Gabriel’s Theorem and the Subspaces Problem

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Outline

1. The Pairs of Subspaces Problem
2. Quiver Representations
3. Gabriel’s Theorem
4. The Triples of Subspaces Problem
The Pairs of Subspaces Problem
One-Subspace Problem

Note

All spaces will be assumed to be complex vector spaces.
One-Subspace Problem
One-Subspace Problem

Idea

Can we classify all cases in which one space is a subspace of another?

Definition (Isomorphism)

We say a vector space $V$ is isomorphic to $V'$ if there exists an isomorphism, defined to be a bijective linear map, from $V$ to $V'$. For pairs $(V, W)$ where $W$ is a subspace of $V$, we define an isomorphism from $(V, W)$ to $(V', W')$ to be an isomorphism from $V$ to $V'$ that takes $W$ to $W'$. 

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- We only care about spaces and their subspaces *up to isomorphism*.
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Definition (Isomorphism)

We say a vector space $V$ is \textit{isomorphic} to $V'$ if there exists an \textit{isomorphism}, defined to be a bijective linear map, from $V$ to $V'$. For pairs $(V, W)$ where $W$ is a subspace of $V$, we define an isomorphism from $(V, W)$ to $(V', W')$ to be an isomorphism from $V$ to $V'$ that takes $W$ to $W'$. 
One-Subspace Problem
One-Subspace Problem

Refinement

Can we classify, up to isomorphism, all pairs of spaces $(V, W)$ where $W$ is a subspace of $V$ and both spaces are finite dimensional?
Solution One-Subspace Problem

1. \( \dim V \geq \dim W \geq 0. \)
2. Let \( \dim W = m \) and \( \dim V = m + n. \)
3. Extend a basis of \( m \) elements for \( W \) to a basis of \( m + n \) elements for \( V \); let the additional \( n \) elements generate \( W' \).
4. Then \( V \cong W \oplus W', W \cong \mathbb{C}^m, W' \cong \mathbb{C}^n. \)
5. Thus, our classification is pairs of the form \( (\mathbb{C}^m \oplus \mathbb{C}^n, \mathbb{C}^n). \)
To make a harder problem, we consider two subspaces instead of just one:

**Problem**

Can we classify up to isomorphism all triples $(V, W_1, W_2)$ of finite-dimensional vector spaces such that $W_1$ and $W_2$ are subspaces of $V$?

We will solve this in a series of steps.
Step 1: Remove Excess

Step 1

Consider the subspace \( W_1 + W_2 \) of \( V \). As we did for the one-subspace problem, get a complement of this subspace in \( V \), \( W_3 \). Then
\[
V = (W_1 + W_2) \oplus W_3.
\]
Step 2: Remove Intersection

Let the intersection of $W_1$ and $W_2$ be $W_0$. Let $W_4$ and $W_5$ be complements of $W_0$ in $W_1$ and $W_2$ respectively. Then $W_1 = W_0 \oplus W_4$, $W_2 = W_0 \oplus W_5$, and $W_1 + W_2 = W_3 \oplus W_4 \oplus W_5$. 
Step 3: Putting it Together

Step 3

\[ V = (W_1 + W_2) \oplus W_3 = W_0 \oplus W_3 \oplus W_4 \oplus W_5, \quad W_1 = W_0 \oplus W_4, \quad \text{and} \]
\[ W_2 = W_0 \oplus W_5. \]

Letting \( a, b, c, d \) be the dimensions of \( W_0, W_3, W_4, W_5 \) respectively, our classification is:

\[
(C^a \oplus C^b \oplus C^c \oplus C^d, C^a \oplus C^c, C^a \oplus C^d)
\]
Quiver Representations
Quivers

Definition

A quiver is a directed graph.
**Quivers**

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A quiver is a directed graph.

**Examples**

\[ \bullet \rightarrow \bullet \]
**Quivers**

**Definition**

A quiver is a directed graph.

