# ORBITS OF STANDARD AND SEMISTANDARD YOUNG TABLEAUX UNDER THE CACTUS GROUP

SOPHIA LIAO

Abstract. The cactus group $J_n$, generated by the Bender–Knuth involutions $t_i$, acts on standard and semistandard Young tableaux by swapping entries of $i$ and $i+1$. The action $J_n$ is a combinatorial abstraction of the problem of finding natural bijections between bases of irreducible representations of the group $S_n$ and the group $S_n \times GL(N)$. We fully classify the orbits of the action of the cactus group on standard Young tableaux and pairs of standard Young tableaux. In particular, we show that the action of $J_n$ is transitive on standard Young tableaux and nearly transitive on pairs, and we conjecture that the image of $J_n$ on standard Young tableaux is either the permutation group or the alternating group. Although standard Young tableaux are transitive under $J_n$, semistandard Young tableaux are not. We establish several invariants, and we find a sufficient condition for one of these invariants to be a complete invariant.

## 1. Introduction

In this paper, we study the orbits of $k$-tuples of standard and semistandard Young tableaux under the cactus group $J_n$. The cactus group $J_n$ is a Coxeter group which can be defined by the generators $s_{i,j}$ where $1 \leq i < j \leq n$ under the following relations:

(1) $s_{i,j}^2 = 1$ for all $1 \leq i < j \leq n$;
(2) $s_{i,j}s_{k,\ell} = s_{k,\ell}s_{i,j}$ if $j < k$;
(3) $s_{i,j}s_{k,\ell}s_{i,j} = s_{i+j-\ell,i+j-k}$ if $i \leq k < \ell \leq j$.

There exists a correspondence $s_{1,i} = t_1(t_2t_1)(t_3t_2t_1)\cdots(t_i\cdots t_1)$ from Schützenberg involutions $s_{1,i}$ to Bender–Knuth involutions $t_i$ as given in [5], and it can be checked that the Bender–Knuth involutions generate $J_n$. A generator $t_i$ acts on a standard Young tableau by permuting the entries $i$ and $i+1$ if the resulting table is a valid standard Young tableau. This action generalizes to semistandard Young tableaux as follows: $t_i$ replaces the $a_k$ free entries of $i$ and $b_k$ free entries of $i+1$ with the $b_k$ entries of $i$ and $a_k$ entries of $i+1$ for every row $k$ in the Young tableau, in the only possible way to preserve horizontally non-decreasing entries. (Under the action $t_i$, an entry $i$ is considered "free" if there is no instance of $i+1$ in the same column; similarly, an entry $i+1$ is considered "free" if there is no instance of $i$ in the same column.)

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The study of the irreducible representations of the symmetric group $S_n$ and the duality between $S_n$ and $\mathrm{GL}(N)$ is widespread and closely related to partitions and their standard and semistandard Young tableaux ([1, 2, 10]). In particular, since the conjugacy classes of a symmetric group $S_n$ correspond to the partitions of $n$ (each part is a disjoint cycle), the number of irreducible representations of a symmetric group $S_n$ equals the number of partitions of $n$. In fact, Schur's construction provides a natural correspondence between these two sets, which preserves several nice properties between the partition and the irreducible representation of $S_n$. In particular, for a partition $\lambda$ of $n$ and the corresponding irreducible representation $V_\lambda$ of $S_n$, the vector space $V_\lambda$ as an $S_{n-1}$ representation is the direct sum of all $V_{\lambda'}$ where $\lambda'$ corresponds to a Young diagram of $\lambda$ with one box removed.

Following [9], the subsequent restriction of an irreducible representation in an inductive chain $S_n \supset S_{n-1} \supset \cdots \supset S_1$ gives rise to a Young–Gelfand–Tsetlin basis. As there are $i$ choices for each restriction $S_i \supset S_{i-1}$, there are in general many different inductive chains, which give rise to distinct bases for a single irreducible representation. However, although there are many ways to relate these bases, there is no canonical bijection between these bases. There are several natural bijections which arise from 1-parametric families of bases connecting different Young–Gelfand–Tsetlin basis, as studied in [4, 7, 12], which generate the cactus group $J_n$ via the correspondence given in [5]. Because each basis element corresponds to a standard Young tableau of the partition associated with the irreducible representation as given by Schur's construction, this problem can be realized purely combinatorially through the action of the cactus group on standard Young tableaux. It is of interest to study the transitivity of this action in order to understand its capability to relate the Young–Gelfand–Tsetlin bases of irreducible representations of $S_n$ to each other.

