Relating Wave and Particle Statistics in Pilot-Wave Hydrodynamics; and an Extension of Floquet's Theorem to Periodically-Driven PDEs

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A number of macroscopic analogues of quantum mechanics have drawn inspiration from the "walking droplet" system of Couder and Fort: a system of oil droplets surfing over a vibrating fluid bath, propelled forward by self-generated waves. In particular, recent authors have observed a correspondence between the probability density function (PDF) of the walking droplet and the mean wave field (MWF) of the underlying bath, analogizing the fundamental relationship between quantum probabilities and the wavefunction. Although this correspondence has been proven in certain single-droplet systems, with particular system geometries and droplet-pressure models, the principle is believed to hold more generally. In this direction, we prove an extension of Floquet's Theorem to periodically-driven PDEs, and we leverage it to rigorously derive the same PDF-MWF correspondence for general pilot-wave models, including droplet systems with arbitrary droplet count, system geometry, and droplet-pressure models. Our work applies functional-analytical methods for non-autonomous equations, demonstrating the utility of such techniques in understanding pilot-wave dynamics at a rigorous level. We demonstrate our results in 1D numerical simulations for various domain and droplet scenarios, and, looking forward, we discuss how they might apply to hydrodynamic analogues of quantum entanglement and measurement.

I. INTRODUCTION

In quantum mechanics, a particle is identified with its spatially-extended *wavefunction*. In the case of a single particle in $\mathbb{R}^d$ under an energy potential $V : \mathbb{R}^d \to \mathbb{R}$, the wavefunction $\psi \in L^2(\mathbb{R}^d; \mathbb{C})$ evolves according to the *Schrödinger equation*:

$$i\partial_t \psi = -\nabla^2 \psi + V\psi. \tag{I.1}$$

In turn, this dynamically-evolving wavefunction yields all the statistics of the quantum particle. For instance, if one were to measure the position of a particle, the probability density ρ of the position $q_p \in \mathbb{R}^d$ would satisfy *Born's rule*:

$$\rho(q_p = q) = |\psi(q)|^2. \tag{I.2}$$

In textbook quantum mechanics, q_p cannot be interpreted as a well-defined particle position, at least until the aforementioned measurement is performed. David Bohm's "pilot-wave" interpretation of quantum mechanics offers a somewhat-more-classical alternative, however, positing that q_p exists at all times as a deterministic, "hidden-variable" position of the particle [1, 2]. In this view, (I.2) becomes a fundamental correspondence between the dynamically-evolving wavefunction and the statistics of a point-particle.

In order to further investigate this relationship, we move from Bohm's quantum pilot-wave theory toward a macroscale, hydrodynamic analogue: the "walking droplet" dynamics of Couder and Fort [3]. In this classical pilot-wave system, a millimetric oil droplet bounces on a vibrating fluid bath of silicone oil, propelled forward by its self-generated waves. The high surface tension and fast vibration of the fluid bath prevent the droplet from coalescing into the bath—instead, the droplet maintains a periodic bouncing motion on the bath's surface. Under the right conditions, the vibration of the system brings the droplet into resonance with its wave field, allowing it to "walk" across the surface of the bath.

The walking droplet system gives rise to classical analogues of several characteristic quantum behaviors: single-particle slit diffraction [3–6], "tunneling" past submerged barriers [7, 8], attaining quantized orbits [9, 10], following "surreal" trajectories characteristic of Bohmian mechanics [11], and displaying quantum-like statistics in an analogue

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of the quantum corral [12]. Motivated by the successes of "hydrodynamic quantum analogues" [13], we would like to see if the Born rule itself (I.2) could be analogized at the macro-scale.

In the walking droplet system, we consider two key statistical quantities: the *mean wave field* (MWF) and the *probability density function* (PDF). More specifically, the MWF is the time-averaged wave field of the underlying bath, analogous to the wavefunction in a stationary solution of the Schrödinger equation (I.1). The PDF is the long-term probability distribution of a walking droplet's position, analogous to the probability distribution of a Bohmian point-particle, as expressed by the Born rule (I.2).

Unlike the linear wave dynamics of quantum mechanics, the nonlinear dynamics of the coupled droplet-wave system blurs any simple relationship between the PDF and MWF. In 2018, Durey, Milewski, and Bush [14] derived a mathematical relationship between the PDF and MWF in particular walking droplet scenarios. They rigorously proved the relationship for a free domain in the droplet models of Milewski et al. [15], Fort et al. [9], and Oza et al. [16], and argued that a similar relationship should occur more broadly. In 2020, by exploiting the radial symmetry of the system, Durey, Milewski, and Wang [17] rigorously outline the case of a single droplet in a circular corral. For these cases, they find that if there exists a stationary probability distribution $\mu(x)$ on a walking droplet domain $\Omega \subset \mathbb{R}^2$, for which the droplet position and the system dynamics are ergodic (or periodic), then the MWF $\bar{\eta}(x)$ satisfies

$$\bar{\eta}(x) = \int_{\Omega} \eta_B(x, y)\mu(y)\, dy, \tag{I.3}$$

where $\eta_B(x, y)$ is the MWF of a stationary, bouncing droplet centered at the position y. Given a PDF and relevant domain parameters, one can use (I.3) to effectively predict the MWFs of applicable walking droplet systems. This correspondence is pictured in Figure 1, albeit, in an elliptical corral, outside the scope of existing results.

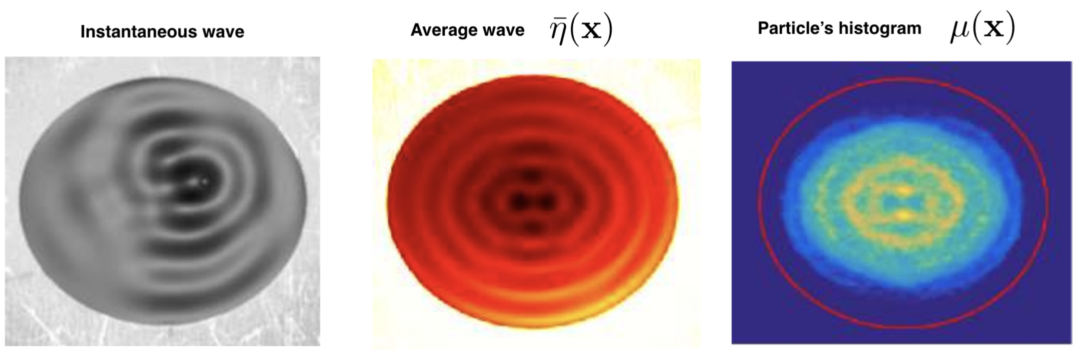


FIG. 1: A walking droplet's instantaneous wave field (left), mean wave field (center), and probability density function (right) in an elliptical corral. A relationship between the probability density and mean wave field has been widely observed, but only proven in particular cases—the elliptical corral not among them. Reproduced from [18] with permission.

As suggested, these results have several limitations. For one, Durey et al. carried out their analysis with an idealized droplet-pressure model, where the droplet and bath make contact only instantaneously in each bouncing period. This assumption allowed them to reduce their wave equation to a homogeneous version; at this point, they could apply Floquet theory to understand the time-averaged dynamics of the wave field. This step introduces another obstacle to generalization: Floquet's theory only necessarily holds for finite-dimensional systems. In both the free domain and the circular corral, Durey et al. were able to diagonalize the full system along the eigenmodes of the Laplacian; this step reduces the PDE to a family of finite-dimensional ODEs, each of which amenable to Floquet-type results. Unfortunately, this property does not carry forward to generic geometries.

