Bounds for Smooth Siegel Theta Sums at Special Rational Parameters

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Abstract

We define inhomogeneous theta sums as exponential sums of the form

$$S_M^f(X;\boldsymbol{\alpha},\boldsymbol{\beta}) := \sum_{\boldsymbol{k}\in\mathbb{Z}^n+\boldsymbol{\beta}} f(M^{-1}\boldsymbol{k})e^{2\pi i\left(\frac{1}{2}\boldsymbol{k}X\,{}^{\mathrm{t}}\boldsymbol{k}+\boldsymbol{k}\,{}^{\mathrm{t}}\boldsymbol{\alpha}\right)},$$

where X is an $n\times n$ symmetric matrix, $\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathbb{R}^n$, and f is a fixed weight function. In recent work, F. Cellarosi and the second named author showed that when $n=1$ and f is a fixed Schwartz function, there exist $\alpha,\beta\in\mathbb{R}$ such that $|S_M^f(x;\alpha,\beta)| \ll_{f,\alpha,\beta} \sqrt{M}$ for every $x\in\mathbb{R}$ and $M\in\mathbb{N}$. We show that this does *not* extend to higher dimensions, i.e. there are no $\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathbb{R}^n$ for which the bound $|S_M^f(X;\boldsymbol{\alpha},\boldsymbol{\beta})| \ll_{f,\boldsymbol{\alpha},\boldsymbol{\beta}} M^{n/2}$ holds for every real symmetric matrix X and every $M\in\mathbb{N}$ when $n>1$.

Contents

1 Introduction

Let $\mathrm{Sym}_n(\mathbb{R})$ denote the set of $n \times n$ symmetric matrices, with real entries. Recall that upon fixing a basis, every quadratic form Q in n variables with real coefficients corresponds to an $X \in \mathrm{Sym}_n(\mathbb{R})$. We define Siegel theta sums as exponential sums of the form

$$S_M(X; \boldsymbol{\alpha}, \boldsymbol{\beta}) = \sum_{\boldsymbol{k} \in (\mathbb{Z}^n + \boldsymbol{\beta}) \cap (0,M]^n} e(\tfrac{1}{2}\boldsymbol{k} X \,{}^{\mathrm{t}}\boldsymbol{k} + \boldsymbol{k} \,{}^{\mathrm{t}}\boldsymbol{\alpha}). \tag{1.1}$$

Such sums are well-studied objects and arise in many modern applications. For instance, estimates for theta sums play an important role in [1, 4, 10, 11] to understand the value distribution of quadratic forms. The goal of this paper is to study estimates for weighted variants of Siegel theta sums, defined in (1.2), when $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{Q}^n$. Throughout, when stating our estimates, we make use of Vinogradov's "$\ll$" notation, which is equivalent to Landau's O-notation. When relevant, we stress the dependence of the implied constants upon certain parameters by writing them as subscripts.

Theta sums in the case when $n = 1$ have received considerable attention over the years. We observe that when x, α, β are rational, then $S_M(x; \alpha, \beta)$ reduces to a quadratic Gauss sum for which various bounds are classical, e.g. if $\gcd(a, q) = 1$ and $N \leq q$ then $|S_M(\frac{a}{q}; 0, 0)| = O(\sqrt{q})$. Further details on Gauss sums can be found in [9, 13]. The detailed study of $S_M(x; \alpha, \beta)$ for $x \in \mathbb{R}$ was initiated by Hardy and Littlewood in [6], who were attracted by its many "interesting and beautiful properties." In particular, they proved an approximate functional equation which they used to obtain various bounds, usually under some stringent Diophantine conditions on x. For instance, for x of bounded type (which is a measure zero condition) and any $\alpha \in \mathbb{R}$, they proved that $S_M(x; \alpha, 0) \ll_x \sqrt{M}$.

We define generalized Siegel theta sums as exponential sums of the form

$$S_M^f(X; \boldsymbol{\alpha}, \boldsymbol{\beta}) := \sum_{\boldsymbol{k} \in \mathbb{Z}^n + \boldsymbol{\beta}} f(M^{-1}\boldsymbol{k}) e(\tfrac{1}{2}\boldsymbol{k} X \,{}^{\mathrm{t}}\boldsymbol{k} + \boldsymbol{k} \,{}^{\mathrm{t}}\boldsymbol{\alpha}), \tag{1.2}$$

where $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^n$, $e(z) := e^{2\pi i z}$, and f is a fixed cut-off function. We observe that if we take $f = \mathbb{1}_{(0,1]^n}$, then we recover S_M defined in (1.1). In what follows, the cut-off function f will be of Schwartz class, $\mathcal{S}(\mathbb{R}^n)$, i.e. the space of smooth functions where all derivatives decay faster than any polynomial. Focusing on such cut-offs guarantees that the sum defining $S_M^f(X, \boldsymbol{\alpha}, \boldsymbol{\beta})$ converges absolutely. We note that by setting $f(\boldsymbol{w}) := e^{-\pi \boldsymbol{w} P \,{}^{\mathrm{t}}\boldsymbol{w}}$, where P is a symmetric, positive-definite matrix, we recover the Siegel theta series associated to the *shifted lattice* $\mathbb{Z}^n + \boldsymbol{\beta}$:

$$\sum_{\boldsymbol{k} \in \mathbb{Z}^n + \boldsymbol{\beta}} e(\tfrac{1}{2}\boldsymbol{k} Z \,{}^{\mathrm{t}}\boldsymbol{k} + \boldsymbol{k} \,{}^{\mathrm{t}}\boldsymbol{\alpha}), \tag{1.3}$$

where $Z = X + i\frac{1}{M^2}P$. We study the size of Siegel theta sums for *fixed* $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathbb{R}^n \times \mathbb{R}^n$ as X varies over some prescribed set.

1.1 Main Results

Recently, Marklof and Welsh [12, Theorem 1.1] showed a general estimate for smooth theta sums. For instance, they showed for any fixed Schwartz cut-off function f and $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^n$ that there is a set of full Lebesgue measure $\mathscr{X}$ such that for any $M \geq 1$ and $X \in \mathscr{X}$, then

$$|S_M^f(X; \boldsymbol{\alpha}, \boldsymbol{\beta})| \ll_{f,X} M^{n/2} \log M. \tag{1.4}$$

In the case where $n = 1$, if we choose α and β to be rationals of a particular form, then the estimate (1.4) was improved significantly by Cellarosi and the second named author [2, Theorem 1.2]. Suppose that $\gcd(a, b, m) = 1$ and $\alpha = \frac{a}{2m}$ and $\beta = \frac{b}{2m}$, where a, b, and m are all odd. Then the bound

$$|S_M^f(x; \alpha, \beta)| \ll_{m,\beta,f} \sqrt{M} \tag{1.5}$$

holds uniformly in $t \in \mathbb{R}$ and $N \in \mathbb{N}$. Conversely, if a bound of the form in (1.5) holds uniformly for $x \in \mathbb{R}$ and $M \in \mathbb{N}$, then we must have that $\alpha = \frac{a}{2m}$ and $\beta = \frac{b}{2m}$, where $\gcd(a, b, m) = 1$ and a, b, and m are all odd.

In other words, it is possible to classify all pairs $(\alpha, \beta) \in \mathbb{Q}^2$ for which the estimate (1.5) holds for *every* one-dimensional quadratic form. We remark that, given generic $(\alpha, \beta) \in \mathbb{R}^2$, a bound of the quality (1.5) typically only holds when there are stringent Diophantine conditions on x: e.g. x of bounded type. We also note that the methods developed in [2] readily extend to obtain bounds for higher-dimensional sums when X is diagonal. More specifically, for fixed $n \geq 1$, there exist vectors $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathbb{Q}^n \times \mathbb{Q}^n$ such that the bound

$$|S_M^f(D; \boldsymbol{\alpha}, \boldsymbol{\beta})| \ll_{f,\boldsymbol{\alpha},\boldsymbol{\beta}} M^{n/2}, \tag{1.6}$$

holds for every diagonal matrix D and every $M \in \mathbb{N}$.

Our main result is that this *cannot* be extended further to every quadratic form X, when $n > 1$.