In Section 2, we recall definitions and terminology relevant to this paper.

In Section 3, we discuss the orbits of $k$-tuples standard Young tableaux of partitions of $n$ under the group action of $J_n$. We first show that single standard Young tableaux are completely transitive under $J_n$. We find, however, that the orbits are more complicated for pairs of standard Young tableaux. In particular, we show that for pairs of certain types of partitions (which we call hook-shaped), there are several orbits, and otherwise, the action of $J_n$ is transitive except for pairs of transposed standard Young tableaux. All of our proofs are constructive, and we demonstrate with an example that the transitivity of pairs of standard Young tableaux is not elementary, as in the case of single standard Young tableaux. We conjecture that the image of $J_n$ on the set of standard Young tableaux of a given partition is either the alternating group or the symmetric group, a result that would imply at least $(N-2)$-transitivity, where $N$ is the number of standard Young tableaux for a particular partition.

In Section 4, we discuss the orbits of semistandard Young tableaux of partitions of $n$ under the group action of $J_n$. We begin by introducing several invariants, and we show that even single semistandard Young tableaux are not transitive under $J_n$, Bender–Knuth involutions preserve the set of counts of each entry in a semistandard Young tableaux (Proposition 4.8). We call a semistandard Young tableaux "semi-transitive" if it is maximally transitive under this invariant. We show that semistandard Young tableaux are also not always semi-transitive, and we illustrate this fact with an example. We present a property of 2-row semistandard Young tableaux which implies semi-transitivity, and we propose a generalization of this property to semistandard Young tableaux with 3 or more rows.

## 2. Preliminaries and Background

In this section, we introduce the notation and terminology that will be used in this paper. Following standard notation, we let $\mathbb{N}$ denote the set of nonnegative integers. For $m, n \in \mathbb{N}$, we set $[\![m, n]\!] := \{k \in \mathbb{N} \mid m \leq k \leq n\}$.

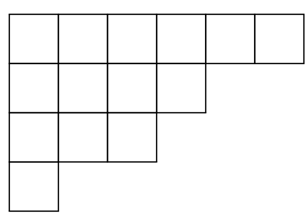
2.1. **Partitions.** A partition $\lambda$ of a positive integer $n$ is a multiset of positive integers $a_1, a_2, \ldots, a_k$ which sum to $n$. We sometimes write the partition partition $\lambda$ as $a_1/a_2/\cdots/a_k$ where $a_1 \geq a_2 \geq \cdots \geq a_k$. When referring to a partition, we interchangeably refer to its Young diagram, using terms such as "rows," "columns," and "boxes" to refer to the coordinates of the Young diagram of the partition. We denote by $r_i(\lambda)$ the number of boxes in row $i$ and by $c_j(\lambda)$ the number of boxes in column $j$ of the standard Young diagram of $\lambda$ (where rows and columns are non-increasing in length).

We denote by $\lambda'$ the transpose partition of $\lambda$ where $r_i(\lambda) = c_i(\lambda')$ for $1 \leq i \leq c_1(\lambda)$ and $c_j(\lambda) = r_j(\lambda')$ for $1 \leq j \leq r_1(\lambda)$.

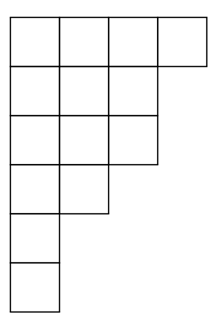
Moreover, a partition $\lambda$ is said to be "hook-shaped" if it does not include a box at (2,2). If $\lambda$ can be written as a hook-shaped partition plus a box at (2,2), it is said to be "almost-hook-shaped."

We show several examples to illustrate these definitions.

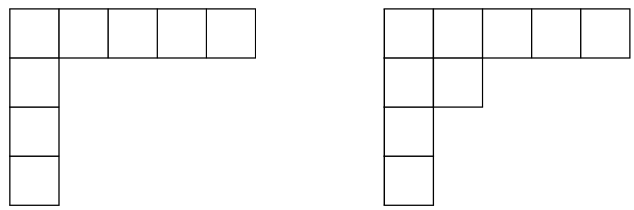
**Example 2.1.** *The following partition $\lambda$ has the shape* 6/4/3/1.



