mathsf{A} \otimes_{\mathbb{Z}} \mathsf{C}$ as a $\mathbb{Z}$-module, and the elements are denoted as $a \bowtie c := a \otimes c$ for $a \in \mathsf{A}$ and $c \in \mathsf{C}$.
(ii) The fixed $\mathbb{Z}$-basis is $B(\mathsf{A} \bowtie \mathsf{C}) = \{a \bowtie c : a \in B(\mathsf{A}), c \in B(\mathsf{C})\}$.
(iii) The multiplication is given by

$$(a \bowtie c)(a' \bowtie c') = a(|c| \triangleright a') \bowtie (c \triangleleft |a'|)c',$$

for all $a, a' \in B(\mathsf{A})$, $c, c' \in B(\mathsf{C})$ and extended $\mathbb{Z}$-linearly.
(iv) The involution $* : \mathsf{A} \bowtie \mathsf{C} \to \mathsf{A} \bowtie \mathsf{C}$ is given by

$$(a \bowtie c)^* = |c|^{-1} \triangleright a^* \bowtie c^* \triangleleft |a|^{-1},$$

for all $a \in B(\mathsf{A})$, $c \in B(\mathsf{C})$ and extended $\mathbb{Z}$-linearly.

Theorem 2.13 ([21] Theorem 3.14)**.** *Let $\mathsf{R} = \mathsf{A} \bullet \mathsf{C}$ be an exact factorization of fusion rings. Then there exists a matched pair of fusion rings between A and C such that $\mathsf{R} \cong \mathsf{A} \bowtie \mathsf{C}$.*

More precisely, an exact factorization $\mathsf{R} = \mathsf{A} \cdot \mathsf{C}$ give rise to a matched pair of fusion rings $(\mathsf{A}, \mathsf{C}, H, K, \blacktriangleright, \blacktriangleleft, \triangleright, \triangleleft)$, where $ca = (|c| \triangleright a)(c \triangleleft |a|)$, with $c \in \mathsf{B}(\mathsf{C}), a \in \mathsf{B}(\mathsf{A})$, such that $\mathsf{R} \simeq \mathsf{A} \bowtie \mathsf{C}$, see [21, Theorem 3.14, Corollary 3.20].

3. Exact Factorization in the Universal Grading Case

In this section, we establish some foundational results concerning exact factorizations of G-crossed braided fusion categories, where G is the universal grading group. One of our main results is the following: if $\mathcal{B}$ is $U(\mathcal{B})$-crossed braided, then both $\mathcal{A}$ and $\mathcal{C}$ inherit $U(\mathcal{A})$-crossed and $U(\mathcal{C})$-crossed braided structures, respectively.

Lemma 3.1. *If $\mathcal{A}$ is a fusion subcategory of $\mathcal{B}$, then $U(\mathcal{A}) \cong U_A(\mathcal{B}) = \{g \in U(\mathcal{B}) \mid \mathcal{A} \cap \mathcal{B}_g \neq 0\} \leq U(\mathcal{B})$.*

Proof. By [23, §2], $\mathcal{A}$ is faithfully graded by the subgroup $U_{\mathcal{A}}(\mathcal{B}) = \{g \in U(\mathcal{B}) \mid \mathcal{A} \cap \mathcal{B}_g \neq 0\} \subseteq U(\mathcal{B})$. Note that $U_{\mathcal{A}}(\mathcal{B})$ is a subgroup of $U(\mathcal{B})$ by Proposition 5.1. By the universal property of $U(\mathcal{A})$, there is a surjective group homomorphism $\varphi_{\mathcal{A}} : U(\mathcal{A}) \to U_{\mathcal{A}}(\mathcal{B})$. By [21, Proposition 3.22], $\varphi_{\mathcal{A}}$ is an isomorphism, and hence $U(\mathcal{A})$ is isomorphic to $U_{\mathcal{A}}(\mathcal{B})$, a subgroup of $U(\mathcal{B})$. $\square$

Remark 3.2. Note that $U(\mathcal{B})$ organizes the fusion category $\mathcal{B}$ into direct summands indexed by elements of $U(\mathcal{B})$. Therefore, any full fusion subcategory $\mathcal{A} \subseteq \mathcal{B}$ containing the adjoint subcategory $\mathcal{B}_{ad}$ can be expressed as $\mathcal{A} \cong \bigoplus_{h\in H} \mathcal{B}_h$, where $H \subseteq U(\mathcal{B})$ is a subgroup by [9, Proposition 2.3]. Furthermore, the group of monoidal automorphisms of the identity functor of $\mathcal{B}$ is canonically isomorphic to $\mathrm{Hom}(U(\mathcal{B}), \Bbbk)$, where $\Bbbk$ is the base field.

Corollary 3.3 ([9] Corollary 2.5)**.** *There is a one-to-one correspondence between fusion subcategories $\mathcal{A} \subseteq \mathcal{B}$ containing $\mathcal{B}_{ad}$ and subgroups $G \subseteq U(\mathcal{B})$, namely $\mathcal{A} \mapsto G_{\mathcal{A}} := \{g \in U(\mathcal{B}) \mid \mathcal{A} \cap \mathcal{B}_g \neq 0\}$ and $G \mapsto \mathcal{A}_G := \bigoplus_{g\in G} \mathcal{B}_g$.*

Theorem 3.4. *Let $\mathcal{B} = \mathcal{A} \bullet \mathcal{C}$ be an exact factorization of fusion categories. If $\mathcal{B}$ is $U(\mathcal{B})$-crossed braided, then $\mathcal{A}$ is $U(\mathcal{A})$-crossed braided and $\mathcal{C}$ is $U(\mathcal{C})$-crossed braided.*

Proof. Since $\mathcal{B}$ is $U(\mathcal{B})$-crossed braided, there is a $U(\mathcal{B})$-action $T : U(\mathcal{B}) \to \mathrm{Aut}_{\otimes}(\mathcal{B})$ such that $T_b(\mathcal{B}_{b'}) \subseteq \mathcal{B}_{bb'b^{-1}}$ for $b, b' \in U(\mathcal{B})$. By Lemma 3.1, $U(\mathcal{A})$ is isomorphic to a subgroup of $U(\mathcal{B})$, and from [21, Proposition 3.24] we have that $U(\mathcal{B}) \simeq U(\mathcal{A}) \cdot U(\mathcal{C})$; hence, T can be restricted to $U(\mathcal{A})$. Consider the restriction $T|_{U(\mathcal{A})} : U(\mathcal{A}) \to \mathrm{Aut}_{\otimes}(\mathcal{B})$. Take $a \in U(\mathcal{A})$. For a simple B in $\mathcal{B}_a$, using Equation (1), we have the isomorphism $B \otimes A \cong T_a(A) \otimes B$. By [13, Theorem 3.8], we write $B \cong A' \otimes C'$, where A' is a simple object in $\mathcal{A}$ and C' is a simple object in $\mathcal{C}$. Then, $|B| = |A'| \cdot |C'|$, and it follows that $|A'| = a$ and $|C'| = e$, where e is the identity element of $U(\mathcal{C})$. Since $T_a(A)$ is a simple in $\mathcal{B}_a$, and tensor autoequivalences send simples to simples, we have $T_a(A) \cong \tilde{A} \otimes \tilde{C}$, where $\tilde{A}$ is a simple object in $\mathcal{A}$ and $\tilde{C}$ is a simple object in $\mathcal{C}$, with $|\tilde{A}| = a|A|a^{-1}$ and $|\tilde{C}| = e$ by definition. Recall that our main equations are $B \otimes A \cong T_a(A) \otimes B$, $B \cong A' \otimes C'$, and $T_a(A) \cong \tilde{A} \otimes \tilde{C}$.