Theorem 1.1. *Fix $f \in \mathcal{S}(\mathbb{R}^n)$ and $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathbb{Q}^n \times \mathbb{Q}^n$. Then, for any $R > 0$, the set*

$$\{X \in \mathrm{Sym}_n(\mathbb{R}) : |S_M^f(X; \boldsymbol{\alpha}, \boldsymbol{\beta})| > RM^{n/2}\} \neq \varnothing, \tag{1.7}$$

for some $M \in \mathbb{N}$.

Theorem 1.1 is an immediate consequence of the following theorem, which shows that the normalized sum $M^{-n/2}S_N^f$ is large relatively often. Identifying $\mathrm{Sym}_n(\mathbb{R})$ with $\mathbb{R}^{n(n+1)/2}$, we equip $\mathrm{Sym}_n(\mathbb{R})$ with the standard Lebesgue measure on $\mathbb{R}^{n(n+1)/2}$.

Theorem 1.2. *Let $\mathscr{A}$ denote the set of real symmetric matrices whose entries are in the interval $[0, 1]$. Fix $f \in \mathcal{S}(\mathbb{R}^n)$ and $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathbb{Q}^n \times \mathbb{Q}^n$. We have that*

$$\lim_{M \to \infty} \mathrm{Leb}(\{X \in \mathscr{A} : M^{-n/2}|S_M^f(X; \boldsymbol{\alpha}, \boldsymbol{\beta})| > R\}) \gg_{f,n} R^{-4n}. \tag{1.8}$$

1.2 Structure of the Paper

In Section 2, we briefly discuss the construction of an automorphic function $|\Theta_f|$, via the Schrödinger-Weil representation of the Jacobi group G on $L^2(\mathbb{R}^n)$. We then show that this function $|\Theta_f|$ agrees with the normalized sum $M^{-n/2}|S_M^f|$ on a special submanifold, $\mathcal{H}_M^{(\boldsymbol{\alpha},\boldsymbol{\beta})}$, known as an expanding horosphere. The function $|\Theta_f|$ possesses many symmetries, which generate a discrete subgroup Γ of the Jacobi group G. Using these symmetries, we may view $|\Theta_f|$ as a function on the quotient $\Gamma\backslash G$. It can be shown that $\mathcal{H}_M^{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ becomes dense in the quotient $\Gamma\backslash G$ as $M \to \infty$. It follows that in order to obtain good estimates for $M^{-n/2}|S_M^f|$, it is enough to estimate $|\Theta_f|$ in $\Gamma\backslash G$. In Section 3, we characterize the region of $\Gamma\backslash G$ that $\mathcal{H}_M^{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ must avoid in order to produce a uniform bound for $|\Theta_f|$. Finally, in Section 4, we show that $\mathcal{H}_M^{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ cannot avoid this region, from which Theorem 1.1 follows.

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2 Preliminaries

2.1 The Jacobi Group

Set

$$J := \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}. \tag{2.1}$$

We define the Heisenberg group $\mathbb{H}(\mathbb{R}^n)$ as the set $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ with group multiplication given by

$$(\boldsymbol{p}, t)(\boldsymbol{p}', t') := (\boldsymbol{p} + \boldsymbol{p}', t + t' + \tfrac{1}{2}\boldsymbol{p} J\, {}^{\mathrm{t}}\boldsymbol{p}'). \tag{2.2}$$

The symplectic group $\mathrm{Sp}(n, \mathbb{R})$ is defined by

$$\mathrm{Sp}(n, \mathbb{R}) := \left\{ g \in \mathrm{GL}_{2n}(\mathbb{R}) : gJ\, {}^{\mathrm{t}}g = J \right\}. \tag{2.3}$$

The following defines a left action of $\mathrm{Sp}(n, \mathbb{R})$ on $\mathbb{H}(\mathbb{R}^n)$ in the following way:

$$g \cdot (\boldsymbol{p}, t) := (\boldsymbol{p} g^{-1}, t), \tag{2.4}$$

where $g \in \mathrm{Sp}(n, \mathbb{R})$, and $(\boldsymbol{p}, t) \in \mathbb{H}(\mathbb{R}^n)$.

The Jacobi group G is then defined as the semi-direct product

$$G := \mathbb{H}(\mathbb{R}^n) \rtimes \mathrm{Sp}(n, \mathbb{R}), \tag{2.5}$$

with multiplication given by

$$(h, g)(h', g') := (h(g \cdot h'), gg'), \tag{2.6}$$

where $g \cdot h'$ is given by (2.4).

2.2 The Schrödinger-Weil Representation

Let $((\boldsymbol{x},\boldsymbol{y}),t) \in \mathbb{H}(\mathbb{R}^n)$. The Schrödinger representation W of $\mathbb{H}(\mathbb{R}^n)$ on $L^2(\mathbb{R}^n)$ by unitary operators is defined by

$$W((\boldsymbol{x},\boldsymbol{y}),t)f(\boldsymbol{w}) = e\big(-t + \tfrac{1}{2}\boldsymbol{x}\,{}^{\mathrm{t}}\boldsymbol{y} + \boldsymbol{y}\,{}^{\mathrm{t}}\boldsymbol{w}\big)\, f(\boldsymbol{w}+\boldsymbol{x}). \tag{2.7}$$

Using the action of $\mathrm{Sp}(n,\mathbb{R})$ on $\mathbb{H}(\mathbb{R}^n)$, for each $g \in \mathrm{Sp}(n,\mathbb{R})$ we may construct a new representation W_g, where

$$W_g(h) = W(g\cdot h). \tag{2.8}$$

By the Stone–von Neumann theorem, W and W_g are unitarily equivalent, i.e. for $g \in \mathrm{Sp}(n,\mathbb{R})$ there exists a unitary operator $R(g)$ such that

$$W_g = R(g)^{-1} W R(g). \tag{2.9}$$

Remark 2.1. Let $\mathrm{U}(L^2(\mathbb{R}^n))$ denote the group of unitary operators on $L^2(\mathbb{R}^n)$. Using Schur's lemma, the map

$$R : \mathrm{Sp}(n,\mathbb{R}) \to \mathrm{U}(L^2(\mathbb{R}^n)), \tag{2.10}$$

$$R : g \mapsto R(g), \tag{2.11}$$

defined implicitly in (2.9), can be shown to be a projective representation of $\mathrm{Sp}(n,\mathbb{R})$. More precisely, there exists a nontrivial, unitary phase cocycle $\rho : \mathrm{Sp}(n,\mathbb{R})\times\mathrm{Sp}(n,\mathbb{R}) \to \mathbb{C}$ such that

$$R(gg') = \rho(g,g')R(g)R(g'). \tag{2.12}$$

This cocycle can be explicitly defined, but is not necessary in what follows. The projective representation R of $\mathrm{Sp}(n,\mathbb{R})$ extends to a true representation $\widetilde{R}$ of its universal cover, $\widetilde{\mathrm{Sp}}(n,\mathbb{R})$. As we note in Remark 2.6, we do not make use of $\widetilde{R}$ directly, but it is needed in order to formally define the Theta function in (2.24).

Set

$$\pi : \widetilde{\mathrm{Sp}}(n,\mathbb{R}) \to \mathrm{Sp}(n,\mathbb{R}) \tag{2.13}$$

to be the standard projection, and for $\tilde{g} \in \widetilde{\mathrm{Sp}}(n,\mathbb{R})$, let $g = \pi(\tilde{g})$. Then we define an action of $\widetilde{\mathrm{Sp}}(n,\mathbb{R})$ on $\mathbb{H}(\mathbb{R}^n)$ as

$$\tilde{g}\cdot(\boldsymbol{p},t) := (\boldsymbol{p}g^{-1},t). \tag{2.14}$$

We then define the universal Jacobi group

$$\mathbb{H}(\mathbb{R}^n) \rtimes \widetilde{\mathrm{Sp}}(n,\mathbb{R}), \tag{2.15}$$

with multiplication

$$(h,\tilde{g})(h',\tilde{g}') = (h(\tilde{g}\cdot h'),\tilde{g}\tilde{g}'), \tag{2.16}$$

where $\tilde{g}\cdot h'$ is as defined in (2.14).