*Its transposed partition $\lambda'$ has the shape* 4/3/3/2/1/1.



**Example 2.2.** *The partition* $5/1/1/1$ *is hook-shaped (left), and* $5/2/1/1$ *is almost-hook-shaped (right).*



2.2. **Standard and Semistandard Young Tableaux.** Let $\lambda$ be an integer partition of $n$, and let $I := \{(i,j) \mid i \in [\![1, c_1(\lambda)]\!], j \in [\![1, i]\!]\}$.A semistandard Young tableau of $\lambda$ with entries up to $N$ is a function $A : I \to [\![1, N]\!]$ where for all $(i, j_1), (i, j_2) \in I$ with $j_1 < j_2$, we have $A(i, j_1) < A(i, j_2)$ and for all $(i_1, j), (i_2, j) \in I$ with $i_1 < i_2$, we have $A(i_1, j) \leq A(i_2, j)$; in other words, the entries in the rows are non-decreasing and the entries in the columns are strictly increasing. A standard Young tableau is a semistandard Young tableau where the final inequality is strict; in other words, the entries in the rows and columns are strictly increasing. Consequently, standard Young tableaux must have exactly one entry of each number, so it follows that $n = N$. We denote by $\mathrm{SYT}(\lambda)$ and $\mathrm{SSYT}(\lambda, N)$ the set of standard Young tableaux of $\lambda$ and the set of semistandard Young tableaux of $\lambda$ with entries up to $N$, respectively.

If $\lambda$ is a partition of $n$, and $A \in \mathrm{SYT}(\lambda)$, we denote by $A \setminus n$ the standard Young tableau of $\lambda$ with the box $A^{-1}(n)$ removed such that $(A \setminus n)^{-1}(k) := A^{-1}(k)$ where $k \in [\![1, n]\!]$. Furthermore, we denote by $A' \in \mathrm{SYT}(\lambda')$ the transpose standard Young tableau of $A$, where $A'(i,j) = A(j,i)$ for all possible $i, j$.

We call a box $(i,j)$ of $\lambda$ a "corner" if there exists a standard Young tableau $A$ of this shape such that $A^{-1}(n) = (i,j)$. We additionally call $(i,j)$ an "extended corner" if there exists a standard Young tableau $A$ of shape $\lambda$ such that $A^{-1}(n) = (i,j)$ and either $A^{-1}(n-1) = (i-1, j)$ or $A^{-1}(n-1) = (i, j-1)$. If $(a,b)$ is a corner of $\lambda$, we denote $\lambda \setminus (a,b)$ as $\lambda$ with the box at $(a,b)$ removed, a partition of $n-1$.

For a semistandard Young tableau $A$, we denote by $r_k(i)_A$ the number of occurrences of $i$ in row $k$. When $A$ is unambiguous, we may simply refer to this value as $r_k(i)$. Moreover, we call $L_k(i)$ the number of locked columns in row $k$ under $t_i$.

## 3. Standard Young Tableaux

In this section, we discuss the nature of the diagonal action of $J_n$ on $k$-tuples of standard Young tableaux. In particular, we show that pairs of standard Young tableaux are nearly transitive, and we classify the orbits when they are not. In addition, we provide an example which illustrates the complexity of this transitivity even in the case of pairs. Finally, we conjecture a that for non-hook-shaped partitions, the image of $J_n$ is either the permutation group or the alternating group, which generalizes our result of 2-transitivity.

Recall that a Bender–Knuth involution $t_i$ swaps the entries $i$ and $i+1$ if the result is a valid standard Young tableau. We begin with the elementary fact that the action of $J_n$ is transitive on the standard Young tableaux of any partition of $n$.