In the fusion ring, by [21, Definition 3.12] and considering a "delifting" that is the converse of the lifting described in [21, Definition 4.4], i.e., a categorification of fusion rings, these isomorphisms translate to the equations:

$$\begin{aligned} BA = T_a(A)(B), \quad B = A' \bowtie C', \quad &\text{and} \quad T_a(A) = \tilde{A} \bowtie \tilde{C} \\ \implies (A' \bowtie C')(A \bowtie 1) &= (\tilde{A} \bowtie \tilde{C})(A' \bowtie C') \\ \implies \quad A'(|C'| \rhd A) \bowtie (C' \lhd |A|) &= (\tilde{A} \bowtie \tilde{C})(A' \bowtie C') \\ \implies \quad A'A \bowtie C' \lhd |A| = \tilde{A}A' \bowtie (\tilde{C} \lhd |A'|)(C'), &\quad \forall A' \text{ simple in } \mathcal{A}_a,\ C' \text{ simple in } \mathcal{C}_e. \end{aligned}$$

Let $C' = 1$, the unit in $\mathcal{C}$, in the above equality, which gives $A'A \bowtie 1 = \tilde{A}A' \bowtie \tilde{C} \lhd |A'|$. Acting from the right with $|A'|^{-1}$ gives

$$A'A \bowtie 1 = \tilde{A}A' \bowtie C \tag{2}$$

In terms of the basis elements A_i of $\mathcal{K}_0(A)$, we get $A'A = \sum_{i=0}^{n} N^i_{A',A} A_i$. Similarly, $\tilde{A}A' = \sum_{i=0}^{n} N^i_{\tilde{A},A'} A_i$. Substituting the basis decompositions yields

$$\sum_{i=0}^{n} N^i_{A',A} A_i \bowtie 1 = \sum_{i=0}^{n} N^i_{\tilde{A},A'} A_i \bowtie \tilde{C} \implies \sum_{i=0}^{n} \left(N^i_{A',A} A_i \bowtie 1\right) = \sum_{i=0}^{n} \left(N^i_{\tilde{A},A'} A_i \bowtie \tilde{C}\right).$$

Let's work in $\mathcal{A} \bowtie \mathcal{C}$, with elements $a \bowtie c := a \otimes c$, with $\mathbb{Z}$-basis $B(\mathcal{A} \bowtie \mathcal{C}) = \{a \bowtie c : a \in B(\mathcal{A}), c \in B(\mathcal{C})\}$, where $1 = b_0 \in B(\mathcal{C})$. For the sake of contradiction, assume $\tilde{C} \neq 1$. Then, since

$\tilde{C}$ is a simple object in the category $\mathcal{C}$, it is an element of the basis $\mathcal{B}(\mathcal{C}) = \{c_0, c_1, \ldots, c_j\}$. By definition, $c_0 = 1$, so assume $\tilde{C} = c_k$ for some $k \in \{1, 2, \ldots, j\}$. Then,

$$\sum_{i=0}^{n} N^i_{A',A} A_i \bowtie 1 = \sum_{i=0}^{n} N^i_{\tilde{A},A'} A_i \bowtie c_k \implies \sum_{i=0}^{n} N^i_{A',A} A_i \bowtie 1 - \sum_{i=0}^{n} N^i_{\tilde{A},A'} A_i \bowtie c_k = 0.$$

Since $\{A_i \bowtie c_j\}_{i=0,\,j=0}^{n,\,m}$ forms a basis of $\mathcal{A} \bowtie \mathcal{C}$, the terms in the sum are basis elements and hence linearly independent. Therefore, $N^i_{A',A} = 0$ and $N^i_{\tilde{A},A'} = 0$ for $i = 0, 1, \ldots, n$. However, this implies $A'A = \tilde{A}A' = 0$, which is false. Hence, we must have $\tilde{C} = 1$.

Since the action T restricts to $U(\mathcal{A})$ and $U(\mathcal{C})$, the grading structure is preserved within both $U(\mathcal{A})$ and $U(\mathcal{C})$. This follows from the fact that T acts as a tensor functor on the larger category $\mathcal{B}$, and thus its restriction to the subcategories $U(\mathcal{A})$ and $U(\mathcal{C})$ naturally maintains the grading. By similar reasoning, since the conditions for the action T hold in the larger category $\mathcal{B}$, they also hold in the subcategory $\mathcal{C}$, as it resides within $\mathcal{B}$.

We need to show that the braiding restricts to $\mathcal{A}$ and $\mathcal{C}$ in a G-crossed braided fusion category. First, note for any $g, h \in G$, the conditions $T_g(\mathcal{C}_h) \subset \mathcal{C}_{ghg^{-1}}$ and the G-braiding isomorphisms must be compatible with the G-action, satisfying $T_g(c_{X,Y}) = c_{T_g(X),T_g(Y)}$ for any $g \in G$ hold because T is closed. Second, we analyze the G-braiding isomorphisms $c_{X,Y} : X \otimes Y \to T_g(Y) \otimes X$ for $X \in \mathcal{C}_g$ and $Y \in \mathcal{C}$. These isomorphisms are functorial in X and Y, meaning the naturality condition holds for morphisms $f : X \to X'$ in $\mathcal{C}_g$ and $h : Y \to Y'$ in $\mathcal{C}$. Since $U(\mathcal{A})$ and $U(\mathcal{C})$are subcategories of $U(\mathcal{B})$ where the G-braiding already satisfies this condition, it naturally extends to these subcategories.

Consider a restriction of the crossed braiding diagrams to the $\mathcal{A}$ subcategory. For the hexagon equation:

$$\begin{tikzcd}
(X \otimes Y) \otimes Z \arrow[r, "\alpha_{X,Y,Z}"] \arrow[d, "c_{X,Y} \otimes 1_Z"'] & X \otimes (Y \otimes Z) \arrow[r, "1_X \otimes c_{Y,Z}"] & X \otimes (T^{\mathcal{A}}(h)(Z) \otimes Y) \arrow[d, "\alpha_{X,T^{\mathcal{A}}(h)(Z),Y}"] \\
(T^{\mathcal{A}}(g)(Y) \otimes X) \otimes Z \arrow[d, "\alpha_{T^{\mathcal{A}}(g)(Y),X,Z}"'] & & (X \otimes T^{\mathcal{A}}(h)(Z)) \otimes Y \arrow[d, "1_X \otimes \alpha_{T^{\mathcal{A}}(h)(Z),X,Y}"] \\
T^{\mathcal{A}}(g)(Y) \otimes (X \otimes Z) \arrow[r, "1_{T^{\mathcal{A}}(g)(Y)} \otimes c_{X,Z}"] & T^{\mathcal{A}}(g)(Y) \otimes (T^{\mathcal{A}}(g)(Z) \otimes X) \arrow[r, "\alpha_{T^{\mathcal{A}}(g)(Y),T^{\mathcal{A}}(g)(Z),X}"] & (T^{\mathcal{A}}(g)(Y) \otimes T^{\mathcal{A}}(g)(Z)) \otimes X
\end{tikzcd}$$

For the compatibility condition:

$$\begin{tikzcd}
T^{\mathcal{A}}(g)(T^{\mathcal{A}}(h)(X)) \arrow[r, "T_2^{\mathcal{A}}(g,h)"] \arrow[d, "c_{g,T^{\mathcal{A}}(h)(X)}"'] & T^{\mathcal{A}}(gh)(X) \arrow[d, "c_{gh,X}"] \\
T^{\mathcal{A}}(g)(X) \otimes 1_h \arrow[r, "c_{g,X} \otimes 1_h"] & 1_g \otimes T^{\mathcal{A}}(h)(X)
\end{tikzcd}$$

These diagrams demonstrate that the $U(\mathcal{A})$-crossed braided structure on $\mathcal{A}$ fulfills the necessary conditions inherited from the $U(\mathcal{B})$-crossed braided structure on $\mathcal{B}$. Similar diagrams can be drawn to show that the same conditions hold for $\mathcal{C}$. $\square$

4. G-crossed Fusion Rings

We generalize the definition for G-crossed braided fusion categories to fusion rings. Then, we impose an exact factorization on such a fusion ring to realize its implications. We prove that if $\mathsf{R} = \mathsf{A} \bullet \mathsf{C}$ is an exact factorization, and R is a $U(\mathsf{R})$-crossed fusion ring, then A is a $U(\mathsf{A})$-crossed fusion ring and C is a $U(\mathsf{C})$-crossed fusion ring. Furthermore, we study the restriction of the $U(\mathsf{R})$-action to fusion subrings A and C in the bicrossed product $\mathsf{R} = \mathsf{A} \bowtie \mathsf{C}$.