The *Schrödinger-Weil representation* of $\mathbb{H}(\mathbb{R}^n) \rtimes \widetilde{\mathrm{Sp}}(n,\mathbb{R})$ is defined as the representation

$$(h,\tilde{g}) \mapsto W(h)\widetilde{R}(\tilde{g}). \tag{2.17}$$

An explicit formula for $R(g)$ for general $g \in \mathrm{Sp}(n,\mathbb{R})$ is complicated, however it is relatively simple to give formulae for $R(g)$ when g belongs to specific subgroups of $\mathrm{Sp}(n,\mathbb{R})$.

Proposition 2.2 ([12], Proposition 2.1)**.** *Let $f \in \mathcal{S}(\mathbb{R}^n)$. Suppose $A \in \mathrm{GL}(n,\mathbb{R})$ and $B \in \mathrm{Sym}_n(\mathbb{R})$. Then*

$$R\left(\begin{pmatrix} A & B \\ 0 & {}^{\mathrm{t}}A^{-1} \end{pmatrix}\right) f(\boldsymbol{w}) = |\det A|^{\frac{1}{2}} \mathrm{e}\big(\tfrac{1}{2}\boldsymbol{w} A\, {}^{\mathrm{t}}B\, {}^{\mathrm{t}}\boldsymbol{w}\big)\, f(\boldsymbol{w}A). \tag{2.18}$$

The group $K := \mathrm{Sp}(n,\mathbb{R}) \cap \mathrm{O}(2n)$ is a maximal compact subgroup of $\mathrm{Sp}(n,\mathbb{R})$. The map

$$Q \mapsto k(Q) := \begin{pmatrix} \Re(Q) & -\Im(Q) \\ \Im(Q) & \Re(Q) \end{pmatrix} \tag{2.19}$$

defines an isomorphism of $\mathrm{U}(n)$ and K. Note that if $Q = iI$, then $k(Q) = J$.

Proposition 2.3 ([3], Theorem 4.53)**.** *If $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}(n,\mathbb{R})$ with $\det B \neq 0$, then, up to the phase cocycle given in* (2.12)*, we have*

$$R(g)f(\boldsymbol{w}) = |\det B|^{-1/2} e\big(-\tfrac{1}{2}\boldsymbol{w} D B^{-1} \boldsymbol{w}\big) \int_{\mathbb{R}^n} e\big(\boldsymbol{v} B^{-1} \boldsymbol{w} - \tfrac{1}{2}\boldsymbol{v} B^{-1} A \boldsymbol{v}\big)\, f(\boldsymbol{v})\, \mathrm{d}\boldsymbol{v}. \tag{2.20}$$

In particular

$$R(J)f = \mathcal{F}f, \tag{2.21}$$

where J is as in (2.1), and $\mathcal{F}$ is the unitary Fourier transform.

For $f \in L^2(\mathbb{R}^n)$ define

$$f_Q := R(k(Q))f. \tag{2.22}$$

The following proposition shows that if $f \in \mathcal{S}(\mathbb{R}^n)$, then so is f_Q.

Proposition 2.4 ([12], Lemma 4.2)**.** *Let $f \in \mathcal{S}(\mathbb{R}^n)$. Then for all $A > 0$ and multi-indices $\alpha \geq 0$, there exists a constant $c_f(\alpha, A)$ such that for all $Q \in \mathrm{U}(n)$,*

$$\left|\left(\frac{\partial}{\partial \boldsymbol{w}}\right)^{\alpha} f_Q(\boldsymbol{w})\right| \leq c_f(\alpha, A)(1 + \|\boldsymbol{w}\|)^{-A}. \tag{2.23}$$

2.3 The Theta Function

For $f \in \mathcal{S}(\mathbb{R}^n)$ we define the theta function $\Theta_f : \mathbb{H}(\mathbb{R}^n) \rtimes \widetilde{\mathrm{Sp}}(n,\mathbb{R}) \to \mathbb{C}$ using the Schrödinger-Weil representation by

$$\Theta_f(h, \tilde{g}) = \sum_{\boldsymbol{m} \in \mathbb{Z}^n} [W(h)\widetilde{R}(\tilde{g})f](\boldsymbol{m}). \tag{2.24}$$

It is not immediately obvious if this series converges for every $(h, \tilde{g}) \in \mathbb{H}(\mathbb{R}^n) \rtimes \widetilde{\mathrm{Sp}}(n,\mathbb{R})$. This will become clear after writing Θ_f in the appropriate coordinates. See Section 2.3.1 for details.

2.3.1 The Theta Function in Iwasawa coordinates

The Iwasawa decomposition of G with respect to the maximal compact subgroup $K = \mathrm{Sp}(n,\mathbb{R}) \cap \mathrm{O}(2n)$ is

$$g = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \begin{pmatrix} Y^{\frac{1}{2}} & 0 \\ 0 & {}^{\mathrm{t}}Y^{-\frac{1}{2}} \end{pmatrix} k(Q), \tag{2.25}$$

where X and Y are $n \times n$ symmetric matrices, Y is positive-definite, and $Q \in \mathrm{U}(n)$.

Let $\pi : \widetilde{\mathrm{Sp}}(n,\mathbb{R}) \to \mathrm{Sp}(n,\mathbb{R})$ be the standard projection as in (2.13). Let

$$\pi(\tilde{g}) = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \begin{pmatrix} Y^{\frac{1}{2}} & 0 \\ 0 & {}^{\mathrm{t}}Y^{-\frac{1}{2}} \end{pmatrix} k(Q) \tag{2.26}$$

be the Iwasawa decomposition for $\pi(\tilde{g})$ and set

$$h = ((\boldsymbol{x}, \boldsymbol{y}), t). \tag{2.27}$$

Let $f \in \mathcal{S}(\mathbb{R}^n)$. Using Propositions 2.2 and 2.4, we write the theta function in 'Iwasawa coordinates', up to the phase cocycle given in (2.12), as

$$\Theta_f(h,\tilde{g}) = (\det Y)^{1/4} e\big(t - \tfrac{1}{2}\boldsymbol{x}\,{}^{\mathrm{t}}\boldsymbol{y}\big) \sum_{\boldsymbol{m}\in\mathbb{Z}^n} f_Q((\boldsymbol{m}+\boldsymbol{x})Y^{1/2}) e\big(\tfrac{1}{2}(\boldsymbol{m}+\boldsymbol{x})X\,{}^{\mathrm{t}}(\boldsymbol{m}+\boldsymbol{x}) + \boldsymbol{m}\,{}^{\mathrm{t}}\boldsymbol{y}\big)\,. \tag{2.28}$$

Remark 2.5. If $f \in \mathcal{S}(\mathbb{R}^n)$, then Θ_f is given by an absolutely convergent series, and so Θ_f is well defined for every $(h,\tilde{g}) \in \mathbb{H}(\mathbb{R}^n) \rtimes \widetilde{\mathrm{Sp}}(n,\mathbb{R})$. To see this, note that

$$|\Theta_f(h,\tilde{g})| \le (\det Y)^{1/4} \sum_{\boldsymbol{m}\in\mathbb{Z}^n} |f_Q((\boldsymbol{m}+\boldsymbol{x})Y^{1/2})| \tag{2.29}$$

$$\underset{\text{Prop 2.4}}{\ll_{f,A}} (\det Y)^{1/4} \sum_{\boldsymbol{m}\in\mathbb{Z}^n} (1 + (\boldsymbol{m}+\boldsymbol{x})Y^{1/2})^{-A}, \tag{2.30}$$

which clearly converges for A sufficiently large.