**Theorem 3.1.** *Let $\lambda$ be an integer partition of $n$. Then, $SYT(\lambda)$ is transitive under $J_n$.*

*Proof.* Let $A \in \mathrm{SYT}(\lambda)$. It suffices to show there exists some $g \in J_n$ such that $gA = T$, where $T(i,j) = j + r_1(\lambda) + r_2(\lambda) + \cdots + r_{i-1}(\lambda)$. Let $(i,j)$ be the first coordinate (ordered lexicographically) where $A(i,j) \neq T(i,j)$. We show that we can always find some $h \in J_n$ such that the first coordinate that $hA$ and $T$ do not agree on is greater than $(i,j)$. Observe that $A(i,j) > T(i,j)$ and so $t_{T(i,j)} \cdots t_{A(i,j)-2} t_{A(i,j)-1} A$ agrees with $T$ at the coordinate $(i,j)$, but leaves all previous coordinates untouched, so the first coordinate they do not agree on must be greater than $(i,j)$. $\square$

We now turn to discuss the action of $J_n$ on pairs of standard Young tableaux of partitions of $n$ (which are not necessarily the same). When the partitions are not both hook-shaped, the action is transitive except in when the standard Young tableaux are the same or transposed. However, if the partitions are both hook-shaped, there are several orbits, as we now describe. We make use of a few lemmas, which we present first.

**Definition 3.2.** *Let $\lambda_1, \lambda_2$ be hook-shaped partitions, and let $A \in SYT(\lambda_1), B \in SYT(\lambda_2)$. We denote by $S_{A,B} = \{k \mid A^{-1}(k) = (a,1), B^{-1}(k) = (b,1)$ for some $a,b\}$ the set of shared values in the first rows of $A$ and $B$.*

**Lemma 3.3.** *The value $I := |S_{A,B}|$ is invariant under $J_n$.*

*Proof.* Let $\lambda_1, \lambda_2$ be hook-shaped partitions, and let $A \in \mathrm{SYT}(\lambda_1), B \in \mathrm{SYT}(\lambda_2)$. It suffices to show that $I$ is invariant under any $t_k$. If $A$ and $B$ are both fixed under $t_k$, clearly $I$ is fixed. If neither $A$ nor $B$ are fixed under $t_k$, then it must be the case that $k$ and $k+1$ are swapped between the first row and the first column: If either $k$ or $k+1$ are shared in the first row, then which of $k$ or $k+1$ is shared is simply swapped in $t_kA$ and $t_kB$; if neither $k$ nor $k+1$ are shared in the first row, then $t_k$ preserves that neither are shared. If $A$ is fixed under $t_k$ and $B$ is not, then either $k$ and $k+1$ are both in the first row of $A$ or both in the first column of $A$. In the former case, $t_k$ maintains that one of $k$ or $k+1$ is shared in the first row, and in the latter case, $t_k$ maintains that neither are shared in the first row. The proof is symmetric for if $B$ is fixed and $A$ is not. $\square$

**Lemma 3.4.** *Let $\lambda_1, \lambda_2$ be hook-shaped partitions of $n$, and let $A \in SYT(\lambda_1), B \in SYT(\lambda_2)$. The value $m(A,B) := \max(S_{A,B})$ exists and is not equal to $n$ if and only if there exists some $g \in J_n$ such that $m(gA, gB) > m(A,B)$.*

*Proof.* The reverse direction follows directly. Now, let $k = m(A,B)$. Clearly, $k+1$ is not in the first row of both $A$ and $B$. If $k+1$ is not in the first row of either, then $m(t_kA, t_kB) = k+1$, since $k+1$ is swapped to the first row of both. Otherwise, if $k+1$ is in the first row of either $A$ or $B$ but not the other, then $t_k$ leaves fixed the standard Young tableau with $k$ and $k+1$ in the first row and swaps $k+1$ to the first row of the other, and hence, $m(t_kA, t_kB) = k+1$. □

We are now in a position to describe the orbits of pairs standard Young tableaux of hook-shaped partitions under $J_n$.

**Theorem 3.5.** *Let $\lambda_1, \lambda_2$ be hook-shaped integer partitions of $n$. If $\lambda_1, \lambda_2$ are both hook-shaped, the set of pairs $(A, B)$ for $A \in SYT(\lambda_1), B \in SYT(\lambda_2)$ under the group $J_n$ has $\min(r_1(\lambda_1), c_1(\lambda_1), r_1(\lambda_2), c_1(\lambda_2))$ orbits.*