To handle the general case, then, one requires an analogue of Floquet's theory that holds for *inhomogeneous*, *infinite-dimensional* systems. Although some authors have made steps in developing Floquet theory for *bounded* operators on a Banach space [19], our wave operators do not fall into this class.

We pursue this program in the present work, recovering a relationship between the PDF and MWF for generic pilot-wave systems—including walking droplet systems with one or more droplets, with arbitrary geometries, and with arbitrary droplet-pressure models. To illustrate our approach, we first derive a finite-dimensional version of our particle-wave correspondence (Theorem IV.3) using classical Floquet theory, lifting the instantaneous-droplet-pressure hypothesis of Durey et al. We then turn to our main result (Theorem IV.9), proving that this correspondence extends to infinite-dimensional systems (and thus arbitrary geometries). Namely, by using the theory of *evolution systems* [20] to develop an infinite-dimensional analogue of Floquet's theorem, we are able to generate abstract solutions to our wave equation and retrieve the relationship (I.3) for periodic, driven-dissipative systems on Banach spaces, with ergodic (or periodic) forcing at each period and only mild regularity conditions. In short, we recover the following result:

Claim I.1. *Suppose we have a 2D domain $\Omega \subset \mathbb{R}^2$, a wave height field $\eta : \Omega \to \mathbb{R}$, and a fixed vibration period $T > T_{Faraday}$. Suppose there are one or more walking droplets in Ω, and that the wave-droplet interaction in a single period of vibration is parametrized (or well-approximated) by a finite collection of parameters $\vec{\alpha} = \{\alpha_1, ..., \alpha_N\}$; for instance, these parameters can quantify droplet positions, velocities, or phase offsets of droplet bounces. Let $\eta_B(x, \vec{\alpha})$ be the mean wave field at $x \in \Omega$ corresponding to a "fixed bouncing configuration", i.e., where $\vec{\alpha}$ is constant in time.*

If the system dynamics are ergodic with a stationary probability distribution $\rho(\vec{\alpha})$, then the mean wave field $\overline{\eta}(x)$ is related to ρ by

$$\overline{\eta}(x) = \int_{\mathbb{R}^N} \eta_B(x, \vec{\alpha})\rho(\vec{\alpha})\, d\vec{\alpha}. \tag{I.4}$$

We formalize this result in Theorem IV.9, and, as an example, we highlight (Example IV.6) how our result applies to the model of Milewski et al. [15] for arbitrary system geometries and droplet-pressure models.

To support our result numerically, we apply the 1D walking droplet model proposed by Nachbin [21]. Along with its non-instantaneous droplet pressure, Nachbin's model is able to exactly resolve the topography of the fluid bath using an appropriate conformal map, allowing us to more-accurately deduce the effect of the system geometry—and putting us outside the scope studied by Durey et al. By simulating Nachbin's walking droplet model in MATLAB, we verify the PDF-MWF relationship for various domain and droplet scenarios.

Looking forward, we show how our results can be applied toward a hydrodynamic analogue of quantum entanglement and measurement. Nachbin [21] has observed correlation between the dynamics of two droplets separated by a submerged barrier in a two-cavity domain. By investigating this analogue of "entanglement" through the lens of the PDF and MWF, we may be able to study (or diagnose) particle entanglement using only their shared wave field. Moreover, we may also observe how particle correlation and statistics collapse when we impose new cavities upon an already-evolving droplet system, serving as a hydrodynamic analogue to quantum measurement.

II. APPLICATIONS

Before delving into our mathematical argument, we highlight various applications of our result to current themes in hydrodynamic pilot-wave research.

a. Multiple Droplets, Arbitrary Pressure Models, Arbitrary Geometries. As suggested above, Claim I.1 directly extends the results of Durey, Milewski, and Bush [14] and Durey, Milewski, and Wang [17] by covering the full range of walking droplet models and settings.

For one, we note that Claim I.1 encompasses systems with several droplets. If $x_1, ..., x_N \in \Omega$ are the positions of N droplets, distributed according to a stationary distribution $\rho = \rho(x_1, ..., x_N)$, the mean wave field $\overline{\eta}$ is given by

$$\overline{\eta}(x) = \int_{\Omega^N} \eta_B(x, x_1, ..., x_N)\rho(x_1, ..., x_N)\, dx_1 \cdots dx_N,$$

where $\eta_B(x, x_1, ..., x_N)$ is the mean wave field corresponding to N stationary bouncers. Investigations of multi-droplet interactions have uncovered a variety of unique behaviors, including orbiting and promenading bound states [22–24], regular lattices [25] and ring structures [26], and long-distance correlations between droplets [21]. We investigate the mean wave field of the latter in Section V.

Likewise, our results place no constraints on the geometry of the bath, putting a number of other interesting droplet phenomena within reach. The MWF-PDF correspondence has already been observed in elliptical corrals, for instance [27], in agreement with Claim I.1.

Finally, although phenomenological and reduced models of walking droplets have employed instantaneous droplet-pressure effects on the wave field [9, 16, 28, 29], more complete models account for the spring-like interaction between the droplet and the wave. With these dynamics in place, the model of Milewski et al. [15] is able to capture important nuances missed by earlier models, such as modulations of the vertical bouncing state [30].

b. Non-Resonant Walking Droplets. Primkulov et al. [31] have recently undertaken a study of droplets bouncing *out of resonance* with the underlying bath. They found that non-resonant bouncing explains a number of subtle details of the droplet system, including the intermittent "backtracking" behavior reported by Perrard et al. [32] and the intermittent mode switching seen in high memory systems.

There are two ways to incorporate such effects into the current theory. First, if an instantaneous pressure model suffices, one can model a single droplet using its position x and its impact phase θ. If both are distributed according to a stationary distribution $\rho = \rho(x, \theta)$, then the mean wave field $\overline{\eta}$ is given by

$$\overline{\eta}(x) = \int_{\Omega \times S^1} \eta_B(x, x', \theta')\rho(x', \theta')\, dx' d\theta',$$

where $\eta_B(x, x', \theta')$ is the mean wave field of a bouncer with a consistent impact phase. If an instantaneous pressure model does *not* suffice, we must account for the fact that a single period $[0, T]$ of wave oscillation could contain effects from *two* droplet bounces: the end of the previous bounce (at $t < 0$) and the start of a new bounce (at $t \in [0, T]$). We could thus model a droplet as carrying two impact phases, θ_1 and θ_2, and proceed as previously. If $\theta_1 = \theta_2$ with high probability, this reduces to the previous equation.

c. Droplet-Inspired Pilot-Wave Theories. Finally, the walking droplet system has inspired a wider investigation of classical pilot-wave systems, both to explore a wider parameter regime than available in the laboratory [33] and to investigate further (and more faithful) classical analogues of quantum phenomena [34–36]. In one parameter regime, the model of Darrow and Bush [36] has been shown to reproduce key features hypothesized in de Broglie's early "double-solution" model of quantum mechanics [37] and to recover a quantitative match to single- and double-slit Fraunhofer patterns [38]; in another parameter regime, it has been proved to converge in the non-relativistic limit to single-particle Bohmian mechanics [39].

In the latter model, the droplet and wave continuously interact, with the wave oscillating regularly over the Compton period $T_c = 2\pi/\omega_c = 2\pi\hbar/mc^2$. Although the "droplet pressure" depends on the full path of the particle over one period, we could approximate it as depending only on the mean position and velocity of the particle over that period:

$$P(x([0, T])) \approx P(\overline{x}, \overline{v}).$$

With this approximation in hand, the mean wave field could be recovered using Claim I.1 and a PDF $\rho(x, v)$ of the particle's position and velocity.