Remark 2.6. It is clear from (2.28) that the unitary cocycle given in (2.12) and t play no role in the size of $|\Theta_f|$. Therefore, $|\Theta_f|$ can be seen as a function over the group $\mathbb{R}^{2n} \rtimes \mathrm{Sp}(n,\mathbb{R})$, where $\mathbb{R}^{2n} \rtimes \mathrm{Sp}(n,\mathbb{R})$ has the multiplication

$$(\boldsymbol{p}, g)(\boldsymbol{p}', g') = (\boldsymbol{p} + \boldsymbol{p}'g^{-1}, gg'). \tag{2.31}$$

Remark 2.7. Taking $Y = \frac{1}{M^2}I$, where I is the $n\times n$ identity matrix, and $(\boldsymbol{\alpha},\boldsymbol{\beta}) \in \mathbb{R}^n \times \mathbb{R}^n$, then we have that

$$M^{-n/2}|S_M^f(X;\boldsymbol{\alpha},\boldsymbol{\beta})| = \left|\Theta_f\left((\boldsymbol{\alpha},\boldsymbol{\beta}); \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \begin{pmatrix} \frac{1}{M}I & 0 \\ 0 & MI \end{pmatrix}\right)\right|. \tag{2.32}$$

2.3.2 Invariance Properties for $|\Theta_f|$

Recall that

$$J := \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}. \tag{2.33}$$

Define the $n \times n$ matrices $M_{k,\ell} = (m_{ij})$, whose entries are

$$m_{ij} = \begin{cases} 1 & \text{if } (i,j) = (k,\ell) \text{ or } (\ell,k), \\ 0 & \text{otherwise.} \end{cases} \tag{2.34}$$

Note that $\{M_{k,\ell} : 1 \leq k \leq \ell \leq n\}$ is a generating set for the additive group of symmetric integral matrices. Let

$$\boldsymbol{s}_k := \tfrac{1}{2}(\delta_{kj})_{1\leq j\leq n} \in \mathbb{R}^n, \tag{2.35}$$

where δ_{kj} is a Kronecker delta. More explicitly, $\boldsymbol{s}_k := (0,\dots,0,\frac{1}{2},0,\dots,0) \in \mathbb{R}^n$, where $\frac{1}{2}$ is in the k^{th} entry of $\boldsymbol{s}_k$.

The following lemma summarises the main invariance formulae for $|\Theta_f|$.

Lemma 2.8 ([5], Lemma 2.8)**.** *Let* $(\boldsymbol{p}, g) \in \mathbb{R}^{2n} \rtimes \mathrm{Sp}(n,\mathbb{R})$. *Define*

$$\Gamma := \left\langle \{((0,0),J)\} \cup \left\{ \left((0, \delta_{k,\ell}\boldsymbol{s}_k), \begin{pmatrix} I & M_{k,\ell} \\ 0 & I \end{pmatrix} \right) : 1 \leq k,\ell \leq n \right\} \right.$$
$$\left. \cup \, \{(2\boldsymbol{s}_k, I) : 1 \leq k \leq n\} \right\rangle. \tag{2.36}$$

Then $|\Theta_f(\gamma\cdot(\boldsymbol{p},g))| = |\Theta_f(g)|$, *for all* $\gamma \in \Gamma$.

We note that the group Γ projects to $\mathrm{Sp}(n,\mathbb{Z})$ under the standard projection $\mathbb{R}^{2n} \rtimes \mathrm{Sp}(n,\mathbb{R}) \to \mathrm{Sp}(n,\mathbb{R})$.

Proposition 2.9 ([8], Proposition 6)**.** *The group* $\mathrm{Sp}(n,\mathbb{Z})$ *is generated by*

$$\left\{ J := \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} I & M_{k,\ell} \\ 0 & I \end{pmatrix} : 1 \leq k \leq \ell \leq n \right\}. \tag{2.37}$$

We also note that the group Γ can be described more explicitly, but this is not necessary for our purposes. See [12] for details.

By Remark 2.6 and Lemma 2.8, it follows that $|\Theta_f|$ may be viewed as a non-negative, real valued function on the quotient $\Gamma\backslash\mathbb{R}^{2n} \rtimes \mathrm{Sp}(n,\mathbb{R})$, i.e.

$$|\Theta_f| : \Gamma\backslash(\mathbb{R}^{2n} \rtimes \mathrm{Sp}(n,\mathbb{R})) \to \mathbb{R}_{\geq 0}. \tag{2.38}$$

2.4 Action on $\mathfrak{H}_n$ and Siegel Sets

The Siegel upper half space is defined by

$$\mathfrak{H}_n := \{X + iY : X, Y \in \mathrm{Sym}_n(\mathbb{R}),\ Y \text{ positive-definite}\}. \tag{2.39}$$

Let $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}(n,\mathbb{R})$. The group $\mathrm{Sp}(n,\mathbb{R})$ acts on $(Z,Q) \in \mathfrak{H}_n \times \mathrm{U}(n)$ via

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot (Z,Q) := ((AZ+B)(CZ+D)^{-1}, Q'), \tag{2.40}$$

for some $Q' \in \mathrm{U}(n)$. We note that it is possible to explicitly describe Q', though it is not necessary for our purposes. We also note that this action is a generalisation of the action of $\mathrm{SL}(2,\mathbb{R})$ on $T^1\mathfrak{H}_1$ by Möbius transformations. That this action is well defined (i.e., that $CZ+D$ is invertible) follows from the fact that $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ is symplectic and $\Im(Z)$ is positive-definite. See [8, Proposition 1.1.1] for details.

We say a subset $S \subseteq \mathfrak{H}_n$ is a *Siegel set* for the action of $\mathrm{Sp}(n,\mathbb{Z})$ if it contains at least one, and at most finitely many, representatives of each orbit, where the number of representatives is bounded independently of the individual orbit.

Given a symmetric, positive-definite matrix Y, we have the decomposition

$$Y = {}^{\mathrm{t}}U \mathrm{diag}(v_1, \ldots, v_n) U, \tag{2.41}$$

where $v_i > 0$ and $U = (u_{k\ell})$ is upper triangular and unipotent. Define

$$\mathcal{D}_n'(t) := \{Y = U\mathrm{diag}(v_1, \ldots, v_n)\, {}^{\mathrm{t}}U : v_i > tv_{i+1}, |u_{k\ell}| < t, \text{ for } k < \ell\}, \tag{2.42}$$

and set

$$\mathcal{D}_n(t) := \left\{X + iY \in H_n : |x_{ij}| < t, Y \in \mathcal{D}_n'(t), v_n > \frac{1}{t}\right\}, \tag{2.43}$$

where v_n is the n^{th} entry of the matrix V.

Sets of this form can be shown to be Siegel sets for the action of $\mathrm{Sp}(n,\mathbb{Z})$ on $\mathfrak{H}_n$, provided t is sufficiently large. This is proven, for instance, in [8, Theorem 1.3.1].

For ease of notation in what follows, we will always take t sufficiently large so that $\mathcal{D}_n(t)$, is a Siegel set, so we define

$$\mathcal{D}_n := \mathcal{D}_n(t). \tag{2.44}$$

Define

$$\mathcal{I}_n := \left[-\frac{1}{2}, \frac{1}{2}\right)^n. \tag{2.45}$$

It follows that the set $\mathcal{I}_{2n} \times \mathcal{D}_n \times \mathrm{U}(n)$ is a Siegel set for the action of Γ on $\mathbb{R}^{2n} \times \mathfrak{H}_n \times \mathrm{U}(n)$.

The advantage of studying $|\Theta_f|$ over $\mathcal{I}_{2n} \times \mathcal{D}_n \times \mathrm{U}(n)$ is that the region is much simpler to work with as opposed to genuine fundamental domains for the action of $\mathrm{Sp}(n,\mathbb{Z})$ on $\mathfrak{H}_n$.

3 A Criterion For A Uniform Bound in Siegel Sets

We define the action of Γ on $\mathcal{I}_{2n}$ as

$$(\boldsymbol{p}, \gamma') \cdot \boldsymbol{r} := \{(\boldsymbol{p} + \boldsymbol{r}(\gamma')^{-1}) + \boldsymbol{m} : \boldsymbol{m} \in \mathbb{Z}^{2n}\} \cap \mathcal{I}_{2n}. \tag{3.1}$$

In other words the action is defined by taking a coset representative, all of whose entries lie in $\mathcal{I}_{2n}$. Let $\mathrm{Orb}_\Gamma(\boldsymbol{p})$ denote the orbit of $\boldsymbol{p}$ under the action of Γ on $\mathcal{I}_{2n}$ as defined in (3.1). Note that if $\boldsymbol{p} \in \mathbb{Q}^{2n}$, then $|\mathrm{Orb}_\Gamma(\boldsymbol{p})|$ is finite.