Definition 4.1. A *G-crossed commutative fusion ring* R is a fusion ring equipped with:

(i) A grading $\mathsf{R} = \bigoplus_{g \in G} \mathsf{R}_g$,
(ii) An action $\rho : G \to \mathrm{Aut}(\mathsf{R})$ of G on R, such that $\rho_g : R \to R$ is an automorphism of fusion rings for each $g \in G$,
(iii) $\rho_g(\mathsf{R}_h) \subseteq \mathsf{R}_{ghg^{-1}}$ for all $g, h \in G$,
(iv) $xy = \rho_g(y)x$, for $x \in \mathsf{R}_g$, $g \in G$, and $y \in \mathsf{R}$.

Remark 4.2. The commutative diagrams that define a braiding in the definition of a G-crossed braided fusion category are redundant under this definition of a G-crossed fusion ring.

Example 4.3. If $\mathcal{C}$ is a G-crossed braided fusion category, then $K_0(\mathcal{C})$ is a G-crossed commutative fusion ring.

Theorem 4.4. *Let $\mathsf{R} = \mathsf{AC}$ be an exact factorization of the fusion ring $(\mathsf{R}, \mathsf{B}(\mathsf{R}))$ into a product of fusion subrings $(\mathsf{A}, \mathsf{B}(\mathsf{A}))$ and $(\mathsf{C}, \mathsf{B}(\mathsf{C}))$. If R is $U(\mathsf{R})$-crossed commutative, then A is $U(\mathsf{A})$-crossed commutative and C is $U(\mathsf{C})$-crossed commutative.*

Proof. Let $\rho : U(\mathsf{R}) \to \mathrm{Aut}(\mathsf{R})$ be the action of $U(\mathsf{R})$ on R. We want to show that $\rho|_{U(\mathsf{A})}(\mathsf{A}) \subseteq \mathsf{A}$ and $\rho|_{U(\mathsf{B})}(\mathsf{B}) \subseteq \mathsf{B}$. By Definition 4.1(iv), we have that $ba = \rho_h(a)b$, for all $h \in U(\mathsf{A})$, $a \in \mathsf{B}(\mathsf{A})$ and $b \in \mathsf{B}(\mathsf{R})_h$. Since R is an exact factorization and $b \in \mathsf{B}(\mathsf{R})$, then $b = a'c'$ and $\rho_h(a) = a''c''$, for uniques $a' \in \mathsf{B}(\mathsf{A})_h$, $a'' \in \mathsf{B}(\mathsf{A})_{h|a|h^{-1}}$ and $c', c'' \in \mathsf{B}(\mathsf{C})_e$. By [21, Corollary 3.20], we have that

$$a'a(c' \triangleleft |a|) = ba \quad = \rho_h(a)b = a''a'(c'' \triangleleft |a'|)c'.$$

Choosing $c' = 1$, we obtain $a'a = a''a'(c'' \triangleleft |a'|)$, for $a' \in \mathsf{B}(\mathsf{A})_h$, $a'' \in \mathsf{B}(\mathsf{A})_{h|a|h^{-1}}$, $c'' \in \mathsf{B}(\mathsf{C})_e$, and for all $a \in \mathsf{B}(\mathsf{A})$, $h \in U(\mathsf{A})$. Let $\mathsf{B}(\mathsf{A}) = \{a_0, a_1, \dots, a_m\}$. Then

$$\sum_{i=1}^{n} N_{a'',a'}^{i} a_i (c'' \triangleleft |a'|) = \sum_{i=0}^{n} N_{a',a}^{i} a_i.$$

Since the basis of R is the product of the basis of A and the basis of C, then $c'' \triangleleft |a'| = 1$ and $c'' = 1$. Therefore, $\rho_h \in \mathrm{Aut}(\mathsf{A})$, for all $h \in U(\mathsf{A})$. Similarly, $\rho_k \in \mathrm{Aut}(\mathsf{C})$, for all $k \in U(\mathsf{C})$. Then it follows that A is $U(\mathsf{A})$-crossed braided and C is $U(\mathsf{C})$-crossed braided. ☐

4.1. Image of the Action.

Let $G = U(\mathsf{R})$, and suppose R is a $U(\mathsf{R})$-crossed commutative ring such that $\mathsf{R} = \mathsf{A} \bowtie \mathsf{C}$ is an exact factorization of fusion rings. Recall multiplication in $\mathsf{R} = \mathsf{A} \bowtie \mathsf{C}$ is given by $(a \bowtie c)(a' \bowtie c') = a(|c| \triangleright a') \bowtie (c \triangleleft |a'|)c'$ by [21, Definition 3.12] and $(a \bowtie c)(a' \bowtie c') = \rho_{|a||c|}(a' \bowtie c')(a \bowtie c)$ by Definition 4.1(iv). Hence, we are interested in finding what $\rho_{|a||c|}(a' \bowtie c')(a \bowtie c)$

looks like - in particular, when it provides a restriction on A or C. To study the restriction, we consider four cases: (i.) $a = 1, c' = 1$, (ii.) $a' = 1, c = 1$, (iii.) $a' = 1, a = 1$, (iv.) $c' = 1, c = 1$.

Let's begin with (i), $a = 1, c' = 1$. Let $\rho_{|c|}(a' \bowtie 1) = (\gamma(a', c) \bowtie \mu(a', c))$, where $\gamma(a', c) \in B(\mathsf{A})$ and $\mu(a', c) \in B(\mathsf{C})$. Then, note

$$\rho_{|c|}(a' \bowtie 1)(1 \bowtie c) = (|c| \triangleright a') \bowtie (c \triangleleft |a'|)$$

$$(\gamma(a', c) \bowtie \mu(a', c))(1 \bowtie c) = (|c| \triangleright a') \bowtie (c \triangleleft |a'|)$$

$$\gamma(a', c) \bowtie \mu(a', c)c = (|c| \triangleright a') \bowtie (c \triangleleft |a'|)$$

Since these are basis elements and form an exact factorization,$\gamma(a', c) = |c| \triangleright a'$ and $\mu(a', c)c = c \triangleleft |a'|$. This implies $\rho_{|c|}(a' \bowtie 1) = (|c| \triangleright a') \bowtie \mu(a', c)$ and $\mu(a', c)(c) = c \triangleleft |a'|$.

Proposition 4.5. *The action ρ restricts on $U(\mathsf{C})$ to A, or $\rho|_{U(\mathsf{C})}(\mathsf{A}) \subseteq \mathsf{A}$, if and only if the right action $\triangleleft : U(\mathsf{A}) \times \mathsf{C} \to \mathsf{C}$ is trivial. That is, $\rho_{|c|}(a' \bowtie 1)$ is in A if and only if $\triangleleft$ is trivial. Moreover, in this case, $\rho_{|c|}(a) = |c| \triangleright a$, for all $|c| \in U(\mathsf{C})$, $a \in \mathsf{A}$.*

Proof. We will first prove that $\rho_{|c|}(a' \bowtie 1) \in \mathsf{A}$ implies $\triangleleft$ is trivial. If $\rho_{|c|}(a' \bowtie 1) \in \mathsf{A}$, then $\rho_{|c|}(a' \bowtie 1) = (|c| \triangleright a') \bowtie \mu(a', c) \in \mathsf{A}$, where $|c| \triangleright a' \in B(\mathsf{A})$ and $\mu(a', c) \in B(\mathsf{C})$. This implies $\mu(a', c) = 1$, and $c = c \triangleleft |a'|$, so $\triangleleft$ is trivial.

