MIT PRIMES STEP Senior Group

Fibonacci Party Tricks

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The Trick

A Demonstration of the Trick

We will need a volunteer from the audience to give us two numbers.

Now, we will sum those two numbers up to make a third number, and sum the second and third term to make a fourth term, etc. until we have a ten-term long Fibonacci-like sequence.

We can sum these 10 numbers in our head without writing all of the terms.

How the trick works

If the first two numbers are *a* and *b*, the 7th element is $5a+8b$, and the sum is $55a+88b$. Thus, this trick will always work, no matter what numbers your friend chooses!

A Whole Family of Tricks

When we experimented with other numbers, we discovered an entire set of tricks similar to this. Here are some examples:

- The sum of the first 14 terms of a Fibonacci-like sequence is the same as the 9th term multiplied by 29.
- The sum of the first 6 terms of a Fibonacci-like sequence is the same as the 5th term multiplied by 4.
- The sum of the first 2 terms of a Fibonacci-like sequence is the same as the 3rd term multiplied by 1.

We then decided to continue the pattern and attempt to find all similar tricks.

We define S_n as the sum of the first n Fibonacci numbers, F_m as the largest possible term that divides S_n, and z as S_n / F_m

Odd Index *m* pattern

Index i is the index number of the Lucas number that Multiplier z is.

Sum of first *n* terms	Index *m*	Multiplier *z*	Index *i*
1	2	1	1
2	3	1	1
3	3	2	0
6	5	4	3
10	7	11	5
14	9	29	7
18	11	76	9

A Pattern?

Rows 1 and 3 of that table are clearly exceptions, so let's delete those.

Now, we have a nice pattern: The sum of the first $4k-2$ terms in a Fibonacci-like sequence is equal to the $2k+1$th term multiplied by the $2k-1$th Lucas number.

Now, let's move on to Fibonacci numbers specifically.

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Pattern for Fibonacci numbers

Fibonacci partial sums table

Sums S_n	1	2	3	4	5	6	7	8	9	10
Index m	2	3	3	2	4	5	4	4	6	7
Multiplier z	1	1	2	7	4	4	11	18	11	11

Sums S_n	11	12	13	14	15	16	17	18	19	20
Index m	6	6	8	9	8	8	10	11	10	10
Multiplier z	29	47	29	29	76	123	76	76	199	377

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Maximality

. . . but what if you don’t want to multiply a term by some random big number in your head?

What is the largest term in the sequence that divides the sum?

An interesting **phenomenon...**

Known Formulae

$$F_{2n} + (-1)^n = F_{n-1}L_{n+1}$$
$$F_{2n} - (-1)^n = F_{n+1}L_{n-1}$$
$$F_{2n+1} + (-1)^n = F_{n+1}L_n$$
$$F_{2n+1} - (-1)^n = F_n L_{n+1}.$$

$x = 2n + 2$

$x = 2n + 1$

$x = 2n + 2$

$x = 2n + 1$

Substitute x

$$F_{4k+2} - 1 = F_{2k}L_{2k+2}$$
$$F_{4k+4} - 1 = F_{2k+3}L_{2k+1}$$
$$F_{4k+3} - 1 = F_{2k+2}L_{2k+1}$$
$$F_{4k+5} - 1 = F_{2k+2}L_{2k+3}$$

Surprise!

$$F_{4k+2} - 1 = F_{2k}L_{2k+2} = S_{4k}$$
$$F_{4k+4} - 1 = F_{2k+3}L_{2k+1} = S_{4k+2}$$
$$F_{4k+3} - 1 = F_{2k+2}L_{2k+1} = S_{4k+1}$$
$$F_{4k+5} - 1 = F_{2k+2}L_{2k+3} = S_{4k+3}.$$

The multipliers are Lucas numbers!
But this might be a coincidence?

...or is it?

Given n and m that satisfy:

1. $n - 3m \leq 4$
2. $n \geq 11$
3. $n - m \geq 0$

We have that $\frac{S_n}{F_m} = L_{n-m+2}$.