*Proof.* We proceed by induction. Suppose without loss of generality that $r_1(\lambda_1) \leq c_1(\lambda_1)$, $r_1(\lambda_2), c_1(\lambda_2)$, and let $A \in \mathrm{SYT}(\lambda_1), B \in \mathrm{SYT}(\lambda_2)$. Let $I_r(A, B)$ denote the number of shared values in the first row of $A$ and the first row of $B$ (that is, the size of the set $\{k \mid A^{-1}(k) = (a, 1), B^{-1}(k) = (b, 1) \text{ for some } a, b\}$). Similarly, let $I_c(A, B)$ denote the number of shared values in the first row of $A$ and the first column of $B$ (that is, the size of the set $\{k \mid A^{-1}(k) = (a, 1), B^{-1}(k) = (1, b) \text{ for some } a, b\}$). Observe that $I_r(A, B) + I_c(A, B) = r_1(\lambda_1)$, and there exist pairs of standard Young tableaux for which $I_r, I_c$ attain values from 1 to $r_1(\lambda_1)$. By Lemma 3.3, since $I_r, I_c$ are invariant under $J_n$, it suffices to show that $I_r$ is a full invariant: Given a fixed value of $I_r$, any two pairs of standard Young tableaux of shapes $\lambda_1, \lambda_2$ with this value are in the same orbit.

Without loss of generality, we assume $I_r(A, B) > 1$ (otherwise, we do a symmetric proof on the row of $A$ and the column of $B$, as $I_c(A, B) > 1$). If $r_1(\lambda_1) = 1$, then observe that $A$ is fixed under any action of $J_n$, and so by Theorem 3.1, there is indeed 1 orbit. Now, suppose $r_1(\lambda_1) > 1$. By Lemma 3.4, there exists $g \in J_n$ such that $(gA)^{-1}(n) = (r_1(\lambda_1), 1), (gB)^{-1}(n) = (r_1(\lambda_2), 1)$. Observe that the number of values shared in the first row of $A \setminus n$ and the first row of $B \setminus n$ is $I_r - 1$, and since this is a full invariant for the shapes $\lambda_1 \setminus (r_1(\lambda_1), 1)$ and $\lambda_2 \setminus (r_1(\lambda_2), 1)$, we have that $I_r$ is also a full invariant. □

When $\lambda_1 = \lambda_2$, we can deduce the sizes of the orbits purely combinatorially, since the orbits are defined by the number of shared entries in the first rows of both standard Young tableaux.

**Remark 3.6.** *Let $\lambda$ be a hook-shaped partition of $n$. The lengths of the orbits of pairs of elements of $SYT(\lambda)$ under the group $J_n$ are $N \cdot \binom{k-1}{0}\binom{\ell-1}{0}, N \cdot \binom{k-1}{1}\binom{\ell-1}{1}, \ldots, N \cdot \binom{k-1}{k-1}\binom{\ell-1}{\ell-k}$, where $k := \min(r_1(\lambda), c_1(\lambda))$, $\ell := \max(r_1(\lambda), c_1(\lambda))$, and $N := |SYT(\lambda)|$.*

We now show that pairs of standard Young tableaux of partitions which are not both hook-shaped are almost transitive under $J_n$.

**Theorem 3.7.** *Let $\lambda_1, \lambda_2$ be integer partitions of $n$ which are not both hook-shaped. The following statements are true about the set of pairs $(A, B)$ for $A \in SYT(\lambda_1), B \in SYT(\lambda_2)$ under the group $J_n$:*

(1) *If $\lambda_1 = \lambda_2$ and $\lambda_1 = \lambda_1'$, there are 3 orbits (where $A = B$, $A' = B$, and $A, A' \neq B$).*
(2) *If $\lambda_1 = \lambda_2$ and $\lambda_1 \neq \lambda_1'$, there are 2 orbits (where $A = B$ and $A \neq B$).*

(3) *If* $\lambda_1 \neq \lambda_2$ *and* $\lambda_1 = \lambda_2'$*, there are 2 orbits (where* $A' = B$ *and* $A' \neq B$*).*
(4) *If* $\lambda_1 \neq \lambda_2$ *and* $\lambda_1 \neq \lambda_2'$*, there is 1 orbit.*

*Proof.* We proceed by induction. We begin by showing that the theorem hold for pairs of partitions $\lambda_1, \lambda_2$ of $n$ such that

(i) both $\lambda_1$ and $\lambda_2$ are almost-hook-shaped or
(ii) either $\lambda_1$ or $\lambda_2$ is almost-hook-shaped and the other partition is hook-shaped.