Reciprocally, we will prove that $\triangleleft$ is trivial implies $\rho_{|c|}(a' \bowtie 1) \in \mathsf{A}$. Recall $\rho_{|c|}(a' \bowtie 1) = (|c| \triangleright a') \bowtie \mu(a', c)$, where $|c| \triangleright a' \in B(\mathsf{A})$, $\mu(a', c) \in B(\mathsf{C})$. If $\triangleleft$ is trivial, then $\mu(a', c)c = c$. Since the right hand side, c, does not depend on a', we have $\mu(a', c) = \mu(1, c)$. Denote this as $\beta(c)$. Take $a' = 1$. Then $\rho_{|c|}(1 \bowtie 1)$ must be $1 \bowtie 1$ since $\rho_{|c|}$ is an automorphism of fusion rings. Note

$$\rho_{|c|}(1 \bowtie 1) = 1 \bowtie 1 = (|c| \triangleright 1) \bowtie \beta(c) = 1 \bowtie \beta(c).$$

Since $1 \bowtie 1$ and $1 \bowtie \beta(c)$ both belong to the basis $B(\mathsf{A} \bowtie \mathsf{C})$, we there must have $\beta(c) = 1$ and $\rho_{|c|}(a' \bowtie 1) = |c| \triangleright a' \bowtie 1$, for all $c \in \mathsf{C}$, $a' \in \mathsf{A}$. □

Case (ii), where, $a' = 1, c = 1$, is similar to (i), and the proof of the below proposition follows by analogous reasoning.

Proposition 4.6. *The action ρ restricts on $U(\mathsf{A})$ to C, or $\rho|_{U(\mathsf{A})}(\mathsf{C}) \subseteq \mathsf{C}$ if and only if the left action $\triangleright : U(\mathsf{C}) \times \mathsf{A} \to \mathsf{A}$ is trivial. That is, $\rho_{|a|}(1 \bowtie c')$ is in C if and only if $\triangleright$ is trivial. Moreover, in this case, $\rho_{|a|}(c) = c \triangleleft |a|$, for all $|a| \in U(\mathsf{A})$, $c \in \mathsf{C}$.*

Note Case (iii), where $a' = 1, a = 1$, and Case (iv), where $c' = 1, c = 1$, follow from Theorem 4.4.

Corollary 4.7. *The action ρ restricts on $U(\mathsf{A})$ to A, or $\rho|_{U(\mathsf{A})}(\mathsf{A}) \subseteq \mathsf{A}$. That is, $\rho_{|a|}(a' \bowtie 1)$ is always in A. Similarly, the action ρ restricts on $U(\mathsf{C})$ to C, or $\rho|_{U(\mathsf{C})}(\mathsf{C}) \subseteq \mathsf{C}$. That is, $\rho_{|c|}(1 \bowtie c')$ is always in C.*

5. Exact Factorization in the General Case

Let $\mathcal{B} = \mathcal{A} \bullet \mathcal{C}$ be an exact factorization. Define $H = \{g \in G, \mathcal{B}_g \cap \mathcal{A} \neq 0\}$ and $K = \{g \in G, \mathcal{B}_g \cap \mathcal{C} \neq 0\}$. Through a series of intermediary propositions, we establish that if $\mathcal{B}_e = \mathcal{A}_e \bullet \mathcal{C}_e$ represents an exact factorization of the trivial components of the gradings, then $G = HK$ is an exact factorization. Consequently, $\mathcal{A}$ must be H-crossed braided and $\mathcal{C}$ must be K-crossed braided.

Proposition 5.1. *H and K are subgroups of G.*

Proof. Note that the identity element e is in H since $\mathcal{B}_e \cap \mathcal{A} = \mathcal{A}$; closure holds as for $h, h' \in H$, $\mathcal{B}_{hh'} \cap \mathcal{A} \supseteq (\mathcal{B}_h \otimes \mathcal{B}_{h'}) \cap \mathcal{A} \supseteq (\mathcal{B}_h \cap \mathcal{A}) \otimes (\mathcal{B}_{h'} \cap \mathcal{A}) \neq 0$; and inverses exist because if $h \in H$, then $\mathcal{B}_{h^{-1}} \cap \mathcal{A} = (\mathcal{B}_h)^* \cap \mathcal{A} \neq 0$. Similarly, K satisfies these criteria. □

Proposition 5.2. *$\mathcal{A}$ has a faithful grading $\mathcal{A} = \bigoplus_{h \in H} \mathcal{A}_h$, where $\mathcal{A}_h = B_h \cap \mathcal{A}$, and $\mathcal{C}$ has a faithful grading $\mathcal{C} = \bigoplus_{k \in K} \mathcal{C}_k$, where $\mathcal{C}_k = B_k \cap \mathcal{C}$.*

Proof. Note that by the definition of H, $\mathcal{A}_h$ consists of objects in $\mathcal{A}$ that also belong to B_h. Consider $\mathcal{A} = \bigoplus_{h \in H} \mathcal{A}_h$. Every object $X \in \mathcal{A}$ can be uniquely expressed as $X = \bigoplus_{h \in H} X_h$ where $X_h \in \mathcal{A}_h$ for each $h \in H$. This is because $\mathcal{A}_h = \mathcal{B}_h \cap \mathcal{A}$, ensuring that $\mathcal{A}$ is decomposed into H-indexed components $\mathcal{A}_h$. Hence, this forms a direct sum decomposition of $\mathcal{A}$ into homogeneous components indexed by H. Next, we seek to prove the faithfulness of the grading. The direct sum $\mathcal{A} = \bigoplus_{h \in H} \mathcal{A}_h$ implies that every object $X \in \mathcal{A}$ belongs to some $\mathcal{A}_h$. $\mathcal{A}_h$ are non-zero because $B_h \cap \mathcal{A} \neq 0$ for each $h \in H$, ensuring $\mathcal{A}_h \neq 0$ and thus $\mathcal{A} = \bigoplus_{h \in H} \mathcal{A}_h$ covers all of $\mathcal{A}$. Therefore, $\mathcal{A}$ admits a faithful grading $\mathcal{A} = \bigoplus_{h \in H} \mathcal{A}_h$, where $\mathcal{A}_h = \mathcal{B}_h \cap \mathcal{A}$.

To show that $\mathcal{A}_h$ are abelian subcategories of $\mathcal{A}$, consider that $\mathcal{B}$ has a faithful G-grading, which means $\mathcal{B} = \bigoplus_{g \in G} \mathcal{B}_g$. Given $\mathcal{A} \subseteq \mathcal{B}$, it follows that: $\mathcal{A}_h \otimes \mathcal{A}_{h'} \subseteq \mathcal{A}_{hh'}$ and $(\mathcal{A}_h)^* \subseteq \mathcal{A}_{h^{-1}}$. This is because $\mathcal{B}_h \otimes \mathcal{B}_{h'} \subseteq \mathcal{B}_{hh'}$ and $(\mathcal{B}_h)^* \subseteq \mathcal{B}_{h^{-1}}$ by the properties of the G-grading of $\mathcal{B}$. So $\mathcal{A}_h$ inherits these properties as intersections with $\mathcal{A}$. The proof is analogous for proving the statement for $\mathcal{C}$. □

Proposition 5.3. *$G = HK$ is a factorization.*

Proof. Since $\pi : U(B) \to G$ is a group epimorphism, every element $g \in G$ corresponds to an element $\tilde{g} \in U(B)$. By the exact factorization $U(B) = U(\mathcal{A}) \cdot U(\mathcal{C})$, every element $\tilde{g} \in U(B)$ can be written as $\tilde{g} = \tilde{h}\tilde{k}$ where $\tilde{h} \in U(\mathcal{A})$ and $\tilde{k} \in U(\mathcal{C})$. Note that $\tilde{h}$ corresponds to some $h \in H$ because $\pi(\tilde{h}) = h$ and $h \in H$ by definition (since $\tilde{h} \in U(\mathcal{A}) \subseteq U(B)$), and $\tilde{k}$ corresponds to some $k \in K$ because $\pi(\tilde{k}) = k$ and $k \in K$ by definition (since $\tilde{k} \in U(\mathcal{C}) \subseteq U(B)$). Therefore, $\tilde{g} = \tilde{h}\tilde{k}$ corresponds to $g = hk$ where $h = \pi(\tilde{h}) \in H$ and $k = \pi(\tilde{k}) \in K$. This shows that every $g \in G$ can indeed be expressed as $g = hk$ for some $h \in H$ and $k \in K$. □

Question 6. Is there an exact factorization of the trivial components of the gradings, $\mathcal{B}_e = \mathcal{A}_e \bullet \mathcal{C}_e$?