In this section, we prove that if there exists a constant K such that

$$|\Theta_f(\boldsymbol{p}; X+iY, Q)| \leq K \tag{3.2}$$

uniformly in $(X+iY, Q) \in \mathcal{D}_n \times \mathrm{U}(n)$, then $\mathrm{Orb}_\Gamma(\boldsymbol{p})$ cannot contain points of the form $(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{I}_n \times \mathcal{I}_n$ with $\boldsymbol{x} = (0, x_2, \ldots, x_n)$. This follows from Proposition 3.10 below.

Recall that given any $n \times n$ positive definite, symmetric matrix, Y we may decompose $Y = UV\,{}^{\mathrm{t}}U$, where V is an $n \times n$ diagonal matrix, and U is an $n \times n$ upper triangular, unipotent matrix.

Theorem 3.1 ([12], Corollary 4.5)**.** *Let $f \in \mathcal{S}(\mathbb{R}^n)$. Then for any $(X+iY, Q) \in \mathcal{D}_n$, and $(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{I}_n \times \mathcal{I}_n$, we have that*

$$|\Theta_f((\boldsymbol{x}, \boldsymbol{y}); X+iY, Q)| \ll_{f,A} (\det V)^{1/4}(1 + \boldsymbol{x}V\,{}^{\mathrm{t}}\boldsymbol{x})^{-A} \tag{3.3}$$

Corollary 3.2. *Suppose $\boldsymbol{q} \in \mathbb{T}^{2n} \cap \mathbb{Q}^{2n}$ such that $\mathrm{Orb}_\Gamma(\boldsymbol{q})$ contains no vector $(\boldsymbol{x}, \boldsymbol{y})$ where $\boldsymbol{x} = (0, x_2, \ldots, x_n)$. Then there exists a constant $K := K(f, \boldsymbol{q})$ for which*

$$\sup_{\boldsymbol{p} \in \mathrm{Orb}_\Gamma(\boldsymbol{q})} |\Theta_f(\boldsymbol{p}; X+iY, Q)| \leq K. \tag{3.4}$$

Proof. In the definition of $\mathcal{D}_n$, we have that $v_1 \geq c_i v_i$, and so it follows immediately from Theorem 3.1 that if $\boldsymbol{x} = (x_1, \ldots, x_n)$, with $x_1 \neq 0$, then

$$|\Theta_f((\boldsymbol{x}, \boldsymbol{y}); X+iY, Q)| \ll_{f,A} (\det V)^{1/4}(1 + \boldsymbol{x}V\,{}^{\mathrm{t}}\boldsymbol{x})^{-A} \leq (\det V)^{1/4}(x_1 v_1)^{-A}. \tag{3.5}$$

Using that $\mathrm{Orb}_\Gamma(\boldsymbol{q})$ is finite in this case, that $v_n \geq C$, and that $v_1 \gg_j v_j$ by the definition of (2.43), we have the result. □

3.1 Theta Function Asymptotics

We now examine the situation when $\mathrm{Orb}_\Gamma(\boldsymbol{p})$ contains a point of the form $(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{I}_n \times \mathcal{I}_n$ with $\boldsymbol{x} = (0, x_2, \ldots, x_n)$.

We begin by examining the growth of $|\Theta_f|$ in a specific part of the 'cusp' of $\mathcal{D}_n$. The following lemma will be useful in many estimates.

Lemma 3.3 ([8], Lemma 2)**.** *Let $Y \in \mathcal{D}'_n$ and $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Write*

$$Y = UV\,{}^{\mathrm{t}}U, \tag{3.6}$$

where $V = \mathrm{diag}(v_1, \ldots, v_n)$, with $v_i > 0$ for $1 \leq i \leq n$, and U is an upper triangular, unipotent matrix. Then

$$\boldsymbol{x}Y\,{}^{\mathrm{t}}\boldsymbol{x} \asymp_{n,t} x_1^2 v_1 + \cdots + x_n^2 v_n. \tag{3.7}$$

Using the previous lemma we are able to determine the asymptotic behaviour of $|\Theta_f|$. This essentially follows from [12, Theorem 4.4], but as the estimate is much simpler in this particular case, we provide a short proof.

Given $\boldsymbol{p} = (\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^n \times \mathbb{R}^n$, where $\boldsymbol{x} = (x_1, \dots, x_n)$, is such that $\boldsymbol{x}^{(\ell)} := (x_1, \dots, x_\ell) \in \mathbb{Z}^\ell$ and $x_{\ell+1} \notin \mathbb{Z}$, we define

$$\begin{aligned} &|\Theta_f^{(\ell)}(\boldsymbol{p}; X + iY, Q)| \\ &:= (v_{\ell+1} \cdots v_n)^{1/4} \left| \sum_{\boldsymbol{m} \in \{-\boldsymbol{x}^{(\ell)}\} \times \mathbb{Z}^{n-\ell}} f((\boldsymbol{m}+\boldsymbol{x})Y^{1/2}) e\left(\tfrac{1}{2}(\boldsymbol{m}+\boldsymbol{x})X\,{}^{\mathrm{t}}(\boldsymbol{m}+\boldsymbol{x}) + \boldsymbol{m}\,{}^{\mathrm{t}}\boldsymbol{y}\right) \right|. \end{aligned} \tag{3.8}$$

Lemma 3.4. *Let $\boldsymbol{p} = (\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^n \times \mathbb{R}^n$, with $\boldsymbol{x} = (x_1, \dots, x_n)$, is such that $\boldsymbol{x}^{(\ell)} := (x_1, \dots, x_\ell) \in \mathbb{Z}^\ell$ and $x_{\ell+1} \notin \mathbb{Z}$. If $v_i \asymp v_{i+1}$ for $1 < i < n$, then for $(X + iY, Q) \in \mathcal{D}_n \times \mathrm{U}(n)$ we have that*

$$\left| |\Theta_f(\boldsymbol{p}; X + iY, Q)| - (v_1 \cdots v_\ell)^{1/4} |\Theta_f^{(\ell)}(\boldsymbol{p}; X + iY, Q)| \right| \ll_{n,f,A} v_1^{-A}, \tag{3.9}$$

for A sufficiently large.

Proof. We have the following estimate:

$$\left| |\Theta_f(\boldsymbol{p}; X + iY, Q)| - (v_1 \cdots v_\ell)^{1/4} |\Theta_f^{(\ell)}(\boldsymbol{p}; X + iY, Q)| \right| \tag{3.10}$$

$$\leq (v_1 \cdots v_\ell)^{1/4} \sum_{\boldsymbol{m} \in \mathbb{Z}^n \setminus \{-\boldsymbol{x}^{(\ell)}\} \times \mathbb{Z}^{n-\ell}} |f_Q((\boldsymbol{m}+\boldsymbol{x})Y^{1/2})| \tag{3.11}$$

$$\underset{\text{Lem. } 3.3_{n,f}}{\ll} (v_1 \cdots v_\ell)^{1/4} \sum_{\boldsymbol{m} \in \mathbb{Z}^n \setminus \{-\boldsymbol{x}^{(\ell)}\} \times \mathbb{Z}^{n-\ell}} \frac{1}{(1 + (\boldsymbol{m}+\boldsymbol{x})V\,{}^{\mathrm{t}}(\boldsymbol{m}+\boldsymbol{x}))^{\frac{A'}{2}}}. \tag{3.12}$$

We have the estimate for A' sufficiently large so that the sum in (3.12) converges. $\square$

Lemma 3.5. *Let $\boldsymbol{p} = (\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^n \times \mathbb{R}^n$, where with $\boldsymbol{x} = (x_1, \dots, x_n)$, is such that $\boldsymbol{x}^{(\ell)} := (x_1, \dots, x_\ell) \in \mathbb{Z}^\ell$ and $x_{\ell+1} \notin \mathbb{Z}$. If $v_i \asymp v_{i+1}$ for $\ell < i < n$, then for $(X + iY, Q) \in \mathcal{D}_n \times \mathrm{U}(n)$ we have that*

$$|\Theta_f^{(\ell)}(\boldsymbol{p}; X + iY, Q)| \ll_{n,f,A} v_{\ell+1}^{-A}. \tag{3.13}$$