**Examples**

- [Graph 1](#)
- [Graph 2](#)
- [Graph 3](#)
**Quivers**

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**Examples**

![Quiver Diagram](image_url)
Dynkin Diagrams

**Definition**

Let $\Gamma$ denote a graph and $R_\Gamma$ be the adjacency matrix. The Cartan matrix of $\Gamma$ is $A_\Gamma = 2I - R_\Gamma$. 
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**Examples**
$A_n$:

```
• —— •  ······· • —— •
```
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**Examples**

- $A_n$:
  
  ![Diagram of $A_n$]

- $D_n$:
  
  ![Diagram of $D_n$]
Dynkin Diagrams

Examples

\[ E_6 : \]

\[
\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet
\]

\[
| \quad \quad \quad \quad \quad \quad \quad \quad |
\]

\[
\bullet
\]
 Dynkin Diagrams

Examples

\[ E_6 : \]
\[ E_7 : \]
Dynkin Diagrams

Examples

$E_6$:

$E_7$:

$E_8$:
Quiver Representation

Quiver Representations can help us solve problems.
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Examples

1. Classifying representations of $A_2$ solves the one subspace problem.
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Examples

1. Classifying representations of $A_2$ solves the one subspace problem.
2. Classifying representations of $A_3$ solves the pair of subspaces problem.
3. Classifying representations of $D_4$ solves the triples of subspaces problem.
Definition

A representation of a quiver $Q$ is an assignment of each vertex $i$ to a vector space $V_i$ and each edge $h_{ij}$ to a linear map $x_{ij} : V_i \to V_j$. 
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Definition

A subrepresentation $(W, y)$ of a quiver $Q$ satisfies $W_i \subset V_i$ for all $i$ and $y_{ij} : W_i \to W_j$ is a linear map.
Quiver Representation

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Definition
A subrepresentation $(W, y)$ of a quiver $Q$ satisfies $W_i \subset V_i$ for all $i$ and $y_{ij}: W_i \to W_j$ is a linear map.

Definition
The direct sum of two representations $(V, x)$ and $(W, y)$ is $(V \oplus W, x \oplus y)$.
Indecomposable Representations

**Definition**

A representation of a quiver is indecomposable if it cannot be written as the direct sum of subrepresentations.

**Example**

The quiver consisting of one vertex and one self-loop has infinitely many indecomposable representations, which are $V = \mathbb{C}^n$ and $f$ is a $n \times n$ Jordan block.

**Remark**

Considering indecomposable representations are helpful because they can be thought of building blocks for all representations.
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Gabriel’s Theorem
Roots

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$A_{\Gamma}$ defines an inner product $B(x, y) = x^T A_{\Gamma} y$. 
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**Definition**

A root is a vector $\alpha$ with integer components which takes the smallest possible value of $B_\Gamma(x, x)$.  

**Proposition**

*There are finitely many roots of $B_\Gamma$.*
Definition

A simple root $\alpha_i$ is in the form $\alpha_i = (0, \ldots, 1, \ldots, 0)$ where the $i$th term is 1.
Definition

A simple root \( \alpha_i \) is in the form \( \alpha_i = (0, \ldots, 1, \ldots, 0) \) where the \( i \)th term is 1.

Lemma

For a root \( \alpha = \sum_{i=1}^{n} k_i \alpha_i \), either all \( k_i \geq 0 \) or all \( k_i \leq 0 \).
Definition
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Lemma
For a root $\alpha = \sum_{i=1}^{n} k_i \alpha_i$, either all $k_i \geq 0$ or all $k_i \leq 0$.

Definition
A root $\alpha = \sum_{i=1}^{n} k_i \alpha_i$ is called a positive root if $k_i \geq 0$ for all $i$. 
Definition
The dimension vector of a representation \( V = (V_1, \ldots, V_n) \) of \( Q \) is

\[
d(V) = (\dim(V_1), \ldots, \dim(V_n)).
\]
Gabriel’s Theorem

Definition

The dimension vector of a representation $V = (V_1, \ldots, V_n)$ of $Q$ is

$$d(V) = (\dim(V_1), \ldots, \dim(V_n)).$$

Theorem

A quiver $Q$ of type $A_n$, $D_n$, $E_6$, $E_7$, $E_8$, has finitely many indecomposable representations. Furthermore, the dimension vector of an indecomposable representation corresponds with a positive root and every positive root corresponds with one indecomposable representation.
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Definition
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A quiver $Q$ of type $A_n$, $D_n$, $E_6$, $E_7$, $E_8$, has finitely many indecomposable representations. Furthermore, the dimension vector of an indecomposable representation corresponds with a positive root and every positive root corresponds with one indecomposable representation.