Corollary 5.4. *Let $\pi : U(\mathcal{B}) \to G$ be the surjective group homomorphism described in Proposition 2.7. Similarly, let $\pi_1 : U(\mathcal{A}) \to H$ and $\pi_2 : U(\mathcal{C}) \to K$ denote the corresponding homomorphisms for $\mathcal{A}$ and $\mathcal{C}$, respectively. Then, $\mathcal{B}_e = \mathcal{A}_e \bullet \mathcal{C}_e$ if and only if $|\ker \pi| = |\ker \pi_1| \cdot |\ker \pi_2|$.*

Proof. Note that $\mathrm{FPdim}(\mathcal{B}_e) = \mathrm{FPdim}(\mathcal{B}_{\mathrm{ad}}) \cdot |\ker(\pi)|$, where $\mathcal{B}_{\mathrm{ad}}$ is the adjoint component. Similarly, $\mathrm{FPdim}(\mathcal{A}_e) = \mathrm{FPdim}(\mathcal{A}_{\mathrm{ad}}) \cdot |\ker(\pi_1)|$ and $\mathrm{FPdim}(\mathcal{C}_e) = \mathrm{FPdim}(\mathcal{C}_{\mathrm{ad}}) \cdot |\ker(\pi_2)|$. Since $\mathrm{FPdim}(\mathcal{B}_{\mathrm{ad}}) = \mathrm{FPdim}(\mathcal{A}_{\mathrm{ad}}) \cdot \mathrm{FPdim}(\mathcal{C}_{\mathrm{ad}})$, we can write $\mathrm{FPdim}(\mathcal{B}_e) = \mathrm{FPdim}(\mathcal{A}_{\mathrm{ad}}) \cdot \mathrm{FPdim}(\mathcal{C}_{\mathrm{ad}}) \cdot |\ker(\pi)|$. It follows that $\mathrm{FPdim}(\mathcal{B}_e) = \mathrm{FPdim}(\mathcal{A}_e) \cdot \mathrm{FPdim}(\mathcal{C}_e) \cdot \frac{|\ker(\pi)|}{|\ker(\pi_1)|\cdot|\ker(\pi_2)|}$. Therefore, to show that $\mathrm{FPdim}(\mathcal{B}_e) = \mathrm{FPdim}(\mathcal{A}_e) \cdot \mathrm{FPdim}(\mathcal{C}_e)$, it suffices to prove that $|\ker(\pi)| = |\ker(\pi_1)| \cdot |\ker(\pi_2)|$, which implies an exact factorization of the kernels $\ker(\pi)$, $\ker(\pi_1)$, and $\ker(\pi_2)$. □

Corollary 5.5. *If $\mathcal{B}_e = \mathcal{A}_e \bullet \mathcal{C}_e$ then $G = HK$ is an exact factorization, implying $\mathcal{A}$ is H-crossed braided and $\mathcal{C}$ is K-crossed braided.*

Proof. It suffices to show that $H \cap K = \{e\}$, i.e, $|\,H \cap K\,| = 1$. Note $\mathrm{FPdim}\ \mathcal{B} = |G|\mathrm{FPdim}\ \mathcal{B}_e$. Since G is a factorization of H and K, we have $|G| = \frac{|H|\cdot|K|}{|\,H\cap K\,|}$. Thus, $\mathrm{FPdim}\ \mathcal{B} = \frac{|H|\cdot|K|}{|\,H\cap K\,|}\mathrm{FPdim}\ \mathcal{B}_e$. By our previous claim, $\mathrm{FPdim}\ \mathcal{B}_e = \mathrm{FPdim}\ \mathcal{A}_e \cdot \mathrm{FPdim}\ \mathcal{C}_e$. Therefore,

$$\mathrm{FPdim}\ \mathcal{B} = \frac{|H| \cdot |K|}{|\,H \cap K\,|}\mathrm{FPdim}\ \mathcal{A}_e \cdot \mathrm{FPdim}\ \mathcal{C}_e.$$

Note that $|H| \cdot \mathrm{FPdim}\ \mathcal{A}_e = \mathrm{FPdim}\ \mathcal{A}$ and $|K| \cdot \mathrm{FPdim}\ \mathcal{C}_e = \mathrm{FPdim}\ \mathcal{C}$. Thus,

$$\mathrm{FPdim}\ \mathcal{B} = \frac{\mathrm{FPdim}\ \mathcal{A} \cdot \mathrm{FPdim}\ \mathcal{C}}{|\,H \cap K\,|}.$$

But since $\mathrm{FPdim}\ \mathcal{B} = \mathrm{FPdim}\ \mathcal{A} \cdot \mathrm{FPdim}\ \mathcal{C}$, it follows that $|\,H \cap K\,| = 1$. □

6. Generalized Semidirect Product

Let $\mathcal{A}$ be a fusion category and G be a finite group. A categorical action by tensor autoequivalences of G on $\mathcal{A}$ is a tensor functor $\rho : \underline{G} \to \mathrm{Aut}_\otimes(\mathcal{A})$. We recall the definition of the *semidirect product* category $\mathcal{A} \rtimes G$, where G acts categorically by tensor autoequivalences on $\mathcal{A}$. As an abelian category $\mathcal{A} \rtimes G$ is the Deligne tensor product $\mathcal{A} \boxtimes \mathrm{Vec}_G$, with tensor product given by

$$(A\#g) \otimes (A'\#g') = A \otimes \rho_g(A')\#gg',\ \ A, A' \in \mathcal{C},\ \ g, h \in G,$$

the unit object is $\mathbf{1}\#e$ and the associativity and unit constraints come from those of $\mathcal{C}$. The semidirect product category is also known as *crossed product* category, see [11, Definition 4.15.5].

Proposition 6.1. *$\mathcal{C} \rtimes G$ is G-crossed braided if and only if $\mathcal{C}$ is braided and ρ_g is isomorphic to $\mathrm{id}_\mathcal{C}$ for all $g \in G$. The action ρ is given by $G \xrightarrow{\rho} \mathrm{Aut}_\otimes(\mathcal{C})$, and $T : G \to \mathrm{Aut}_\otimes(\mathcal{C} \rtimes G)$.*

Proof. We first prove the forward direction. Since $\mathcal{C} \rtimes G$ is G-crossed braided, then $(\mathcal{C} \rtimes G)_e = \mathcal{C} \otimes e \cong \mathcal{C}$ is braided. Now, consider the natural isomorphism μ_x:

$$\begin{aligned}
x &\xrightarrow{\mu_x} x \boxtimes e \cong (x \otimes 1) \boxtimes e \cong (x \otimes \rho_g(1)) \boxtimes e \cong (x \boxtimes g) \otimes (1 \boxtimes g^{-1}) \\
&\xrightarrow{r} [(x \otimes 1) \boxtimes g] \otimes (1 \boxtimes g^{-1}) \cong [(x \otimes \rho_e(1)) \boxtimes g] \otimes (1 \boxtimes g^{-1}) \\
&\cong [(x \boxtimes e) \otimes (1 \boxtimes g)] \otimes (1 \boxtimes g^{-1}) \cong [T_e(1 \boxtimes g) \otimes (x \boxtimes e)] \otimes (1 \boxtimes g^{-1}) \\
&\xrightarrow{T_e \cong \mathrm{id}} [(1 \otimes \rho_g(x)) \boxtimes g] \otimes (1 \boxtimes g^{-1}) \cong (\rho_g(x) \boxtimes g) \otimes (1 \boxtimes g^{-1}) \\
&\cong (\rho_g(x) \otimes \rho_g(1)) \boxtimes e \cong (\rho_g(x) \otimes 1) \boxtimes e \cong \rho_g(x) \boxtimes e \cong \rho_g(x). \quad \square
\end{aligned}$$