Proof. Similar reasoning to the previous lemma gives

$$|\Theta_f^{(\ell)}(\boldsymbol{p}; X + iY, Q)| \ll_{n,f} (v_{\ell+1} \cdots v_n)^{1/4} \sum_{\boldsymbol{m} \in \{\boldsymbol{x}^{(\ell)}\} \times \mathbb{Z}^{n-\ell}} \frac{1}{(1 + (\boldsymbol{m}+\boldsymbol{x})V\,{}^{\mathrm{t}}(\boldsymbol{m}+\boldsymbol{x}))^{A'/2}} \tag{3.14}$$

$$\ll_{n,f} (v_{\ell+1} \cdots v_n)^{1/4} \sum_{\boldsymbol{m}' \in \mathbb{Z}^{n-\ell}} \frac{1}{(1 + (\boldsymbol{m}'+\boldsymbol{x}')\mathrm{diag}(v_{\ell+1}, \cdots, v_n)\,{}^{\mathrm{t}}(\boldsymbol{m}'+\boldsymbol{x}'))^{A'/2}} \tag{3.15}$$

$$\ll_{n,f,A'} v_{\ell+1}^{-\frac{A'}{2}+\frac{1}{4}}. \tag{3.16}$$

$\square$

3.2 Limit Theorems and Tail Estimates

Let μ denote the Haar measure on $\mathrm{Sp}(n,\mathbb{R})$, normalised so that it descends to a probability measure $\mu_{\mathrm{Sp}(n,\mathbb{Z})}$ on the quotient $\mathrm{Sp}(n,\mathbb{Z})\backslash\mathrm{Sp}(n,\mathbb{R})$. The measure μ in terms of the Iwasawa decomposition

$$g = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \begin{pmatrix} UV^{\frac{1}{2}} & 0 \\ 0 & {}^{\mathrm{t}}U^{-1}V^{-\frac{1}{2}} \end{pmatrix} k(Q) \tag{3.17}$$

is given by

$$\mu := \frac{1}{\mathcal{V}_n} \left(\prod_{1\le i\le j\le n} \mathrm{d}x_{ij} \right) \left(\prod_{1\le i<j\le n} \mathrm{d}u_{ij} \right) \left(\prod_{1\le j\le n} v_j^{-n+j-2}\, \mathrm{d}v_j \right) \mathrm{d}Q. \tag{3.18}$$

where $\mathrm{d}Q$ is the *normalised* Haar measure on $\mathrm{U}(n)$; $\mathrm{d}x_{ij}$, $\mathrm{d}u_{ij}$ and $\mathrm{d}v_{jj}$ are the Lebesgue measures on the entries of X, U, and V respectively; and $\mathcal{V}_n$ is a normalising constant so that μ descends to a probability measure $\mu_{\mathrm{Sp}(n,\mathbb{Z})}$ on $\mathrm{Sp}(n,\mathbb{Z})\backslash\mathrm{Sp}(n,\mathbb{R})$.

For $\boldsymbol{q} \in \mathbb{Q}^{2k}$, we define the measure $\mu^{\boldsymbol{q}}$ on $\mathbb{R}^{2n} \rtimes \mathrm{Sp}(n,\mathbb{R})$ as

$$\mu^{\boldsymbol{q}} := \frac{1}{|\mathrm{Orb}_\Gamma(\boldsymbol{q})|} \left(\sum_{\boldsymbol{k}\in\mathbb{Z}^n} \sum_{\boldsymbol{p}\in\mathrm{Orb}_\Gamma(\boldsymbol{q})} \delta_{\boldsymbol{p}+\boldsymbol{k}} \right) \times \mu, \tag{3.19}$$

where $\delta_{\boldsymbol{v}}$ denotes a delta mass at the point $\boldsymbol{v} \in \mathbb{R}^n$.

Lemma 3.6 ([5], Lemma 2.11)**.** *The measure $\mu^{\boldsymbol{q}}$ is invariant under the action of Γ defined in* (3.1)*, i.e. for any measurable subset A we have*

$$\mu^{\boldsymbol{q}}(\gamma \cdot A) = \mu^{\boldsymbol{q}}(A). \tag{3.20}$$

The above lemma implies that the measure

$$\mu^{\boldsymbol{q}}_\Gamma := \frac{1}{|\mathrm{Orb}_\Gamma(\boldsymbol{q})|} \left(\sum_{\boldsymbol{p}\in\mathrm{Orb}_\Gamma(\boldsymbol{q})} \delta_{\boldsymbol{p}} \right) \times \mu_{\mathrm{Sp}(n,\mathbb{Z})} \tag{3.21}$$

can be viewed as a measure on the quotient $\Gamma\backslash(\mathbb{R}^{2n} \rtimes \mathrm{Sp}(n,\mathbb{R}))$. The measure $\mu_{\mathrm{Sp}(n,\mathbb{Z})}$ is the Haar measure μ on $\mathrm{Sp}(n,\mathbb{R})$, descended to the quotient $\mathrm{Sp}(n,\mathbb{Z})\backslash\mathrm{Sp}(n,\mathbb{R})$, normalised to be a probability measure.

Let $\mathrm{Sym}_n(\mathbb{R})$ denote the space of $n \times n$ symmetric matrices, with real entries.

Theorem 3.7. *Let λ be a probability measure, absolutely continuous with respect to the Lebesgue measure on $\mathrm{Sym}_n(\mathbb{R})$, and let $\boldsymbol{q} := (\boldsymbol{\alpha},\boldsymbol{\beta}) \in \mathbb{Q}^n \times \mathbb{Q}^n$. Then for $R > 0$, we have that*

$$\begin{aligned} \lim_{M\to\infty} \lambda\{X \in \mathrm{Sym}_n(\mathbb{R}) : M^{-n/2}|S^f_M(X;\boldsymbol{\alpha},\boldsymbol{\beta})| > R\} \\ = \mu^{\boldsymbol{q}}_\Gamma\{(\boldsymbol{p}; X+iY, Q) \in \mathcal{I}_{2n} \times \mathcal{F}_n \times \mathrm{U}(n) : |\Theta_f(\boldsymbol{p}; X+iY, Q)| > R\}, \end{aligned} \tag{3.22}$$

where $\mathcal{F}_n$ is a fixed fundamental domain for the action of Γ on $\mathfrak{H}_n$.

Proof. This follows from the equidistribution of Horospheres [7, Theorem 2.2.1] and the Portmanteau theorem of probability. The proof mimics the proof in [5, Theorem 3.5]. □

Remark 3.8. We note that if $S_M^f(X;\boldsymbol{\alpha},\boldsymbol{\beta}) \ll_{n,f} M^{n/2}$ for every M, then there would exist an R_0 such that for all $R > R_0$ the set

$$\{X \in \mathrm{Sym}_n(\mathbb{R}) : M^{-n/2}|S_M^f(X;\boldsymbol{\alpha},\boldsymbol{\beta})| > R\} = \varnothing, \tag{3.23}$$

and so, by Theorem 3.7, for $R > R_0$ we would have

$$\mu_\Gamma^{\boldsymbol{q}}\{(\boldsymbol{p}; X+iY, Q) \in \mathcal{I}_{2n} \times \mathcal{F}_n \times \mathrm{U}(n) : |\Theta_f(\boldsymbol{p}; X+iY, Q)| > R\} = 0. \tag{3.24}$$

Remark 3.9. As $\mathcal{D}_n$ is a Siegel set, it must contain a fundamental domain $\mathcal{F}_n$. Furthermore, there exist $\gamma_1,\ldots,\gamma_k \in \mathrm{Sp}(n,\mathbb{Z})$ such that $\mathcal{D}_n \subseteq \bigcup_i^k \gamma_i\mathcal{F}_n$. It follows that

$$\begin{aligned}\mu_\Gamma^{\boldsymbol{q}}\{(\boldsymbol{p}, Z, Q) \in \mathcal{I}_{2n} \times \mathcal{F}_n \times \mathrm{U}(n) : |\Theta_f(\boldsymbol{p}, Z, Q)| > R\} \geq \\ \frac{1}{k}\,\mu^{\boldsymbol{q}}\{(\boldsymbol{p}, Z, Q) \in \mathcal{I}_{2n} \times \mathcal{D}_n \times \mathrm{U}(n) : |\Theta_f(\boldsymbol{p}, Z, Q)| > R\},\end{aligned} \tag{3.25}$$

and so to prove our characterization, it is enough to lower bound the right hand side of (3.25) by a power of R.