III. BACKGROUND ON PREVIOUS RESULTS

Durey, Milewski, and Bush [14], and Durey, Milewski, and Wang [17] prove (I.3) in the case of a free domain and circular corral, respectively, using the trajectory equation of Molaçek and Bush [28] to model the droplet and the quasi-potential flow equations of Milewski et al. [15] to model the surface of the fluid bath:

$$\begin{aligned} \phi_t &= -G(1 - \Gamma\cos(4\pi t))\eta + Bo\Delta\eta + 2Re^{-1}\Delta\phi - P_0, \\ \eta_t &= \mathrm{DtN}(\phi) + 2Re^{-1}\Delta\eta, \end{aligned} \tag{III.1}$$

with appropriate (homogeneous) boundary conditions on ϕ and η. Here, η is the free-surface displacement, ϕ is the velocity potential, Re is the Reynolds number, Bo is the Bond number, and $P_0 = P_0(x, t)$ is the droplet pressure; in the studies of Durey, Milewski, and Bush and Durey, Milewski, and Wang, the forcing pressure is taken to be instantaneous and fully local, i.e., $P_0 \sim \delta^2(x - x_0)\delta(t - t_0)$. The coefficient $-G(1 - \Gamma\cos(4\pi t))$ represents the vertical shaking of the bath, with gravitational constant G, amplitude Γ, and period $1/2$ corresponding to the $(2, 1)$ walking mode [28]. Finally, the operator DtN is the geometry-dependent Dirichlet-to-Neumann map from the velocity potential ϕ to the vertical velocity ϕ_z at the surface of the bath.

In both the free domain and the circular corral, the result (I.3) was obtained by decomposing $L(t)$ along a basis of Bessel functions, thereby diagonalizing both Δ and DtN. This procedure reduces (III.1) to a family of finite-dimensional ODEs. Moreover, by supposing that P_0 acts instantaneously, these equations can be further reduced to *homogeneous* ODEs over each bouncing period. These hypotheses thus allowed Durey et al. to apply Floquet's theory to analyze the time-averaged wave field, recovering (I.3) explicitly.

As we show in Section IV A, their analysis can be extended to generic finite-dimensional systems using classical techniques, thus lifting their hypotheses on P_0 and on the number of droplets. For an infinite-dimensional function space, however, it requires strict hypotheses on the wave operator $L(t) : (\phi, \eta) \mapsto (\phi_t, \eta_t)$. For instance, if $L(t)$ is normal, or if the Dirichlet-to-Neumann map commutes with the Laplacian in (III.1) (as it did in Durey et al.'s case), then $L(t)$ can be decomposed into finite-dimensional operators as before. Alternatively, if $L(t)$ is bounded, one can use the extended Floquet theory of Albasrawi [19] to retrieve the result. However, diagonalizability does not hold for arbitrary system geometries, and boundedness does not generally hold for any differential operator, forcing us to look elsewhere for a more general proof of (I.3). We carry out such a procedure in the next section.

IV. RELATING THE MEAN WAVE FIELD TO THE PROBABILITY DENSITY

In order to relate the MWF and PDF of the droplet system, we model the wave field as an infinite-dimensional vector in a *Banach space* $\mathcal{B}$. We recall:

Definition IV.1. A *Banach space* $\mathcal{B}$ is a real or complex vector space, equipped with a norm $\|\cdot\|$ and with the property of "completeness". That is, if $u_k \in \mathcal{B}$ is a sequence for which $\|u_k - u_\ell\| \to 0$ as $k, \ell \to \infty$, then the sequence converges to a limit $u \in \mathcal{B}$.

Vector spaces of physical interest typically fall into this class, such as:

1. If $\Omega \subset \mathbb{R}^d$ is a fixed domain, the space $L^\infty(\Omega; \mathbb{R}^m)$ of bounded vector fields on Ω is a Banach space. For any bounded vector field $f : \Omega \to \mathbb{R}^m$, we define the norm $\|f\|_\infty \doteq \sup_{x\in\Omega} |f(x)|$.

2. In the same setting, the space $W^{\infty,k}(\Omega; \mathbb{R}^m)$ of vector fields with bounded k^{th} derivatives is a Banach space, with norm $\|f\|_{\infty,k} = \sum_{\ell\le k} \|\nabla^{\otimes \ell} f\|_\infty$.

3. Hilbert spaces—used widely in quantum mechanics—are special cases of Banach spaces. If $\langle \cdot, \cdot \rangle$ is the inner product on a Hilbert space H, the norm $\|\psi\|_H \doteq \sqrt{\langle \psi, \psi \rangle}$ furnishes H with a Banach space structure.

Notably, we can identify $L^\infty = W^{\infty,0}$. The derivative ∇ can be seen as a linear map from $W^{\infty,k}$ to $W^{\infty,k-1}$ for any k. To study a wave equation like ours (III.1), one might be interested in the case $\mathcal{B} = W^{\infty,2}$, for which our solution has two well-defined spatial derivatives.

With this in mind, we write our wave equation (III.1) as an abstract differential equation on a Banach space $\mathcal{B}$:

$$\dot{y}(t) = L(t)y + b(t, X(t)). \tag{IV.1}$$

Here, $y(t)$ is our time-evolving wave field, $L(t)$ is our wave operator, $X(t)$ is the position of the droplet on impact, and $b(t, X(t))$ is the pressure applied by the droplet on y. In the droplet setting, we identify $y = (\eta, \phi)$ and, as before, we identify $L(t)$ with the linear wave operator of (III.1).

Remark. It is important to note that L is *not* generally well-defined as a map from $\mathcal{B}$ to $\mathcal{B}$. For instance, if $\mathcal{B} = W^{\infty,2}$ and L involves two spatial derivatives, we can generically only say that $Ly \in L^\infty$ for any $y \in W^{\infty,2}$. This is a typical feature of differential equations; such an operator is said to be *unbounded* on $\mathcal{B}$. Conversely, if $Ly \in \mathcal{B}$ for all $y \in \mathcal{B}$, the operator L is said to be *bounded*.

Even with an unbounded operator, one often wants to keep working within the setting of $\mathcal{B}$. To this end, we associate to any unbounded operator L a subspace $\mathrm{dom}(L) \subset \mathcal{B}$ such that, if $y \in \mathrm{dom}(L)$, we have $Ly \in \mathcal{B}$; this subspace is known as the *domain* of L. We can resolve the case discussed above, then, by taking $\mathcal{B} = L^\infty$ and $\mathrm{dom}(L) = W^{\infty,2} \subset L^\infty$.

In this case (and those we study below), the domain $\mathrm{dom}(L)$ has the property that it is "dense" in $\mathcal{B}$. That is, for any $y \in \mathcal{B}$, there is a sequence $y_k \in \mathrm{dom}(L)$ such that $y_k \to y$ as $k \to \infty$. This property is critical; it means that, even if L cannot be applied everywhere in $\mathcal{B}$, it can be applied *nearly* everywhere.

A. The Finite-Dimensional Case

Before proving our main result, we take a step back from operator theory and look at the case where $\mathcal{B} = \mathbb{R}^d$ (or $\mathbb{C}^d$) is finite-dimensional. This case can be handled using classical methods of linear algebra, and helps to illustrate our technique before moving to the infinite-dimensional setting. Physically, the finite-dimensional case represents a fixed numerical discretization of our wave equation.