We now prove the reverse implication. Consider $\tilde{\rho} : G \to \mathrm{Aut}_{\otimes}(\mathcal{C} \rtimes G)$ by $g \to \tilde{\rho}(g) : \mathcal{C} \rtimes G \to \mathcal{C} \rtimes G$, with $\tilde{\rho}_g$ given by $x \boxtimes h \mapsto \rho(g)(x) \boxtimes ghg^{-1}$. We first show that $\tilde{\rho}$ and $\tilde{\rho}_g$ are tensor functors.

$$\begin{aligned}
\tilde{\rho}(gh)(x \boxtimes l) &\to \rho(gh)(x) \boxtimes ghl(gh)^{-1} \cong \rho(g)(\rho(h)(x)) \boxtimes ghlh^{-1}g^{-1} \\
&\cong \tilde{\rho}(g)(\rho(h)(x) \boxtimes hlh^{-1}) = (\tilde{\rho}(g) \cdot \tilde{\rho}(h))(x \boxtimes l) \\
&\cong \tilde{\rho}(g)((x \otimes \rho(h)(y)) \boxtimes hl) \cong \rho(g)(x \otimes \rho(h)(y)) \boxtimes ghlg^{-1} \\
&\xrightarrow{\text{by } \rho \text{ action}} \rho(g)(x) \otimes \rho(ghg^{-1})(\rho(g)(y)) \boxtimes ghg^{-1}glg^{-1} \\
&\cong (\rho(g)(x) \boxtimes ghg^{-1}) \otimes (\rho(g)(y) \boxtimes glg^{-1}) \\
&\cong \tilde{\rho}(g)(x \boxtimes h) \otimes \tilde{\rho}(g)(y \boxtimes l) \cong (\tilde{\rho}(g) \circ \tilde{\rho}(h))(x \boxtimes l) \quad \square
\end{aligned}$$

Note $\mathcal{C} \rtimes G = \bigoplus_{g \in G} (\mathcal{C} \rtimes g)$. Hence, $\rho(g) \cong \mathrm{id}_{\mathcal{C}}$ implies $\mathcal{C} \rtimes G$ is isomorphic to $\mathcal{C} \boxtimes \mathrm{Vec}_G$. It can be seen that $\mathcal{C}_{x \boxtimes g, y \boxtimes h} : (x \boxtimes g) \otimes (y \boxtimes h) \to (y \boxtimes ghg^{-1}) \otimes (x \boxtimes g) = (y \otimes x) \boxtimes (gh)$. Also, note that $(x \boxtimes g) \otimes (y \boxtimes h)$ is equal to $(x \otimes y) \boxtimes gh$ and hence

$$(x \boxtimes g) \otimes (y \boxtimes h) \xrightarrow{C_{x,y} \otimes \mathrm{id}} (y \otimes x) \boxtimes gh$$

is a G-crossed braiding. $\square$

The associativity in the semidirect product can be twisted by a 3-cocycle ω of G, see [10, Definition 2.10]. In this case, the category is denoted by $\mathcal{A} \rtimes^{\omega} G$. We offer an alternative construction of this category in Proposition 6.2; the proof is straightforward.

Proposition 6.2. *Let G be a finite group acting by tensor autoequivalences on a fusion category $\mathcal{A}$. Let $\mathcal{C}$ be the tensor subcategory of $(\mathcal{A} \rtimes G) \boxtimes \mathrm{Vec}_G^{\omega}$ generated by the elements*

$$(A\#g) \boxtimes g, \qquad\qquad A \in \mathrm{Irr}(\mathcal{A}), g \in G.$$

Then $\mathcal{C}$ is tensor equivalent to $\mathcal{A} \rtimes^{\omega} G$.

The previous proposition give us a way to generalize $\mathcal{A} \rtimes^{\omega} G$, as it suffices to change Vec_G^{ω} to another faithfully G-graded fusion category.

Definition 6.3. Let $\mathcal{A}$ and $\mathcal{C}$ be fusion categories and G be a finite group. Assume that G acts categorically by tensor autoequivalences on $\mathcal{A}$ and $\mathcal{C}$ has a faithful G-grading. We define the *generalized semidirect product* of fusion categories $\mathcal{A} \rtimes \mathcal{C}$ as the fusion subcategory of $(\mathcal{A} \rtimes G) \boxtimes \mathcal{C}$ generated by the elements

$$(A\#|C|) \boxtimes C, \qquad\qquad A \in \mathrm{Irr}(\mathcal{A}), C \in \mathrm{Irr}(\mathcal{C}).$$

Remark 6.4. Notice that $\mathcal{A} \rtimes \mathcal{C}$ is as an abelian category $\mathcal{A} \boxtimes \mathcal{C}$. Hence, we can denote the simple objects of $\mathcal{A} \rtimes \mathcal{C}$ by $A\#C$, $A \in \mathrm{Irr}(\mathcal{A})$, $C \in \mathrm{Irr}(\mathcal{C})$. The tensor product is then given by

$$A\#C \otimes A'\#C' = A \otimes_{\mathcal{A}} (|C| \rightharpoonup A')\#C \otimes_{\mathcal{C}} C'.$$

Remark 6.5. The semidirect product category $\mathcal{A} \rtimes \mathcal{C}$ does not depend of the choice of grading group G as we can always choose $G = U(\mathcal{C})$. Indeed, since there exists an epimorphism $\phi \colon U(\mathcal{C}) \to G$ [14], the G-action on $\mathcal{A}$ induces a $U(\mathcal{C})$-action on $\mathcal{A}$ via ϕ. The semidirect product defined with this

$U(\mathcal{C})$-action coincides with the semidirect product defined with the G-action. So, the semidirect product depends only on the action, not the grading group.

Proposition 6.6. *Let $\mathcal{B} = \mathcal{A} \bowtie \mathcal{C}$. Suppose $\mathcal{B}$ is $U(\mathcal{B})$-crossed braided, $\triangleleft\colon H^{op} \to Aut(\mathcal{C})$ acts by tensor autoequivalence, $v_h : - \triangleleft h \cong id_{\mathcal{C}}$ for all $h \in H$ are monoidal natural isomorphisms, and $T|_{U(\mathcal{C})}|_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$. Then, $\mathcal{A}$ is $U(\mathcal{A})$-crossed braided, $\mathcal{C}$ is $U(\mathcal{C})$-crossed braided, and $\mathcal{B}$ is monoidally equivalent to $\mathcal{A} \ltimes \mathcal{C}$.*

Proof. It suffices to show that there exists a monoidal functor $(F, \mathbf{F}^2)$ from $\mathcal{A} \ltimes \mathcal{C}$ to $\mathcal{B}$. Define $F : \mathcal{A} \ltimes \mathcal{C} \to \mathcal{B}$ on objects as $F(A\#C) = A \otimes C$ for $A \in \mathrm{Irr}(\mathcal{A})$, $C \in \mathrm{Irr}(\mathcal{C})$ and on morphisms as $F(f\#g) = f \otimes g$. It is straightforward to verify that since c satisfies the braiding axioms, $\mathbf{F}^2_{(A\#C),(A'\#C')}$ defined as follows satisfies the monoidal functor axioms.

$$\begin{tikzcd}
F(A\#C) \otimes F(A'\#C') \arrow[r, "\mathbf{F}^2_{A\#C,A'\#C'}"] \arrow[d, dash, "="'] & F((A\#C) \otimes (A'\#C')) \arrow[d, dash, "="] \\
(A \otimes C) \otimes (A' \otimes C') \arrow[dr, "\mathrm{id}"'] & F((A \otimes T_{|c|}(A'))\#(C \otimes C')) \arrow[d, dash, "="] \\
& (A \otimes T_{|c|}(A')) \otimes (C \otimes C')
\end{tikzcd}$$

$\square$

7. Related Factorizations

In this section, we study implications of a G-crossed braided structure when considering other factorization structures. For the Deligne product, we prove that if $\mathcal{B} = \mathcal{A} \boxtimes \mathcal{C}$, where $\mathcal{A}$ is H-crossed braided and $\mathcal{C}$ is K-crossed braided, then $\mathcal{B}$ is $H \times K$-crossed braided. We then recall the definition of the fiber product and prove that if $\mathcal{A}$ and $\mathcal{C}$ are braided G-crossed braided fusion categories, then the fiber product $\mathcal{A} \boxtimes_G \mathcal{C}$ is G-crossed braided.