Remark
The proof of this theorem involves looking at reflection functors, which preserves indecomposable representations and dimension.
The Triples of Subspaces Problem
The Triples of Subspaces Problem

The Problem

Consider the quiver $D_4$ with the following orientation of arrows and labeling of vertices.

We want to find indecomposable representations of the above quiver.

While we could do a similar process as with the pairs of subspaces problem, the process is much more complicated.
The Triples of Subspaces Problem

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Consider the quiver $D_4$ with the following orientation of arrows and labelling of vertices.
The Triples of Subspaces Problem

The Problem

Idea

Consider the quiver $D_4$ with the following orientation of arrows and labelling of vertices.

\[ \begin{array}{c}
2 & \rightarrow & 1 & \leftarrow & 4 \\
\uparrow & & & & \\
\downarrow & & & & \\
3 & & & &
\end{array} \]
The Triples of Subspaces Problem

The Problem

Idea

Consider the quiver $D_4$ with the following orientation of arrows and labelling of vertices.

```
2 ---> 1  <--- 4
   |    ↑
  3
```

We want to find indecomposable representations of the above quiver.
Consider the quiver $D_4$ with the following orientation of arrows and labelling of vertices.

We want to find indecomposable representations of the above quiver. While we could do a similar process as with the pairs of subspaces problem, the process is much more complicated.
Using Gabriel’s Theorem

Gabriel’s Theorem states that the dimension vectors of the indecomposable representations and the positive roots of $\Gamma$ have a 1-to-1 correspondence. If we can find the positive roots of $\Gamma_{D_4}$, we can match these with indecomposable representations of $D_4$. 

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Gabriel’s Theorem states that the dimension vectors of the indecomposable representations and the positive roots of $B_Γ$ have a 1-to-1 correspondence.

If we can find the positive roots of $B_{D_4}$, we can match these with indecomposable representations of $D_4$. 
Solving the Problem

To compute the positive roots of $B^D_4$, we first compute the adjacency matrix $R^D_4$ as follows.

$$R^D_4 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

We can then compute $A^D_4$ by using the formula $A^D_4 = 2I_4 - R^D_4$.

$$A^D_4 = \begin{pmatrix} 2 & -1 & -1 & -1 \\ -1 & 2 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ -1 & 0 & 0 & 2 \end{pmatrix}$$
Solving the Problem

**Solution**

To compute the positive roots of $B_{D_4}$, we first compute the adjacency matrix $R_{D_4}$ as follows.

\[
R_{D_4} = \begin{pmatrix}
0 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}
\]

We can then compute $A_{D_4}$ by using the formula $A_{D_4} = 2I - R_{D_4}$.
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To compute the positive roots of $B_{D_4}$, we first compute the adjacency matrix $R_{D_4}$ as follows.

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1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}$$
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Solution

To compute the positive roots of $B_{D_4}$, we first compute the adjacency matrix $R_{D_4}$ as follows.

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We can then compute $A_{D_4}$ by using the formula $A_{D_4} = 2I_d - R_{D_4}$. 
Solving the Problem

Solution

To compute the positive roots of $B_{D_4}$, we first compute the adjacency matrix $R_{D_4}$ as follows.

$$R_{D_4} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

We can then compute $A_{D_4}$ by using the formula $A_{D_4} = 2\text{Id} - R_{D_4}$.

$$A_{D_4} = \begin{pmatrix} 2 & -1 & -1 & -1 \\ -1 & 2 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ -1 & 0 & 0 & 2 \end{pmatrix}$$
Solving the Problem

Solution

Let $B(x, x) = x^T A_{D_4} x = 2$ where $x$ is a root and let $x$ be some vector in $\mathbb{Z}^4$ such that $x = (a, b, c, d)$ for some $a, b, c, d \in \mathbb{Z}$. Then in order for $x$ to be a positive root, we want $a, b, c, d \geq 0$. 