As a starting point, for a fixed period $T > 0$, consider the differential equation

$$\dot{y} = L(t)y + b(t), \tag{IV.2}$$

where $L(t) \in \mathbb{C}^{d\times d}$ and $b(t) \in \mathbb{C}^d$ are continuous and T-periodic. Comparing against (IV.1), we see that this equation corresponds to a stationary droplet $X(t) = \mathit{const}$. The following result shows how to apply Floquet's theory to an inhomogeneous equation, thus lifting the instantaneous-droplet-pressure hypothesis employed by Durey et al. [14]:

Lemma IV.2. *Suppose $L(t)$ and $b(t)$ are T-periodic and continuous. Then, any solution $y(t)$ to* (IV.2) *satisfies*

$$y((n + 1)T + t_0) = C(t_0)y(nT + t_0) + R(t_0), \tag{IV.3}$$

for any $n \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$ *and* $t_0 \in [0, T]$*, where* $C(t_0) = Q(t_0)e^{BT}Q^{-1}(t_0)$ *and*

$$R(t_0) = e^{B(t_0+T)} \int_{t_0}^{t_0+T} e^{-Bs}Q^{-1}(s)b(s)\ ds \tag{IV.4}$$

are dependent only on t_0*.*

Proof. Let $Y(t)$ be a fundamental matrix of the homogeneous equation, i.e., a matrix consisting of n linearly independent solutions to $\dot{y}(t) = L(t)y(t)$. By Floquet's theorem [40], we can write $Y(t) = Q(t)e^{Bt}$ for a continuously differentiable, T-periodic matrix $Q \in \mathbb{C}^{d\times d}$ with $Q(0) = 1$ and a constant matrix $B \in \mathbb{C}^{d\times d}$. Substituting this expression into $\dot{Y}(t) = L(t)Y(t)$ and removing a common factor of e^{Bt}, we have

$$\dot{Q} + QB = LQ. \tag{IV.5}$$

Separately, let $z(t) = Q^{-1}(t)y(t)$ for a given solution $y(t)$ to (IV.2). We have

$$\dot{Q}z + Q\dot{z} = LQz + b. \tag{IV.6}$$

After right multiplying (IV.5) by z, we combine (IV.5) and (IV.6) to obtain

$$\dot{z} = Bz + Q^{-1}b, \tag{IV.7}$$

which is solved by

$$z(t) = e^{Bt}z_0 + e^{Bt}\int_0^t e^{-Bs}Q^{-1}(s)b(s)\ ds,$$

for a given $z_0 = z(0) = y(0)$. Setting $t = nT + t_0$, we find

$$\begin{aligned} z(t+T) &= e^{B(t+T)}z_0 + e^{B(t+T)}\int_0^{t+T} e^{-Bs}Q^{-1}(s)b(s)\ ds \\ &= e^{B(t+T)}z_0 + e^{B(t+T)}(e^{-Bt}z(t) - z_0) + e^{B(t+T)}\int_t^{t+T} e^{-Bs}Q^{-1}(s)b(s)\ ds \\ &= e^{BT}z(t) + e^{B(t+T)}\int_t^{t+T} e^{-Bs}Q^{-1}(s)b(s)\ ds \\ &= e^{BT}z(t) + e^{B(t_0+T)}\int_{t_0}^{t_0+T} e^{-Bs}Q^{-1}(s)b(s)\ ds, \end{aligned}$$

applying the T-periodicity of Q and b. The lemma follows. □

Now, the (discretized) droplet system *nearly* obeys an equation of the form (IV.2), except that the forcing $b(t)$, which here would represent the droplet pressure, can generally vary with the droplet position from one period of vibration to the next. Even still, we can use a technique reminiscent of Durey et al. [14] to characterize the mean dynamics of such a system:

Theorem IV.3. *Suppose* $X(t) \in \mathbb{C}^m$ *is piecewise constant on each period* $[k, k+T)$*, for* $k \in T\mathbb{N}_0 = \{0, T, 2T, ...\}$*, and undergoes an ergodic process with stationary probability density* μ *on* $\mathbb{C}^m$*. Suppose that* $y(t) \in \mathbb{C}^d$ *solves*

$$\dot{y} = L(t)y + b(t, X(t)), \tag{IV.8}$$

where $L(t) \in \mathbb{C}^{d\times d}$ *is* T*-periodic with eigenvalues of strictly negative real part, and* $b(t, x)$ *is* T*-periodic in* t *for each fixed value* $x \in \mathbb{C}^m$*. Further suppose that* $x \mapsto \int_0^T |b(t, x)|\, dt$ *is uniformly bounded for* $x \in \mathbb{C}^m$*. Then, the mean field* $\langle y\rangle \doteq \lim_{N\to\infty} N^{-1}\sum_{n=1}^N y(nT)$ *is given by*

$$\langle y\rangle = \int_{\mathbb{R}} \mu(\mathcal{X})y_0(\mathcal{X})d\mathcal{X},$$

where $y_0(\mathcal{X}) \in \mathbb{C}^n$ *is the mean field when* $X(t) = \mathcal{X}$ *is constant.*

Remark. Here, the term $y_0(\mathcal{X}) \in \mathbb{C}^d$ represents the wave field generated by a fixed bouncer at $\mathcal{X} \in \mathbb{C}^m$; note that this term can be computed numerically for a given droplet model. The hypothesis that $L(t)$ carries only negative eigenvalues corresponds to a subcritical forcing frequency $\Gamma < \Gamma_F$ in the droplet system, and the (relatively mild) integral hypothesis on b simply guarantees that $R(t_0, x)$ is well-defined and uniformly bounded in x.

Proof. Recall that the fundamental matrix of the homogeneous equation can be expressed as $Y(t) = Q(t)e^{Bt}$, from Floquet's theorem [40]. Viewing the restriction $y|_{[t,t+T]}$ as a solution to (IV.8) with periodic forcing, Lemma IV.2 implies

$$y(t+T) = e^{BT}y(t) + R(\mathcal{X}(t)), \tag{IV.9}$$

noting that the dependence of b on $\mathcal{X}$ confers the same to R. We can now take the mean value of both sides, noting (from our integrability assumptions on b) that R is uniformly bounded in $\mathcal{X}$; subsequently using the ergodic theorem [41] to simplify, we have

$$(1 - e^{BT})\langle y\rangle = \int_{\mathbb{R}} R(\mathcal{X})\mu(\mathcal{X})\, d\mathcal{X}.$$

That $L(t)$ has eigenvalues of negative real part means that $\|e^{BT}\| < 1$; hence, $1 - e^{BT}$ is invertible, and

$$\langle y\rangle = \int_{\mathbb{R}} (1 - e^{BT})^{-1} R(\mathcal{X})\mu(\mathcal{X})\, d\mathcal{X}.$$

The stationary bouncing droplet corresponds to a Dirac measure $\mu(\mathcal{X}) = \delta(x - \mathcal{X})$. Thus, $y_0(x, \mathcal{X}) = (1 - e^{BT})^{-1}R(\mathcal{X})$, proving the theorem. □

B. The Infinite-Dimensional case

Broadly speaking, the main difficulty in the infinite-dimensional case is that Floquet's Theorem [40] does not generally hold on a Banach space, preventing us from recovering an equivalent of Lemma IV.2 directly. As discussed above, one could proceed by hypothesizing that $L(t)$ decomposes along finite-dimensional subspaces of our Banach space $\mathcal{B}$. This was the approach taken previously to study the free particle and the circular corral [14, 17], made possible by the strong symmetries of both systems; in both cases, the wave equation (III.1) decomposes along eigenmodes of the Laplacian. In general, however, we require an analogue of Floquet's theory appropriate to the present setting. For this, we use the following notation:

Definition IV.4. Following Pazy [20, Definition 5.3], we define an *evolution system* on a Banach space $\mathcal{B}$ as a two-parameter family of bounded linear operators $U(t, s) : \mathcal{B} \to \mathcal{B}$, for values $0 \leq s \leq t \leq T$, satisfying

($\mathbf{E_1}$) $U(s, s) = 1$;

($\mathbf{E_2}$) $U(t, r)U(r, s) = U(t, s)$ for $s \leq r \leq t$; and

($\mathbf{E_3}$) For any $x \in \mathcal{B}$, the map $(t, s) \mapsto U(t, s)x$ is continuous.