Proposition 7.1. *Suppose $\mathcal{B} = \mathcal{A} \boxtimes \mathcal{C}$, where $\mathcal{A}$ is H-crossed braided and $\mathcal{C}$ is K-crossed braided. Then $\mathcal{B}$ is $H \times K$-crossed braided.*

Proof. Let $\mathcal{A}$ have H-grading as $\mathcal{A} = \bigoplus_{h \in H} \mathcal{A}_h$. and let $\mathcal{C}$ have K-grading as $\mathcal{C} = \bigoplus_{k \in K} \mathcal{C}_k$. Define the grading on $\mathcal{B}$ by setting $\mathcal{B}_{(h,k)} = \mathcal{A}_h \boxtimes \mathcal{C}_k$ for each $(h, k) \in H \times K$. Consequently, $\mathcal{B}$ can be decomposed as

$$\mathcal{B} = \bigoplus_{(h,k) \in H \times K} \mathcal{B}_{(h,k)} = \bigoplus_{(h,k) \in H \times K} (\mathcal{A}_h \boxtimes \mathcal{C}_k).$$

Given that $\bigoplus_{h \in H} \mathcal{A}_h = \mathcal{A}$ and $\bigoplus_{k \in K} \mathcal{C}_k = \mathcal{C}$, it follows that

$$\bigoplus_{(h,k) \in H \times K} \mathcal{B}_{(h,k)} = \bigoplus_{h \in H} \mathcal{A}_h \boxtimes \bigoplus_{k \in K} \mathcal{C}_k = \mathcal{A} \boxtimes \mathcal{C} = \mathcal{B}.$$

Next, we consider the tensor products of the graded components. For $\mathcal{B}_{(h,k)}$ and $\mathcal{B}_{(h',k')}$, we have:

$$\mathcal{B}_{(h,k)} \otimes \mathcal{B}_{(h',k')} = (\mathcal{A}_h \boxtimes \mathcal{C}_k) \otimes (\mathcal{A}_{h'} \boxtimes \mathcal{C}_{k'}) = (\mathcal{A}_h \otimes \mathcal{A}_{h'}) \boxtimes (\mathcal{C}_k \otimes \mathcal{C}_{k'}).$$

Since $\mathcal{A}$ and $\mathcal{C}$ are graded by H and K respectively, we have $\mathcal{A}_h \otimes \mathcal{A}_{h'} \subseteq \mathcal{A}_{hh'}$ and $\mathcal{C}_k \otimes \mathcal{C}_{k'} \subseteq \mathcal{C}_{kk'}$. Therefore,

$$(\mathcal{A}_h \otimes \mathcal{A}_{h'}) \boxtimes (\mathcal{C}_k \otimes \mathcal{C}_{k'}) \subseteq \mathcal{A}_{hh'} \boxtimes \mathcal{C}_{kk'} = \mathcal{B}_{(hh',kk')}.$$

To determine the duals, we note that the dual of $\mathcal{B}_{(h,k)}$ is given by:

$$(\mathcal{B}_{(h,k)})^* = (\mathcal{A}_h \boxtimes \mathcal{C}_k)^* \cong (\mathcal{A}_h)^* \boxtimes (\mathcal{C}_k)^*.$$

Since the dual of a graded component is the graded component indexed by the inverse element, we have $(\mathcal{A}_h)^* \cong \mathcal{A}_{h^{-1}}$ and $(\mathcal{C}_k)^* \cong \mathcal{C}_{k^{-1}}$. Thus,

$$(\mathcal{B}_{(h,k)})^* \cong \mathcal{A}_{h^{-1}} \boxtimes \mathcal{C}_{k^{-1}} = \mathcal{B}_{(h^{-1},k^{-1})}.$$

Next, we define the action $T : H \times K \to \mathrm{Aut}(\mathcal{A} \boxtimes \mathcal{C})$ by $T_{(h,k)} = T_1(h) \boxtimes T_2(k)$, where $T_1 : H \to \mathrm{Aut}_{\otimes}(\mathcal{A})$ and $T_2 : K \to \mathrm{Aut}_{\otimes}(\mathcal{C})$ are the actions associated with the H-crossed braiding on $\mathcal{A}$ and the K-crossed braiding on $\mathcal{C}$, respectively. Then, for $\mathcal{B}_{(h',k')} = \mathcal{A}_{h'} \boxtimes \mathcal{C}_{k'}$,

$$T_{(h,k)}(\mathcal{B}_{(h',k')}) = (T_1(h) \boxtimes T_2(k))(\mathcal{A}_{h'} \boxtimes \mathcal{C}_{k'}) = T_1(h)(\mathcal{A}_{h'}) \boxtimes T_2(k)(\mathcal{C}_{k'}).$$

Since $T_1(h)$ acts on $\mathcal{A}$ and $T_2(k)$ acts on $\mathcal{C}$, we have $T_1(h)(\mathcal{A}_{h'}) \subseteq \mathcal{A}_{hhh'^{-1}}$ and $T_2(k)(\mathcal{C}_{k'}) \subseteq \mathcal{C}_{kkk'^{-1}}$. Therefore,

$$T_{(h,k)}(\mathcal{B}_{(h',k')}) \subseteq \mathcal{A}_{hhh'^{-1}} \boxtimes \mathcal{C}_{kkk'^{-1}} = \mathcal{B}_{(hhh'^{-1},kkk'^{-1})}.$$

Hence, the decomposition $\mathcal{B} = \bigoplus_{(h,k)\in H\times K} \mathcal{B}_{(h,k)}$ respects the $H \times K$-grading, and the action $T : H \times K \to \mathrm{Aut}(\mathcal{B})$ defined by $T_{(h,k)} = T_1(h) \boxtimes T_2(k)$ satisfies the required properties for $\mathcal{B}$ to be $H \times K$-crossed braided. Thus, we have shown that $\mathcal{B}$ is $H \times K$-crossed braided. $\square$

We now study the implications of a G-crossed braided structure in the fiber product, as defined in [25, §2.2]:

Definition 7.2. The *fiber product* of $\mathcal{A}$ and $\mathcal{C}$ is the fusion category $\mathcal{A} \boxtimes_G \mathcal{C} := \bigoplus_{g\in G} \mathcal{A}_g \boxtimes \mathcal{C}_g$.

Proposition 7.3. *Let $\mathcal{A}$ and $\mathcal{C}$ be G-crossed braided fusion categories. Then, the fiber product $\mathcal{A} \boxtimes_G \mathcal{C}$ is G-crossed braided.*

Proof. Firstly, consider the grading of the fiber product. Clearly, $\mathcal{A} \boxtimes_G \mathcal{C}$ is a fusion subcategory of $\mathcal{A}\boxtimes\mathcal{C}$ graded by G, where each component $\mathcal{A}_g \boxtimes \mathcal{C}_g$ corresponds to an element $g \in G$. This naturally provides a G-grading for the fiber product, as the grading is inherited from the G-gradings of $\mathcal{A}$ and $\mathcal{C}$.

Next, we define the G-action on the fiber product $\mathcal{A} \boxtimes_G \mathcal{C}$. The actions of G on $\mathcal{A}$ and $\mathcal{C}$ gives rise to an action of G on $\mathcal{A} \boxtimes_G \mathcal{C}$, defined by $g(X \boxtimes Y) = g(X) \boxtimes g(Y)$ for $X \in \mathcal{A}_h$, $Y \in \mathcal{C}_h$, and $g \in G$. This action is well-defined because both $\mathcal{A}$ and $\mathcal{C}$ are G-crossed categories, meaning that $g(\mathcal{A}_h) \subset \mathcal{A}_{ghg^{-1}}$ and $g(\mathcal{C}_h) \subset \mathcal{C}_{ghg^{-1}}$. Therefore, the action on the fiber product respects the grading.