Proposition 3.10. *Let $\boldsymbol{p} = (\boldsymbol{x},\boldsymbol{y}) \in \mathbb{R}^n \times \mathbb{R}^n$, where with $\boldsymbol{x} = (x_1,\ldots,x_n)$, is such that $\boldsymbol{x}^{(\ell)} := (x_1,\ldots,x_\ell) \in \mathbb{Z}^\ell$ and $x_{\ell+1} \notin \mathbb{Z}$. Then*

$$\mu(\{(X+iY, Q) \in \mathcal{F} : |\Theta_f(\boldsymbol{p}; X+iY, Q)| > R\}) \gg R^{-4n}, \tag{3.26}$$

as $R \to \infty$.

Proof. By Remark 3.9 we have that

$$\begin{aligned}&\mu(\{(X+iY, Q) \in \mathcal{F}_n : |\Theta_f(\boldsymbol{p}; X+iY, Q)| > R\}) &(3.27)\\ &\gg \mu(\{(X+iY, Q) \in \mathcal{D}_n : |\Theta_f(\boldsymbol{p}; X+iY, Q)| > R\}). &(3.28)\end{aligned}$$

Let $\mathcal{A}$ be a positive measure subset of $\mathcal{D}_n$ such that $v_1 > C_1 v_2$ and $C_1 v_{i+1} < v_i < C_2 v_{i+1}$ for $1 < i < n$. For $(X+iY, Q) \in \mathcal{A} \times \mathrm{U}(n)$ we have, by Lemma 3.4, that

$$\begin{aligned}&\{(X+iY, Q) \in \mathcal{D}_n \times \mathrm{U}(n) : |\Theta_f(\boldsymbol{p}; X+iY, Q)| > R\} \\ &\supseteq \{(X+iY, Q) \in \mathcal{A} \times \mathrm{U}(n) : (v_1\cdots v_\ell)^{1/4}|\Theta_f^{(\ell)}(\boldsymbol{p}; X+iY, Q)| > T_R\},\end{aligned} \tag{3.29}$$

where $T_R := R - R^{-B}$ for some $B > 0$. We note that this B can be found explicitly, but has no effect on our calculation. Now,

$$\begin{aligned}&\mu(\{(X+iY, Q) \in \mathcal{D}_n \times \mathrm{U}(n) : |\Theta_f(\boldsymbol{p}; X+iY, Q)| > R\}) \\ &\geq \mu\{(X+iY, Q) \in \mathcal{A} \times \mathrm{U}(n) : (v_1\cdots v_\ell)^{1/4}|\Theta_f^{(\ell)}(\boldsymbol{p}; X+iY, Q)| > T_R\}.\end{aligned} \tag{3.30}$$

Using (3.18), our problem reduces to estimating an integral over $\mathcal{A}$ from below. We note that in $\mathcal{A}$, the X and Q variables are constrained to pre-compact regions. We therefore isolate the integral over the v_i coordinates, and so we consider the integral

$$\int_{1/t}^{\infty}\cdots\int_{C_1v_3}^{C_2v_3}\int_{C_1v_2}^{\infty}\mathbb{1}\{(v_1\cdots v_\ell)^{1/4}|\Theta_f^{(\ell)}(\boldsymbol{p};X+iY,Q)|>T_R\}\frac{\mathrm{d}v_1\,\mathrm{d}v_2\,\cdots\,\mathrm{d}v_n}{v_1^{n+1}v_2^n\cdots v_n^2} \tag{3.31}$$

$$=\int_{1/t}^{\infty}\cdots\int_{C_1v_3}^{C_2v_3}\min\left\{\frac{1}{C_1^nv_2^n},\frac{(v_2\cdots v_\ell)^n|\Theta_f^{(\ell)}(\boldsymbol{p};X+iY,Q)|^{4n}}{T_R^{4n}}\right\}\frac{\mathrm{d}v_2\,\cdots\,\mathrm{d}v_n}{v_2^n\cdots v_n^2}, \tag{3.32}$$

where t is as in (2.43). Taking A sufficiently large in Lemma 3.5, and using the fact that in the region of integration $v_i\asymp v_{i+1}$ for $1<i<n$, we have that the inequality

$$\frac{1}{C_1^nv_2^n}<\frac{(v_2\cdots v_\ell)^n|\Theta_f^{(\ell)}|^{4n}}{T_R^{4n}} \tag{3.33}$$

cannot hold for R sufficiently large. Therefore, we have the lower bound

$$\int_{1/t}^{\infty}\cdots\int_{C_1v_3}^{C_2v_3}\min\left\{\frac{1}{C_1v_2^n},\frac{(v_2\cdots v_\ell)^n|\Theta_f^{(\ell)}|^{4n}}{T_R^{4n}}\right\}\frac{\mathrm{d}v_2\,\cdots\,\mathrm{d}v_n}{v_2^n\cdots v_n^2} \tag{3.34}$$

$$\gg T_R^{-4n}\int_{1/t}^{\infty}\cdots\int_{C_1v_3}^{C_2v_3}(v_2\cdots v_\ell)^n|\Theta_f^{(\ell)}|^{4n}\frac{\mathrm{d}v_2\,\cdots\,\mathrm{d}v_n}{v_2^n\cdots v_n^2}. \tag{3.35}$$

In the region $\mathcal{A}$, we have that $v_{\ell+1}\asymp v_i$, and so $(v_2\cdots v_\ell)^n|\Theta_f^{(\ell)}|^{4n}$ is bounded but non-zero in some positive measure subset. Finally, we note that

$$T_R^{-4n}=R^{-4n}\frac{1}{(1-R^{-(B+1)})^{4n}}, \tag{3.36}$$

and so we have the result. □

4 Main Results

In this section, we prove that if $\boldsymbol{q}\in\mathbb{Q}^{2n}$, then there is a point $(\boldsymbol{x},\boldsymbol{y})\in\mathrm{Orb}_\Gamma(\boldsymbol{q})$ where $\boldsymbol{x}=(0,x_2,\ldots,x_n)$. This result, along with the results proven in Section 3, leads to the proofs of Theorem 1.1 and Theorem 1.2.

4.1 Orbit Structures

Let

$$\gamma_{k,\ell}=\left(\delta_{k,\ell}\boldsymbol{s}_k,\begin{pmatrix}I & M_{k,\ell}\\ 0 & I\end{pmatrix}\right)\ \text{and}\ \bar{J}=(\boldsymbol{0},J)\,. \tag{4.1}$$

Note that $\gamma_{k,\ell}$ and $\bar{J}$ are a subset of the generators of Γ as defined in (2.36).

Lemma 4.1. *Let $(\boldsymbol{x},\boldsymbol{y}) = (x_1,\cdots,x_n,y_1,\cdots,y_n) \in \mathbb{Q}^n \times \mathbb{Q}^n$. If there exists an i such that x_i or y_i is 0, then $\mathrm{Orb}_\Gamma(\boldsymbol{x},\boldsymbol{y})$ contains a point whose first coordinate is 0.*

Proof. Suppose there exists an i such that $x_i = 0$. Then $\gamma_{1,i}\bar{J}\gamma_{1,i}\cdot(\boldsymbol{x},\boldsymbol{y})$ has a 0 in the first coordinate. Otherwise, suppose there exists an i such that $y_i = 0$. Then $\gamma_{1,i}\bar{J}\gamma_{1,i}\bar{J}\cdot(\boldsymbol{x},\boldsymbol{y})$ has a 0 in the first coordinate. $\square$

Let $\xi(x)$ denote the distance of $x \in \mathbb{R}$ to the nearest integer. Explicitly, we have $\xi\colon \mathbb{R} \to \left[0,\frac{1}{2}\right]$ as $\xi(x) := \min\left(x - \lfloor x \rfloor, \lceil x \rceil - x\right)$. Note that $\xi(x) = \xi(-x)$. We state the following elementary lemma without proof.