The primary utility of evolution systems is in providing solutions to differential equations with time-dependent coefficients. For a Banach space $\mathcal{B}$ and a family of operators $L(t) : D \to \mathcal{B}$ on a uniform, dense domain $D \subset \mathcal{B}$, consider the initial value problem

$$\dot{y}(t) = L(t)y, \qquad y(0) = y_0 \in D. \tag{IV.10}$$

We say that (IV.10) "admits an evolution system $U(t, s)$" if, for any $x \in D$, we have

$$\frac{\partial}{\partial t}U(t, s)x = L(t)U(t, s)x, \qquad \frac{\partial}{\partial s}U(t, s)x = -U(t, s)L(s)x, \tag{IV.11}$$

for $0 \leq s \leq t \leq T$ [20, Theorem 5.2]. In this setting, we say that $y(t) = U(t, 0)y_0$ is a "classical solution" to (IV.10). Following Pazy, we propose the following criteria for the equation (IV.10) to admit a unique evolution system:

Proposition IV.5. *The equation* (IV.10) *admits a unique evolution system* $U(t, s)$ *if the following conditions are met:*

($\mathbf{P_1}$) *The domain D is dense in $\mathcal{B}$ and independent of t.*

($\mathbf{P_2}$) *There are constants $c, M > 0$ such that for all $t \in [0,T]$ and for all λ with Re $\lambda \geq c$, the resolvent $R(\lambda : L(t)) \doteq (L(t)-\lambda)^{-1}$ exists and has operator norm [42] bounded as*

$$\|R(\lambda : L(t))\| \leq \frac{M}{|\lambda - c| + 1}.$$

($\mathbf{P_3}$) *The map $t \mapsto L(t)$ is Hölder continuous in operator norm [43].*

Remark. The requirement ($\mathbf{P_3}$) can be significantly weakened, although it is not necessary for the application to walking droplet dynamics. In general, the proposition still holds if ($\mathbf{P_3}$) is weakened to

$$\|(\tilde{L}(t) - \tilde{L}(s))\tilde{L}(\tau)^{-1}\| \leq C|t-s|^{\alpha},$$

for constants $C > 0$ and $\alpha \in (0,1]$ and all $t, s, \tau \in [0,T]$. Here, $\tilde{L}(t) \doteq L(t) - c$.

Proof. Define $\tilde{L}(t) = L(t) - c$, which carries the same domain D as $L(t)$. From ($\mathbf{P_2}$), note that $\tilde{L}(t)$ has a resolvent $R(\zeta : \tilde{L}(t))$ defined for all $\zeta \geq 0$. So, for all $\zeta = \lambda - c \geq 0$, we have

$$(\mathbf{P_2})' : \quad \|R(\zeta : \tilde{L}(t))\| = \|R(\lambda : L(t))\| \leq \frac{M}{|\zeta| + 1},$$

for some constant $M > 0$. Now, ($\mathbf{P_3}$) implies that $\tilde{L}(t)$ is also Hölder continuous, so we have

$$(\mathbf{P_3})' : \quad \|(\tilde{L}(t) - \tilde{L}(s))\tilde{L}(\tau)^{-1}\| \leq \|\tilde{L}(t) - \tilde{L}(s)\|\|\tilde{L}(\tau)^{-1}\| \leq CM|t-s|^{\alpha},$$

for constants $C > 0$ and $s, t, \tau \in [0,T]$. The conditions ($\mathbf{P_1}$), ($\mathbf{P_2}$)$'$, and ($\mathbf{P_3}$)$'$ imply that $\tilde{L}(t)$ admits a unique evolution system $\tilde{U}(t,s)$ [20, Theorem 6.1].

Now, let $U(t,s) = e^{c(t-s)}\tilde{U}(t,s)$; we now show that $U(t,s)$ is the unique evolution system that $L(t)$ admits. Note that ($\mathbf{E_1}$) is satisfied since $U(s,s) = \tilde{U}(s,s) = I$, ($\mathbf{E_2}$) is satisfied since $U(t,r)U(r,s) = e^{c(t-r)}e^{c(r-s)}\tilde{U}(t,r)\tilde{U}(r,s) = e^{c(t-s)}\tilde{U}(t,s) = U(t,s)$, and ($\mathbf{E_3}$) is satisfied since strong continuity is preserved when multiplying by the continuous scalar function $e^{c(t-s)}$. Moreover,

$$\frac{\partial}{\partial t}U(t,s) = e^{c(t-s)}\frac{\partial}{\partial t}\tilde{U}(t,s) + ce^{c(t-s)}\tilde{U}(t,s) = (\tilde{L}(t) + c)e^{c(t-s)}\tilde{U}(t,s) = L(t)U(t,s),$$

$$\frac{\partial}{\partial s}U(t,s) = e^{c(t-s)}\frac{\partial}{\partial s}\tilde{U}(t,s) - ce^{c(t-s)}\tilde{U}(t,s) = -e^{c(t-s)}\tilde{U}(t,s)(\tilde{L}(s) + c) = -U(t,s)L(s),$$

so $L(t)$ indeed admits the evolution system $U(t,s)$. That $U(t,s)$ is unique follows from running the same logic backwards; if there were a distinct evolution system $U'(t,s)$ for $L(t)$, we could construct a distinct evolution system $e^{-c(t-s)}U'(t-s) \neq \tilde{U}(t-s)$ for $\tilde{L}(t)$. ∎

The conditions in IV.5 cover a wide range of physically-relevant wave operators. As an example, we consider the operator found in the model of Milewski et al. [15].

Example IV.6. In the model (IV.12) of Milewski et al. [15], the wave operator is given as follows:

$$L(t) = \begin{pmatrix} 2\mathrm{Re}^{-1}\Delta & -G(1-\Gamma\cos(4\pi t)) \\ \mathrm{DtN} & 2\mathrm{Re}^{-1}\Delta \end{pmatrix}. \tag{IV.12}$$

If we consider the case $\mathcal{B} = L^{\infty}(\mathbb{R}^2;\mathbb{R}^2)$ of bounded flow fields on $\mathbb{R}^2$, the domain of $L(t)$ would be $D = W^{\infty,2}(\mathbb{R}^2;\mathbb{R}^2)$, independent of t. As discussed before, this subspace is dense in $\mathcal{B}$, so it satisfies ($\mathbf{P_1}$). The spectral condition ($\mathbf{P_2}$) follows from splitting $L(t)$ into a time-independent component $L_0 = L(\cos 4\pi t = 0)$ and a time-dependent component $L_1(t) = L(t) - L_0$. The spectrum of the former—which corresponds to a non-vibrating fluid bath—is supported in the negative half-plane, so L_0 satisfies ($\mathbf{P_2}$) with $c = 0$. Applying the perturbation result [44, Section 4, Theorem 3.17], one can see that $L(t)$ satisfies ($\mathbf{P_2}$) with a potentially larger c. The equation satisfies ($\mathbf{P_3}$) because the only time dependent term is uniformly (and thus Hölder) continuous; hence, $L(t)$ itself is Hölder continuous.