The G-crossed braidings c_1 and c_2 of $\mathcal{A}$ and $\mathcal{C}$ can similarly be combined to define a G-crossed braiding on $\mathcal{A} \boxtimes_G \mathcal{C}$. Specifically, we define the braiding isomorphisms $c_{X\boxtimes Y, Z\boxtimes W}$ as $c_{X\boxtimes Y, Z\boxtimes W} = c^1_{X,Z} \boxtimes c^2_{Y,W}$, where $c^1_{X,Z}$ and $c^2_{Y,W}$ are the G-braiding isomorphisms in $\mathcal{A}$ and $\mathcal{C}$ respectively. These isomorphisms are functorial, as they inherit the functoriality from the individual G-braidings in $\mathcal{A}$

and $\mathcal{C}$. Furthermore, the isomorphisms are compatible with the G-action due to the compatibility of the braidings in the original categories.

Finally, we verify the duality structure of $\mathcal{A} \boxtimes_G \mathcal{C}$. The dual of the graded component $\mathcal{B}_{(h,k)} = \mathcal{A}_h \boxtimes \mathcal{C}_k$ is given by $(\mathcal{B}_{(h,k)})^* \cong (\mathcal{A}_h)^* \boxtimes (\mathcal{C}_k)^*$, where $(\mathcal{A}_h)^* \cong \mathcal{A}_{h^{-1}}$ and $(\mathcal{C}_k)^* \cong \mathcal{C}_{k^{-1}}$. This implies that $(\mathcal{B}_{(h,k)})^* \cong \mathcal{B}_{(h^{-1},k^{-1})}$, ensuring that the duality structure of $\mathcal{A} \boxtimes_G \mathcal{C}$ is compatible with its G-grading. Hence, the fiber product $\mathcal{A} \boxtimes_G \mathcal{C}$ indeed inherits the structure of a G-crossed braided fusion category from $\mathcal{A}$ and $\mathcal{C}$. □

8. General Braiding

In this section, we study the more general braided fusion categories and their natural analog in rings, commutative fusion rings. In Proposition 8.1, we prove that $\mathsf{A} \bowtie \mathsf{C}$ is a commutative ring if and only if A and C are commutative rings, and the actions $\triangleleft$ and $\triangleright$ are trivial. In Proposition 8.2, we prove that if $\mathcal{B} = \mathcal{A} \bullet \mathcal{C}$ is an exact factorization of fusion categories, $\mathcal{B}$ is braided if and only if $\mathcal{A}$ and $\mathcal{C}$ are braided and $\mathcal{B} \cong \mathcal{A} \boxtimes \mathcal{C}$.

Proposition 8.1. *$\mathsf{A} \bowtie \mathsf{C}$ is commutative if and only if A and C are commutative, and $\triangleleft$ and $\triangleright$ are trivial.*

Proof. Note that if $\mathsf{A} \bowtie \mathsf{C}$ is commutative, then for any $a, a' \in B(\mathsf{A})$, we have $(a \bowtie 1)(a' \bowtie 1) = (aa' \bowtie 1)$ and $(a' \bowtie 1)(a \bowtie 1) = (a'a \bowtie 1)$ which implies $a'a = aa'$, so A is commutative. Similarly, C is commutative.

For any $c \in B(\mathsf{C})$ and $a \in B(\mathsf{A})$, we have $(1 \bowtie c)(a \bowtie 1) = (|c| \triangleright a) \bowtie (c \triangleleft |a|)$. Also, since $\mathsf{A} \bowtie \mathsf{C}$ is commutative, $(1 \bowtie c)(a \bowtie 1) = (a \bowtie 1)(1 \bowtie c) = (a \bowtie c)$. Note that $(|c| \triangleright a)$ and a are in $B(\mathsf{A})$, and $(c \triangleleft |a|)$ and c are in $B(\mathsf{C})$. Since $\mathsf{A} \bowtie \mathsf{C} = \mathsf{A} \cdot \mathsf{C}$ is an exact factorization, we must have

$$(|c| \triangleright a) = a \quad \text{and} \quad (c \triangleleft |a|) = c \text{ for all } c \in B(\mathsf{C}) \text{ and } a \in B(\mathsf{A}),$$

implying that $\triangleleft$ and $\triangleright$ are trivial.

For the other direction, assume A and C are commutative and $\triangleleft$ and $\triangleright$ are trivial. Then, $(a \bowtie c)(a' \bowtie c') = a(|c| \triangleright a') \bowtie (c \triangleleft |a'|)c' = aa' \bowtie cc' = a'a \bowtie c'c = (a' \bowtie c')(a \bowtie c)$ □

Proposition 8.2. *Let $\mathcal{B} = \mathcal{A} \bullet \mathcal{C}$ be an exact factorization. Then, $\mathcal{B}$ is braided if and only if $\mathcal{A}$ and $\mathcal{C}$ are braided and $\mathcal{B} \cong \mathcal{A} \boxtimes \mathcal{C}$.*

Proof. Suppose $\mathcal{B}$ is braided and has braiding c. Then, the restriction $c^{\mathcal{A}} := c|_{\mathcal{A}}$ implies $\mathcal{A}$ is braided and the restriction $c^{\mathcal{C}} := c|_{\mathcal{C}}$ implies $\mathcal{C}$ is braided. By [13, Corollary 3.9], $\mathcal{B} \cong \mathcal{A} \boxtimes \mathcal{C}$.

Suppose $\mathcal{A}$ and $\mathcal{C}$ are braided and $\mathcal{B} \cong \mathcal{A} \boxtimes \mathcal{C}$. It is straightforward to check $\tilde{c}_{A \boxtimes C, A' \boxtimes C'}$, defined as follows, is a braiding in $\mathcal{B} \cong \mathcal{A} \boxtimes \mathcal{C}$:

$$\begin{tikzcd}
(A \boxtimes C) \otimes (A' \boxtimes C') \arrow[r, "\tilde{c}_{A\boxtimes C, A'\boxtimes C'}"] \arrow[d, "\cong"'] & (A' \boxtimes C') \otimes (A \boxtimes C) \arrow[d, "\cong"] \\
(A \otimes A') \boxtimes (C \otimes C') \arrow[r, "c^{A}_{A,A'} \boxtimes c^{C}_{C,C'}"] & (A' \otimes A) \boxtimes (C' \otimes C)
\end{tikzcd}$$

□

Proposition 8.3. *$\mathcal{A} \bowtie \mathcal{C}$ is braided if and only if $\mathcal{A}$ and $\mathcal{C}$ are braided, $\triangleright$ and $\triangleleft$ are actions by tensor autoequivalence, and $u_k : k \triangleright - \to id_{\mathcal{A}}$ and $v_h : - \triangleleft h \to id_{\mathcal{C}}$ are monoidal natural isomorphisms.*

Proof. The reverse direction follows by [21, Proposition 4.11]. For the first direction, define the braiding in $\mathcal{A}$ as, for all A, A' simple in $\mathcal{A}$, $C_{A,A'} := C_{A \bowtie 1, A' \bowtie 1}$. Define a braiding on $\mathcal{C}$ analogously.

Note $(A \bowtie 1)(1 \bowtie C) \cong A \bowtie C$ by [21, Proposition 4.7], so $(|C| \triangleright A) \bowtie (C \triangleleft |A|) \cong A \bowtie C$. Hence, $|C| \triangleright A \bowtie C \triangleleft |A| \cong A \bowtie C$. via a natural isomorphism. This implies $|C| \triangleright A \cong A$, which we denote as the functor $u_{|C|}(A)$, and $C \triangleleft |A| \cong C$, which we denote as the functor $v_{|A|}(C)$. It is straightforward to check that the actions $\triangleleft$ and $\triangleright$ are monoidal functors, as this would imply they act by tensor autoequivalence, and to verify that $u_{|C|}$ and $v_{|A|}$ are monoidal natural isomorphisms. □

Acknowledgements

I kindly thank my PRIMES mentors, Prof. Julia Plavnik and Prof. Monique Müller, for their patience, guidance, and support throughout this research. I am also grateful to the MIT PRIMES-USA program for making this research possible.

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