Lemma 4.2. *Let a and b be real numbers that are not integers. Then there exists a nonnegative integer k such that $\xi(a+kb) < \xi(b)$.*

Proposition 4.3. *For any vector $\boldsymbol{v} = (v_1,\cdots,v_{2n}) \in \mathbb{Q}^{2n}/\mathbb{Z}^{2n}$, there exists a vector in its orbit under the action of Γ whose first coordinate is 0.*

Proof. Let $\boldsymbol{v} = (\boldsymbol{x},\boldsymbol{y})$ such that $(\boldsymbol{x},\boldsymbol{y}) \in (\mathbb{Q}^n/\mathbb{Z}^n) \times (\mathbb{Q}^n/\mathbb{Z}^n)$. Write $\boldsymbol{x} = (x_1,\cdots,x_n)$ and $\boldsymbol{y} = (y_1,\cdots,y_n)$. Now let $v_1 = x_1 = a$ and $v_{2n} = y_n = b$. If either a or b is 0, then directly applying Lemma 4.1 gives the result. Now suppose both a and b are nonzero. We apply a version of the Euclidean Algorithm on a and b to obtain a 0.

By Lemma 4.2 there exists a nonnegative integer q_0 such that $\xi(a+q_0b) < \xi(b)$. Then let $m_0 = a + q_0b$, so that

$$\begin{aligned}
\gamma_{1,n}^{q_0}\cdot(\boldsymbol{x},\boldsymbol{y}) &= ((x_1+q_0y_n,\cdots,x_n+q_0y_1),(y_1,\cdots,y_n)) && (4.2)\\
&= ((m_0,\cdots,c_1),(d_1,\cdots,b)). && (4.3)
\end{aligned}$$

Here, and for the rest of this proof, c_i and d_i represent real numbers that are not significant for our proof. Now $\xi(m_0) < \xi(b)$, so we apply $\bar{J}$ to get

$$\bar{J}\cdot((m_0,\cdots,c_1),(d_1,\cdots,b)) = ((d_2,\cdots,-b),(m_0,\cdots,c_2)). \tag{4.4}$$

By Lemma 4.2 again, there exists a nonnegative integer q_1 such that $\xi(-b+q_1m_0) < \xi(m_0)$, so let $m_1 = -b + q_1m_0$. Then

$$\begin{aligned}
\gamma_{1,n}^{q_1}\cdot((d_2,\cdots,-b),(m_0,\cdots,c_2)) &= ((d_3,\cdots,-b+q_1m_0),(m_0,\cdots,c_3)) && (4.5)\\
&= ((d_3,\cdots,m_1),(m_0,\cdots,c_3)). && (4.6)
\end{aligned}$$

Now $\xi(m_1) < \xi(m_0)$, so we apply $\bar{J}$ to get

$$\bar{J}\cdot((d_3,\cdots,m_1),(m_0,\cdots,c_3)) = ((-m_0,\cdots,c_4),(d_4,\cdots,m_1)). \tag{4.7}$$

We can apply Lemma 4.2 to find a nonnegative integer q_2 such that $\xi(-m_0+q_2m_1) < \xi(m_1)$, and let $m_2 = -m_0 + q_2m_1$. Then

$$\begin{aligned}
\gamma_{1,n}^{q_2}\cdot((-m_0,\cdots,c_4),(d_4,\cdots,m_1)) &= ((-m_0+q_2m_1,\cdots,c_5),(d_5,\cdots,m_1)) && (4.8)\\
&= ((m_2,\cdots,c_5),(d_5,\cdots,m_1)). && (4.9)
\end{aligned}$$

In general, we can recursively define q_k to be a nonnegative integer such that $\xi(-m_{k-2}+q_k m_{k-1}) < \xi(m_{k-1})$, and let $m_k = -m_{k-2} + q_k m_{k-1}$. Then the vector

$$\left(\prod_{i=0}^{k} \bar{J}\gamma_{1,n}^{q_i}\right) \cdot \boldsymbol{v} \tag{4.10}$$

contains m_k and $-m_{k-1}$.

Thus, we have constructed a sequence $\boldsymbol{m} = (\xi(m_i))_{i\geq 0}$ such that $\xi(m_0) > \xi(m_1) > \xi(m_2) > \cdots > \xi(m_i) > \cdots$. Since each m_i was recursively constructed from $\mathbb{Z}$-linear combinations of a and b, the denominators (in lowest terms) in $\boldsymbol{m}$ are upper bounded by a constant. As $\boldsymbol{m}$ is also strictly decreasing and nonnegative, it must reach 0. Therefore there exists a nonnegative integer N such that $\xi(m_N) = 0$, so

$$\left(\prod_{i=0}^{N} \bar{J}\gamma_{1,n}^{q_i}\right) \cdot \boldsymbol{v} \tag{4.11}$$

contains $m_N = 0$. Then applying Lemma 4.1 gives the result. $\square$

4.2 Proofs of Main Theorems

We begin with the proof of a slightly more general version of Theorem 1.2.

Theorem 4.4. *Fix $f \in \mathcal{S}(\mathbb{R}^n)$ and $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathbb{Q}^n \times \mathbb{Q}^n$. Let λ be a probability measure on $\mathrm{Sym}_n(\mathbb{R})$ absolutely continuous with respect to Lebesgue measure. Then*

$$\lim_{M\to\infty} \lambda(\{X \in \mathrm{Sym}_n(\mathbb{R}) : M^{-n/2}|S_M^f(X;\boldsymbol{\alpha},\boldsymbol{\beta})| > R\}) \gg_{f,n} R^{-4n}. \tag{4.12}$$

Proof. Let $\boldsymbol{q} = (\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathbb{R}^n \times \mathbb{R}^n$. By Theorem 3.7, we get

$$\begin{aligned}\lim_{M\to\infty} &\lambda\{X \in \mathrm{Sym}_n(\mathbb{R}) : M^{-n/2}|S_M^f(X;\boldsymbol{\alpha},\boldsymbol{\beta})| > R\} \\ &= \mu_\Gamma^{\boldsymbol{q}}\{(\boldsymbol{p}; X+iY, Q) \in \mathcal{I}_{2n} \times \mathcal{F}_n \times \mathrm{U}(n) : |\Theta_f(\boldsymbol{p}; X+iY, Q)| > R\}.\end{aligned} \tag{4.13}$$

By Proposition 4.3, there exists an element γ of Γ such that $\gamma \cdot \boldsymbol{q}$ is of the form $(0, q_2, \cdots, q_{2n})$. Then using Proposition 3.10 on $\gamma \cdot \boldsymbol{q}$ yields

$$\begin{aligned}&\mu_\Gamma^{\boldsymbol{q}}\{(\boldsymbol{p}; X+iY, Q) \in \mathcal{I}_{2n} \times \mathcal{F}_n \times \mathrm{U}(n) : |\Theta_f(\boldsymbol{p}; X+iY, Q)| > R\} \\ &\qquad \geq \mu(\{(X+iY, Q) \in \mathcal{F}_n \times \mathrm{U}(n) : |\Theta_f(\gamma\cdot\boldsymbol{q}; X+iY, Q)| > R\}) \gg R^{-4n}.\end{aligned} \tag{4.14}$$

$\square$

Lastly, we prove Theorem 1.1 as a consequence of Theorem 4.4.

Proof of Theorem 1.1. It directly follows from Theorem 4.4 that for any given R, there exists an $M \in \mathbb{N}$ such that

$$\lim_{M\to\infty} \lambda\{X \in \mathrm{Sym}_n(\mathbb{R}) : M^{-n/2}|S_M^f(X;\boldsymbol{\alpha},\boldsymbol{\beta})| > R\} \gg R^{-4n}, \tag{4.15}$$

so the set

$$\{X \in \mathrm{Sym}_n(\mathbb{R}) : M^{-n/2}|S_M^f(X;\boldsymbol{\alpha},\boldsymbol{\beta})| > R\} \tag{4.16}$$

must be nonempty. $\square$

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