In this setting, we can construct an infinite-dimensional analogue of Floquet's theorem:

Lemma IV.7 (Floquet form on a Banach space)**.** *Suppose the differential equation* (IV.10) *admits a unique evolution system $U(t,s)$, and suppose the operators $L(t)$ are T-periodic. Then, for any $n \in \mathbb{N}_0$ and $t_0 \in [0,T]$, we have*

$$U(nT + t_0, 0) = C(t_0)^n U(t_0, 0),$$

for a fixed, bounded operator $C(t_0)$ on $\mathcal{B}$. Equivalently, $y(nT+t_0) = C(t_0)^n y(t_0)$ for any classical solution y.

Proof. Consider the family of operators $U(t+T, s+T)$. Since $L(t)$ is T-periodic, $U(t+T,s+T)$ satisfies (IV.11), so $U(t+T,s+T)$ is an evolution system of (IV.10). However, our evolution system is unique by hypothesis, so we have $U(t,s) = U(t+T,s+T)$. Hence,

$$\begin{aligned} U((n+1)T + t_0, 0) &= U((n+1)T + t_0, nT + t_0) U(nT + t_0, 0) \\ &= U(T + t_0, t_0) U(nT + t_0, 0) \\ &= C(t_0) U(nT + t_0, 0), \end{aligned}$$

noting that $C(t_0) \doteq U(t_0 + T, t_0)$ depends only on t_0. The lemma follows inductively. □

As in the finite-dimensional case, we can leverage the Floquet form provided by Lemma IV.7 to characterize solutions of the inhomogeneous equation:

Lemma IV.8. *Suppose $L(t)$ is T-periodic and satisfies the criteria of Proposition IV.5, and that $b(t) \in D$ is a T-periodic, Hölder continuous family of vectors. Then there is a unique solution to* (IV.10)*, and it satisfies*

$$y\left((n+1)T + t_0\right) = C(t_0) y(nT + t_0) + R(t_0) \tag{IV.13}$$

for any $n \in \mathbb{N}_0$ and $t_0 \in [0,T]$; here, $C(t_0)$ is as given in Lemma IV.7 and $R(t_0) \in \mathcal{B}$ is a constant vector.

Proof. From [20, Theorem 4.1], the solution to the initial value problem

$$\dot{y}(t) = L(t)y + b(t), \qquad y(0) = y_0 \in D$$

exists, is unique, and is given by

$$y(t) = U(t,0)y_0 + \int_0^t U(t,r)b(r)\ dr.$$

Setting $t = nT + t_0$, we have

$$\begin{aligned} y\left(t+T\right) &= U\left(t+T, 0\right) y_0 + \int_0^{t+T} U(t+T, r) b(r)\ dr \\ &= C(t_0) U(t, t_0) y(t_0) + C(t_0) \int_0^t U(t,r) b(r)\ dr + C(t_0) \int_t^{t+T} U(t,r) b(r)\ dr \\ &= C(t_0) y(t) + C(t_0) \int_{t_0}^{t_0+T} U(t_0, r) b(r)\ dr, \end{aligned}$$

noting that $U(t+T, r) = C(t_0)U(t,r)$ and $U(t,s) = U(t+T, s+T)$. □

Finally, we recover our primary result:

Theorem IV.9. *Suppose $X(t) \in \Omega \subset \mathbb{C}^m$ is piecewise constant on each period $[k, k+T)$, for $k \in T\mathbb{N}_0 = \{0, T, 2T, ...\}$, and undergoes an ergodic process with stationary probability density μ. Suppose that $y(t) \in \mathcal{B}$ solves*

$$\dot{y} = L(t)y + b(t, X(t)), \tag{IV.14}$$

where $L(t)$ is a T-periodic operator satisfying the conditions of Proposition IV.5, and $b(t,x)$ is T-periodic and Hölder continuous in t for each fixed value $x \in \Omega$. Further suppose that the map $x \mapsto \int_0^T \|b(t,x)\|\, dt$ is uniformly bounded over Ω, and that the map $C(0) : y(0) \mapsto y(T)$ has norm $\|C(0)\| < 1$. Then, the mean field $\langle y \rangle \doteq \lim_{N\to\infty} N^{-1} \sum_{n=1}^N y(nT)$

is given by

$$\langle y\rangle = \int_\Omega \mu(\mathcal{X}) y_0(\mathcal{X}) d\mathcal{X},$$

where $y_0(\mathcal{X}) \in \mathcal{B}$ *is the mean field when* $X(t) = \mathcal{X}$ *is constant.*

Proof. Let $C \doteq C(0)$. As in the finite-dimensional case, Lemma IV.8 gives us

$$y(t+T) = Cy(t) + R(\mathcal{X}(t)). \tag{IV.15}$$

We can now take the mean value of both sides, noting that R is uniformly bounded in $\mathcal{X}$ from our uniform integrability assumption on b. Using the ergodic theorem to simplify, we have

$$(1-C)\langle y\rangle = \int_\Omega R(\mathcal{X})\mu(\mathcal{X})\ d\mathcal{X}$$

Since $\|C\| < 1$, we know that $1 - C$ is invertible with continuous inverse. Hence,

$$\langle y\rangle = \int_\Omega (1-C)^{-1} R(\mathcal{X})\mu(\mathcal{X})\ d\mathcal{X}.$$

The stationary case $X(t) = \mathcal{X}$ corresponds to the Dirac measure $\mu(\mathcal{X}) = \delta(x - \mathcal{X})$, or $y_0(x, \mathcal{X}) = (1 - C)^{-1}R(\mathcal{X})$. Thus, $y_0(\mathcal{X}) = (1 - C)^{-1}R(\mathcal{X})$, proving the theorem. □

V. NUMERICAL SIMULATION OF THE WALKING DROPLET MODEL

We explore the PDF-MWF relationship within the 1-dimensional model of Nachbin [21], which exactly resolves the topography of the fluid bath using an appropriate conformal map. This allows us to observe the walking droplet system on domains that do not conform to the hypotheses of Durey et al. [14]. Nachbin reduces the Faraday wave model of Milewski et al. [15] to the 1-dimensional case as follows:

$$\phi_t = -g(t)\eta + 2v^*\phi_{xx} + \frac{\sigma}{\rho}\eta_{xx} \quad \text{and} \quad \eta_t = \mathrm{DtN}(\phi) + 2v^*\eta_{xx}.$$

Here, v^*, σ, and ρ represent viscosity, surface tension, and density, respectively. The function $g(t) = g(1+\Gamma\sin(\omega_0 t))$ represents vertical acceleration of the vibrating bath for frequency ω_0 and amplitude Γ. Unlike in the 2-dimensional case, the Dirichlet-to-Neumann operator can be exactly expressed using a conformal map, which transforms a non-trivial geometry into a domain of constant depth, simplifying numerical computation.

Now that we have the appropriate equations and method to simulate the wave field, we apply Nachbin's equation for the horizontal position x_p of the particle:

$$m\frac{d^2x_p}{dt^2} + 0.01F(t)\frac{dx_p}{dt} = -F(t)\frac{\partial\eta}{\partial x}(x_p, t),$$

where m is the drop mass and $F(t)$ is the force exerted on the droplet by the bath: constant while the two are in

Variable	Value	Description
g	981 cm s^{-2}	Gravity
ω_0	$80 \times 2\pi$ s^{-1}	Vibration Frequency ($\times 2\pi$)
T_f	0.025 s	Faraday period
σ	20.9 dyn cm^{-1}	Surface tension
ρ	0.95 g cm^{-3}	Fluid density
ν	0.16 cm^2 s^{-1}	Fluid viscosity
ν_p	0.01 g cm^{-1}	Stokes' drag coefficient
R_0	0.035 cm	Droplet radius

TABLE I: Physical parameters used for the simulation, following Nachbin [21].