By a few approximations, we get:

$$S_n = F_{n+2} - 1 \approx \frac{\phi^{n+2}}{\sqrt{5}}$$

$$F_m \approx \frac{\phi^m}{\sqrt{5}}$$

$$\frac{S_n}{F_m} \approx \phi^{n-m+2} \approx L_{n-m+2}$$

Products of a Fibonacci and a Lucas Number

We found that $F_aL_b = F_cL_d$ is true for only a few cases:

1. $a = c, b = d \Rightarrow F_aL_b = F_aL_b$
2. $a = c = 0 \Rightarrow F_0L_b = F_0L_d = 0$
3. $a = 1, c = 2, b = d \Rightarrow F_1L_b = F_2L_b = L_b$
4. $a = 2, c = 1, b = d \Rightarrow F_1L_b = F_2L_b = L_b$
5. $a = b = k, c = 2k, d = 1 \Rightarrow F_kL_k = F_{2k}$
6. $(a, b, c, d) = (1, 3, 3, 0), (2, 3, 3, 0), (1, 0, 3, 1), (2, 0, 3, 1), (3, 2, 4, 0)$

Proving Maximality

- Suppose there exists a **greater solution** m' such that $F_{m'} \mid S_n$.
- We can check that m' satisfies the conditions necessary for $S_n / F_{m'} = L_{n-m'+2}$.
- Since $S_n = F_{m'} L_{n-m'+2} = F_m L_{n-m+2}$, we conclude that for most cases, $m' = m$.
- Contradiction!

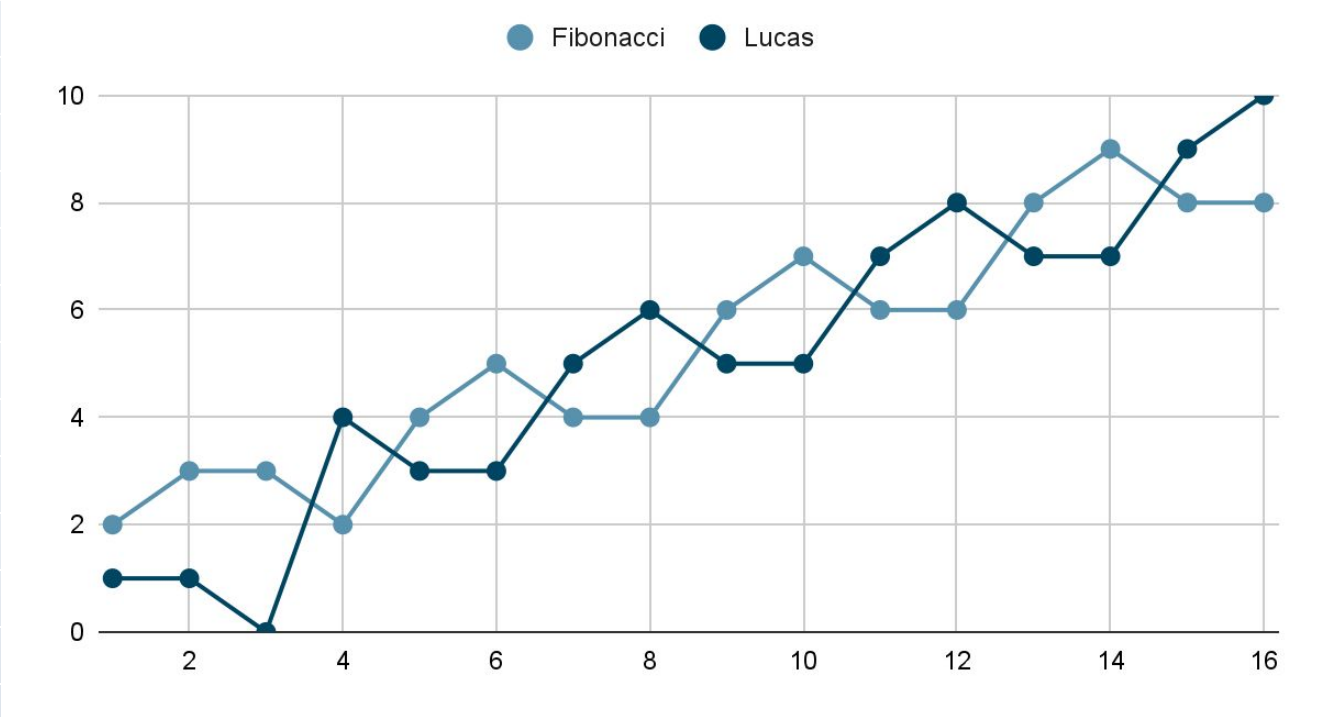
$$S_{4n} = F_{2n} L_{2n+2}$$
$$S_{4n+1} = F_{2n+2} L_{2n+1}$$
$$S_{4n+2} = F_{2n+3} L_{2n+1}$$
$$S_{4n+3} = F_{2n+2} L_{2n+3}$$