Carrying out the multiplication we get

$$B(x, x) = (a, b, c, d) \begin{pmatrix} 2 & -1 & -1 & -1 \\ -1 & 2 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ -1 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = 2a^2 + 2b^2 + 2c^2 + 2d^2 - 2ab - 2ac - 2ad = 2.$$
Solving the Problem

Solution

Let $B(x, x) = x^T A_{D_4} x = 2$ where $x$ is a root and let $x$ be some vector in $\mathbb{Z}^4$ such that $x = (a \ b \ c \ d)$ for some $a, b, c, d \in \mathbb{Z}$. Then in order for $x$ to be a positive root, we want $a, b, c, d \geq 0$.

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$$B(x, x) = (a \ b \ c \ d) \begin{pmatrix} 2 & -1 & -1 & -1 \\ -1 & 2 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ -1 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$
Solving the Problem

Solution

Let $B(x, x) = x^TA_{D_4}x = 2$ where $x$ is a root and let $x$ be some vector in $\mathbb{Z}^4$ such that $x = (a \ b \ c \ d)$ for some $a, b, c, d \in \mathbb{Z}$. Then in order for $x$ to be a positive root, we want $a, b, c, d \geq 0$.

Carrying out the multiplication we get

\[
B(x, x) = (a \ b \ c \ d) \begin{pmatrix} 2 & -1 & -1 & -1 \\ -1 & 2 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ -1 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = 2a^2 + 2b^2 + 2c^2 + 2d^2 - 2ab - 2ac - 2ad = 2
\]
Solving the Problem

Solution

So we must solve for

\[ a^2 + b^2 + c^2 + d^2 - ab - ac - ad = 1 \]
Solving the Problem

Solution

So we must solve for

\[ a^2 + b^2 + c^2 + d^2 - ab - ac - ad = 1 \]

It turns out that there are only 12 solutions to this equation where \( a, b, c, d \geq 0 \). These solutions are:

- \((1, 0, 0, 0)\)
- \((0, 1, 0, 0)\)
- \((0, 0, 1, 0)\)
- \((0, 0, 0, 1)\)
- \((1, 1, 0, 0)\)
- \((1, 0, 1, 0)\)
- \((1, 0, 0, 1)\)
- \((1, 1, 1, 0)\)
- \((1, 1, 0, 1)\)
- \((1, 0, 1, 1)\)
- \((1, 1, 1, 1)\)
- \((2, 1, 1, 1)\)
Solving the Problem

Solution

So we must solve for

\[ a^2 + b^2 + c^2 + d^2 - ab - ac - ad = 1 \]

It turns out that there are only 12 solutions to this equation where \( a, b, c, d \geq 0 \). These solutions are:

\[
\begin{align*}
(1 & 0 & 0 & 0) & (0 & 1 & 0 & 0) & (0 & 0 & 1 & 0) & (0 & 0 & 0 & 1) \\
(1 & 1 & 0 & 0) & (1 & 0 & 1 & 0) & (1 & 0 & 0 & 1) & (1 & 1 & 1 & 0) \\
(1 & 1 & 0 & 1) & (1 & 0 & 1 & 1) & (1 & 1 & 1 & 1) & (2 & 1 & 1 & 1)
\end{align*}
\]
The Solution