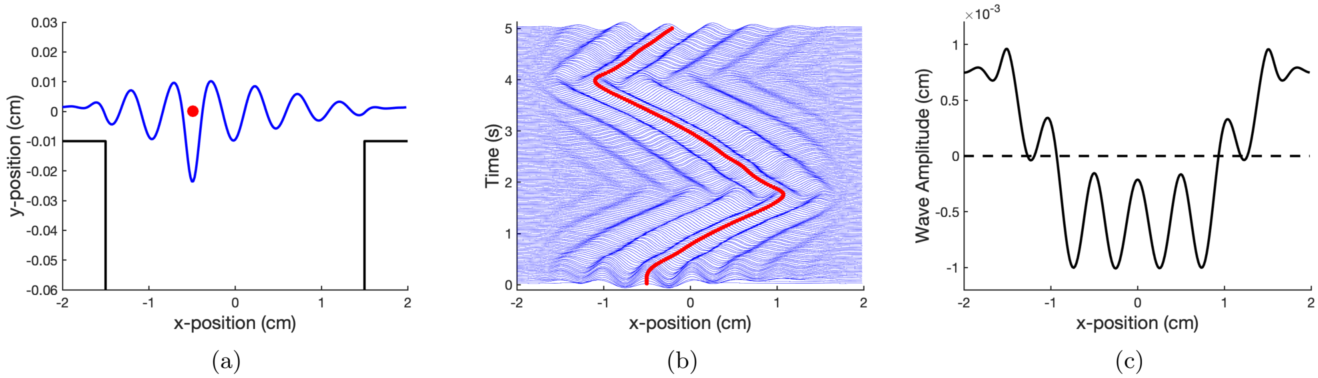


FIG. 2: **(a)** A snapshot of the droplet model, showing the bath topography (black, partially shown), the wave field (blue), and the droplet (red). **(b)** A droplet (red) starts at -0.5cm and traverses the 3cm-wide domain for ~ 5s seconds, or 200 Faraday periods; the memory parameter is $\Gamma/\Gamma_F = 0.85$. Snapshots of the underlying wavefield (blue) are shown at each Faraday period. **(c)** The long-term mean wave field (black) is calculated by averaging the value of the wave field at the start of each Faraday period.

contact, and 0 otherwise. In our simulation, the contact time is a quarter of the bath's *Faraday period*, which is the period of the droplet's bouncing motion. We use the physical parameters given in Table I.

We first run the simulation on two rectangular domains: a 3cm domain from $x = -1.5$cm to $x = 1.5$cm, and a 4cm domain from $x = -2$cm to 2cm. In both cases, the depth is 5cm within the cavity and 0.01cm outside, and the boundary extends 0.5cm past the edges of the cavity. The variables η and ϕ are discretized with 256 horizontal nodes for the 3cm domain and 384 nodes for the 4cm domain. We run the simulation with a minimum timestep of $\Delta t = 1.25 \times 10^{-5}s = 5 \times 10^{-4}T_f$.

An important parameter for our model is the amplitude of vibration Γ. We consider the Γ as a proportion of the Faraday threshold Γ_F, which is the smallest value of Γ such that standing waves on the free surface become unstable. The proportion $\Gamma/\Gamma_F < 1$ acts as the "memory" value of the system, corresponding (inversely) to the dissipation rate of the wave field [13].

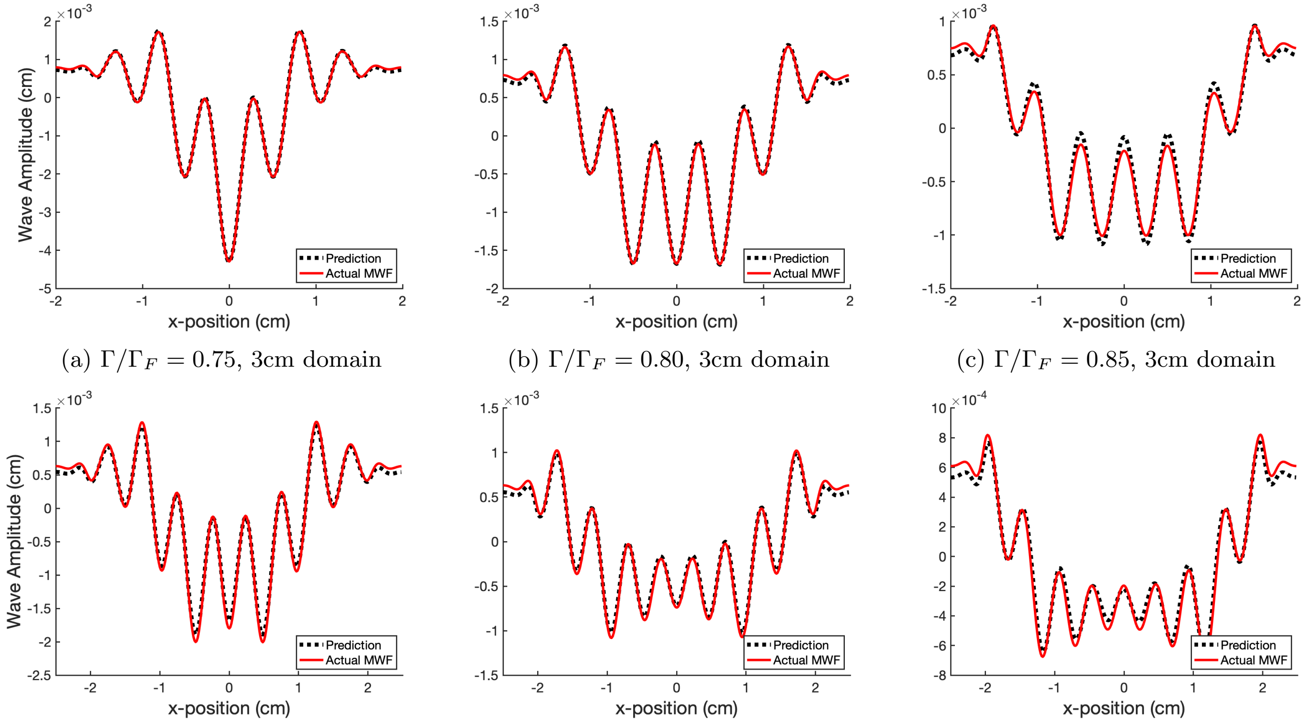
In order to verify the result in Theorem IV.9, we run a long-time simulation (~ 500 seconds, or 2×10^4 Faraday periods) of the walking droplet model, recording the instantaneous wave fields and droplet position once every Faraday period (example shown in Figure 2a). The droplet generates a time-varying wave field as it moves about the cavity (as shown in Figure 2b), and we aggregate these to generate our MWF. An example MWF is shown in Figure 2c.

Separately, we numerically compute $y_0(\mathcal{X})$ for each $\mathcal{X}$ in the domain by allowing the MWF to converge for fixed, bouncing droplets at each position in the domain. Finally, we integrate the computed kernel $y_0(\mathcal{X})$ against the probability density of our simulation, in accordance with Theorem IV.9, to retrieve a predicted MWF. We compare these predictions against the true MWFs in various cases, shown in Figure 3.