If n and m satisfy:

- ❑ $F_m \mid S_n$
- ❑ $n - 3m \leq -4$
- ❑ $n \geq 11$
- ❑ $n - m \geq 0$

Then $S_n / F_m = L_{n-m+2}$.

Indices of Divisors, Graphed



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A Second Trick?

Fibonacci and Trigonometry

When analyzing trigonometric identities, we find analogous identities using Fibonacci and Lucas numbers. The study of these similarities was dubbed "Fibonometry" in Conway and Ryba's original paper, a combination of "Fibonacci" and "Trigonometry". For example, note the similarities between

$$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha) \quad \text{and} \quad F_{2n} = F_n L_n$$

$$\cos(2\alpha) = \cos(\alpha)^2 - \sin(\alpha)^2 \quad \text{and} \quad 2L_{2n} = L_n^2 + 5F_n^2$$

We can see similarities between sine and the Fibonacci numbers as we do with the cosine and Lucas numbers. We apply this pattern in the rules to follow.

Fibonometry Rules

Conway and Ryba came up with the following rule for converting a Fibonacci Identity to a Trigonometric Identity.

Fibonometry rule. Replace an angle $\theta = p\alpha + q\beta + r\gamma + \cdots$ with a subscript $n = pa + qb + rc + \cdots$. Then replace $\sin\theta$ with $\frac{i^n F_n}{2}$ and $\cos\theta$ with $\frac{i^n L_n}{2}$. Finally, insert a factor -5 for any square of sines; thus, insert $(-5)^k$ for any term that contains $2k$ or $2k+1$ sines.

We came up with another rule, that only uses one step and doesn't have to look at the square of sines.

One-step Fibonometry rule. Replace an angle θ with a subscript n as before. Then, replace

$$\sin\theta \quad \text{with} \quad \frac{\sqrt{5}}{2} i^{n-1} F_n \quad \text{and} \quad \cos\theta \quad \text{with} \quad \frac{1}{2} i^n L_n.$$

Why do these rules work?

These rules work because of the similarities between the formulas of the Fibonacci and Lucas numbers and the cosine and sine formulas.

It is well known that the closed form for Fibonacci and Lucas Numbers are:

$$F_n = \frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}} \quad \text{and} \quad L_n = \varphi^n + (-\varphi)^{-n}$$

By Euler's Formula, we also have the following formulas for the cosine and sine functions.

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \quad \text{and} \quad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}.$$

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Generalizing Fibonometry

Lucas Sequences

We define Lucas sequences of the first kind as $U_n(P, Q)$. We have the following rules:

$$U_0(P,Q) = 0,$$
$$U_1(P,Q) = 1,$$
$$U_n(P,Q) = P \cdot U_{n-1}(P,Q) - Q \cdot U_{n-2}(P,Q) \text{ for } n > 1$$

We similarly define the Lucas sequence of the second kind as $V_n(P, Q)$. We also have the following rules:

$$V_0(P,Q) = 2,$$
$$V_1(P,Q) = P,$$
$$V_n(P,Q) = P \cdot V_{n-1}(P,Q) - Q \cdot V_{n-2}(P,Q) \text{ for } n > 1$$

Fibonometry with Lucas Sequences

Using similar rules to what we did previously, we derived the following rules that convert trigonometric identities to Lucas sequence identities.