Solution

These solutions correspond to the following indecomposable representations.
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These solutions correspond to the following indecomposable representations:

\[
\begin{array}{cccccccccccc}
0 \rightarrow & 1 & \leftarrow & 0 & 1 \rightarrow & 0 & \leftarrow & 0 & 0 \rightarrow & 0 & \leftarrow & 0 & 0 \rightarrow & 0 & \leftarrow & 1 \\
\uparrow & & & & \uparrow & & & & \uparrow & & & & \uparrow & & & & \uparrow \\
0 & & & & 0 & & & & 1 & & & & 0 & & & & 0 \\
1 \rightarrow & 1 & \leftarrow & 0 & 0 \rightarrow & 1 & \leftarrow & 0 & 0 \rightarrow & 1 & \leftarrow & 1 & 1 \rightarrow & 1 & \leftarrow & 0 \\
\uparrow & & & & \uparrow & & & & \uparrow & & & & \uparrow & & & & \uparrow \\
0 & & & & 1 & & & & 0 & & & & 1 & & & & 1 \\
1 \rightarrow & 1 & \leftarrow & 1 & 0 \rightarrow & 1 & \leftarrow & 1 & 1 \rightarrow & 1 & \leftarrow & 1 & 1 \rightarrow & 1 & \leftarrow & 2 & \leftarrow & 1 \\
\uparrow & & & & \uparrow & & & & \uparrow & & & & \uparrow & & & & \uparrow \\
0 & & & & 1 & & & & 1 & & & & 1 & & & & 1 \\
\end{array}
\]
The Triples of Subspaces Problem

The Solution

Solution

Note that 3 of these solutions are not injective and thus, cannot contribute to our triples of subspaces problem. Specifically, these are the following indecomposable representations:
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\[
\begin{array}{c}
1 \\
\uparrow \\
0
\end{array}
\quad
\begin{array}{c}
0 & 0 & 0 \\
\uparrow & \uparrow & \\
0 & 0 & 0
\end{array}
\quad
\begin{array}{c}
0 \\
\uparrow \\
1
\end{array}
\quad
\begin{array}{c}
0 \\
\uparrow \\
0
\end{array}
\quad
\begin{array}{c}
1
\end{array}
\]
Relating Back to the Triples of Subspaces Problem
The Problem

Can we classify up to isomorphism all quadruples \((V, W_1, W_2, W_3)\) of finite-dimensional vector spaces such that \(W_1, W_2, W_3\) are subspaces of \(V\)?
Relating Back to the Triples of Subspaces Problem

**The Problem**

Can we classify up to isomorphism all quadruples \((V, W_1, W_2, W_3)\) of finite-dimensional vector spaces such that \(W_1, W_2, W_3\) are subspaces of \(V\)?

We can relate this to the quivers we found by letting the numbers at each vertex represent the dimensions of \(V, W_1, W_2,\) and \(W_3\).
Relating Back to the Triples of Subspaces Problem

Using a method similar to the beginning of the presentation, paired with the indecomposable representations, we can deduce the following.

\[ W_1 \oplus W_2 \oplus W_3 \cong L_{m_1} I_{m_1} \] where the \( I \) are the indecomposable representations and \( m_I \) is its multiplicity.
Solution

Using a method similar to the beginning of the presentation, paired with the indecomposable representations, we can deduce the following.
Relating Back to the Triples of Subspaces Problem

Solution

Using a method similar to the beginning of the presentation, paired with the indecomposable representations, we can deduce the following.

\[ W_1 \rightarrow V \leftarrow W_3 \]

\[ W_2 \]

is \( \bigoplus_I m_I \cdot I \) where the \( I \) are the indecomposable representations and \( m_I \) is its multiplicity.
Relating Back to the Triples of Subspaces Problem
Solution

\[ 0 \rightarrow 1 \leftarrow 0 \]

For this representation, the multiplicity of it is equal to the dimension of the complement of \( W_1 + W_2 + W_3 \) in \( V \).
Relating Back to the Triples of Subspaces Problem

Solution

0 → 1 ← 0

For this representation, the multiplicity of it is equal to the dimension of the complement of $W_1 + W_2 + W_3$ in $V$.

1 → 1 ← 0

For this case, if we let $W_4$ be the intersection of $W_1$, $W_2$, and $W_3$ and $W_5$ be the complement of $W_4$ in $W_1$, then the multiplicity is the dimension of the direct sum of the complement of $W_1 + W_2 + W_3$ in $V$ and $W_5$. 
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