We also apply our analysis to a two-droplet simulation, considering the joint PDF $\mu : \mathbb{R}^2 \to [0, \infty)$ of two droplets in our 1D pilot-wave system. We conduct experiments in two different variations of the two-cavity domain of Nachbin [21], which we refer to as the "uncorrelated" and "correlated" systems, respectively. In both, the left cavity is 1cm wide, and the right cavity is 1.1cm wide. For the uncorrelated case, the two droplets are separated by a wide, shallow region, such that their wave fields interact minimally. We run the simulation for $1.8 \times 10^4 T_f$. For the correlated case, the central barrier is replaced by another cavity, giving room for the each droplet to affect the other through their shared wave field. Even with the droplet correlation, we find that the predicted PDF-MWF relationship holds closely. Results from both systems are displayed in Figure 4.

VI. CONCLUSION AND PERSPECTIVES

In this work, we have derived a correspondence between the mean wave field (MWF) and probability density function (PDF) of a general pilot-wave system, recovering a fundamental relationship analogous to that between the wavefunction and probability density of a quantum particle. In particular, we used the theory of evolution systems [20] to derive a Floquet-like result on Banach spaces, allowing us to treat the solution of the wave equation directly. This result extends previous work on the PDF-MWF relationship [14, 17] by rigorously proving the relationship in non-simple geometries, for non-instantaneous pressures, and for more than one droplet at a time.



(a) $\Gamma/\Gamma_F = 0.75$, 3cm domain

(b) $\Gamma/\Gamma_F = 0.80$, 3cm domain

(c) $\Gamma/\Gamma_F = 0.85$, 3cm domain

(d) $\Gamma/\Gamma_F = 0.75$, 4cm domain

(e) $\Gamma/\Gamma_F = 0.80$, 4cm domain

(f) $\Gamma/\Gamma_F = 0.85$, 4cm domain

FIG. 3: Measured and predicted MWFs from single-droplet experiments, where the droplet moves around within a single, rectangular cavity. Each is calculated over 2×10^4 vibration periods for both 3cm and 4cm domains and with a variety of memory parameters. In all cases, we note a strong correspondence between the measured MWF and the prediction from Claim I.1.

We numerically verify this relationship by simulating the walking droplet system on 1-dimensional rectangular domains, retrieving the expected PDF-MWF correspondence. We conclude that a strong relationship between the MWF and PDF is characteristic of pilot-wave systems, and, given system parameters, one can recover particle and field statistics from one another.

Looking forward, it might be interesting to apply our results to hydrodynamic analogues of quantum entanglement and measurement. For a hydrodynamic entanglement analogue, we consider "correlating" two droplets in a two-cavity system, as seen in V. The droplets move within their own, separated cavities, but their shared wave field allows communication back and forth. Closing off the central separator allows for the observation of how the droplet correspondence changes; as suggested by Theorem IV.9, one should be able to detect entanglement by observing just the MWF, similar to the role of the wavefunction in quantum mechanics. Moreover, it might be interesting to impose new cavities in each of the separated domains, analogous to imposing a particle detector in quantum measurement, and potentially observe the decoherence of the particle statistics in the same way.

More generally, we believe the functional analysis techniques applied in this work may be useful in further studies of classical pilot-wave theories and may provide a new, analytical perspective on wave-particle relationships in these systems.

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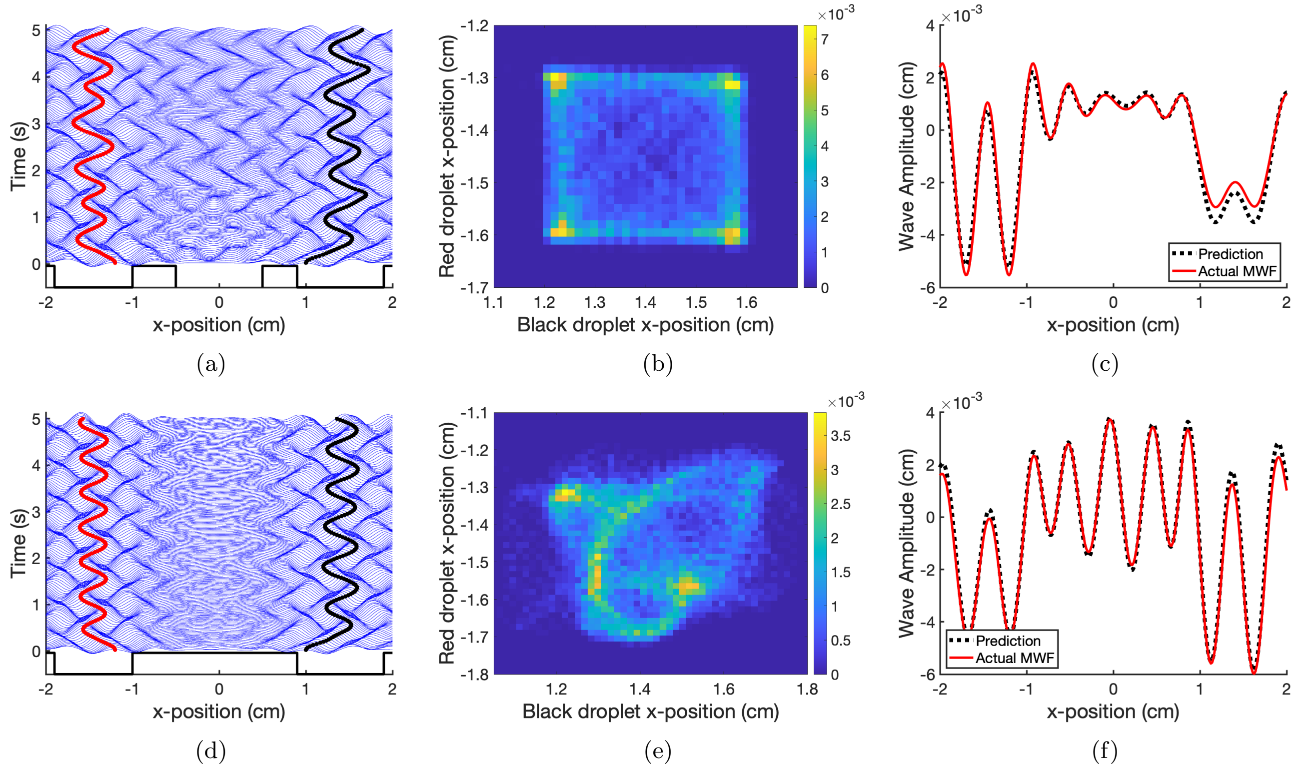

FIG. 4: Two droplets move in each of the asymmetrical "uncorrelated" and "correlated" systems, following Nachbin [21]. The uncorrelated domain consists of two cavities separated by a wide central barrier, preventing the droplets from communicating with one another; the correlated domain replaces the barrier with another cavity, such that the droplets interact through their shared wave field. One droplet starts at -1.20cm (red) and another starts at 1.00cm (black). The vertical acceleration is given by $\Gamma = 4.8$. **(a,d)** The uncorrelated and correlated two droplet systems, respectively, during the first 200 Faraday periods. **(b,e)** The PDF of the systems after 1.8×10^4 Faraday periods. **(c,f)** The comparison between the actual and predicted MWFs.

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[42] Recall that the *operator norm* of a bounded operator $L : \mathcal{B} \to \mathcal{B}$ is the smallest $C > 0$ such that, for all $y \in \mathcal{B}$ with $\|y\| \le 1$, we have $\|Ly\| \le C$.
[43] Specifically, this means that constants $C, \alpha > 0$ exist such that, for any times t_1 and t_2, the operator $L(t_2) - L(t_1)$ has operator norm bounded by $\|L(t_2) - L(t_1)\| \le C|t_2 - t_1|^\alpha$.
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