Beyond Fibonometry rule. Replace an angle θ with a subscript n as before. Then, replace

$$\sin\theta \text{ with } \frac{\sqrt{D}}{2i}\left(\frac{-1}{\sqrt{Q}}\right)^n U_n \quad \text{and} \quad \cos\theta \text{ with } \frac{1}{2}\left(\frac{-1}{\sqrt{Q}}\right)^n V_n.$$

In this rule, we have that $D = P^2 - 4Q$. We see that this closely resembles our previous Fibonometry rules, where $D = 5$ and $Q = -1$.

One-step Fibonometry rule. Replace an angle θ with a subscript n as before. Then, replace

$$\sin\theta \quad \text{with} \quad \frac{\sqrt{5}}{2} i^{n-1} F_n \quad \text{and} \quad \cos\theta \quad \text{with} \quad \frac{1}{2} i^n L_n.$$

Example of Fibonometry with Lucas Sequences

We consider the identity $\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$.

We now convert this into an identity with the Lucas sequences. By using the rule on the previous slide we get that:

$$\frac{\sqrt{D}}{2i}\left(\frac{-1}{\sqrt{Q}}\right)^{m} U_m + \frac{\sqrt{D}}{2i}\left(\frac{-1}{\sqrt{Q}}\right)^{n} U_n = 2\frac{\sqrt{D}}{2i}\left(\frac{-1}{\sqrt{Q}}\right)^{\frac{m+n}{2}} U_{\frac{m+n}{2}} \frac{1}{2}\left(\frac{-1}{\sqrt{Q}}\right)^{\frac{m-n}{2}} V_{\frac{m-n}{2}}$$

This can be simplified to get:

$$U_m + (-\sqrt{Q})^{m-n} U_n = U_{\frac{m+n}{2}} V_{\frac{m-n}{2}}$$

Notice that we must have that *m* and *n* are of the same parities. So we can simplify the equation into:

$$U_m + Q^{\frac{m-n}{2}} U_n = U_{\frac{m+n}{2}} V_{\frac{m-n}{2}}.$$

Table of Converted Trig to Lucas Sequence Identities

Below is a table of trigonometric identities and there converted Lucas sequence identities.

Trigonometry	Lucas Sequence
$\sin^2\alpha + \cos^2\alpha = 1$	$V_n^2 - DU_n^2 = 4Q^n$
$\sin(-\alpha) = -\sin\alpha$	$-Q^nU_{-n} = U_n$
$\cos(-\alpha) = \cos\alpha$	$V_n = Q^nV_{-n}$
$\sin 2\alpha = 2\cos\alpha\sin\alpha$	$U_{2n} = U_nV_n$
$\cos 2\alpha = 2\cos^2\alpha - 1$	$V_{2n} = V_n^2 - 2Q^n$
$\cos 2\alpha = 1 - 2\sin^2\alpha$	$V_{2n} = 2Q^n - DU_n^2$
$\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$	$U_{3n} = 3Q^nU_n + DU_n^3$
$\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$	$V_{3n} = V_n^3 - 3Q^nV_n$
$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$	$2U_{m+n} = U_mV_n + U_nV_m$
$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$	$2V_{m+n} = V_mV_n + DU_mU_n$
$\sin\alpha\sin\beta = \frac{\cos(\alpha-\beta)-\cos(\alpha+\beta)}{2}$	$DU_mU_n = V_{m+n} - Q^nV_{m-n}$
$\cos\alpha\cos\beta = \frac{\cos(\alpha+\beta)+\cos(\alpha-\beta)}{2}$	$V_mV_n = V_{m+n} + Q^nV_{m-n}$
$\sin\alpha\cos\beta = \frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{2}$	$U_mV_n = U_{m+n} + Q^nU_{m-n}$
$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$	$U_m + Q^{\frac{m-n}{2}}U_n = U_{\frac{m+n}{2}}V_{\frac{m-n}{2}}$
$\cos\alpha - \cos\beta = -2\sin(\frac{\alpha+\beta}{2})\sin(\frac{\alpha-\beta}{2})$	$V_m - Q^{\frac{m-n}{2}}V_n = DU_{\frac{m+n}{2}}U_{\frac{m-n}{2}}$

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Any Questions?

Thank you!