Let's assume that $\lambda_1$ is almost-hook-shaped and $\lambda_2$ is either almost-hook-shaped or hook-shaped. If we have $\lambda_1 = \lambda_2 = 2/2$, or we have $\lambda_1 = 3/2$ and $\lambda_2 = 3/2$ $\lambda_2 = 3/1/1$, $\lambda_2 = 2/1/1/1$, or $\lambda_2 = 1/1/1/1/1$, or we have $\lambda_1 = 3/2/1$ and $\lambda_2 = 4/2$, $\lambda_2 = 4/1/1$, $\lambda_2 = 3/2/1$, $\lambda_2 = 3/1/1/1$, $\lambda_2 = 2/2/1/1$, $\lambda_2 = 2/1/1/1/1$, or $\lambda_2 = 1/1/1/1/1/1$, we can manually verify that the theorem holds.

Otherwise, observe that $\max(r_1(\lambda_1), c_1(\lambda_1)) > 3$, and suppose that the theorem holds for pairs of partitions of $n-1$ which satisfy (i) or (ii). Let $A \in \mathrm{SYT}(\lambda_1), B \in \mathrm{SYT}(\lambda_2)$, and without loss of generality, suppose $r_1(\lambda_1) \geq c_1(\lambda_1)$ and $r_1(\lambda_2) \geq c_1(\lambda_2)$. It suffices to show that there exists some $g \in J_n$ such that $(gA)^{-1}(n) = (r_1(\lambda_1), 1)$ and $(gB)^{-1}(n) = (r_1(\lambda_2), 1)$. By Theorem 3.1, there exists some $h \in J_n$ such that $(hA)^{-1}(n) = (r_1(\lambda_1), 1)$. If $(hB)^{-1}(n) = (r_1(\lambda_2), 1)$, we are done. Otherwise, observe that since $r_1(\lambda_1) > 3$, it follows that $\lambda_1$ with the box $(hA)^{-1}(n)$ taken away is either a hook-shaped partition or an almost-hook-shaped partition with $\max(r_1(\lambda_1), c_1(\lambda_1)) \geq 3$ and $\lambda_2$ with the box $(hB)^{-1}(n)$ taken away is hook-shaped, so there exists some $h' \in J_n$ such that $(h'hA)^{-1}(n-1) = (r_1(\lambda_1)-1, 1)$ and $(h'hB)^{-1}(n-1) = (r_1(\lambda_2), 1)$. Then, $(t_{n-1}h'hA)^{-1}(n) = (r_1(\lambda_1), 1)$ and $(t_{n-1}h'hB)^{-1}(n) = (r_2(\lambda_1), 1)$, as desired.

Now, let $\lambda_1, \lambda_2$ be partitions of $n$ which are not both hook-shaped, and let $A \in \mathrm{SYT}(\lambda_1), B \in \mathrm{SYT}(\lambda_2)$. We assume statement (2) holds for all pairs of partitions of $n-1$. It is clear that if $A = B$ (or $A' = B$) then $gA = gB$ (resp., $gA' = gB$) for all $g \in J_n$. Furthermore, by Lemma 3.1, if there exist $C \in \mathrm{SYT}(\lambda_1), D \in \mathrm{SYT}(\lambda_2)$ such that $C = D$ (resp., $C' = D$), there exists some $g \in J_n$ such that $gA = C, gB = D$.

We now show that all other cases are in one orbit; that is, for any $C \in \mathrm{SYT}(\lambda_1), D \in \mathrm{SYT}(\lambda_2)$ where $A \neq B, A' \neq B$ and $C \neq D, C' \neq D$, there exists some $g \in J_n$ such that $gA = C, gB = D$.

We now show that there exists $h_1 \in J_n$ such that $h_1A \setminus n \neq h_1B \setminus n$ and $(h_1A \setminus n)' \neq h_1B \setminus n$.

**Case 1:** $\lambda_1 = \lambda_2$ or $\lambda_1' = \lambda_2$. If $A \setminus n = B \setminus n$, then clearly $A = B$, and if $(A \setminus n)' = B \setminus n$, then clearly $A' = B$, so we have that $A \setminus n \neq B \setminus n$ and $(A \setminus n)' \neq B \setminus n$, and we are done.

**Case 2:** $\lambda_1 \neq \lambda_2$. Observe that either $\lambda_1$ or $\lambda_2$ must have at least 2 corners—without loss of generality, suppose it is $\lambda_1$. By Theorem 3.1, there exists some $h \in J_n$ such that $(hA)^{-1}(n-1)$ is a corner. Let $\lambda_1^*$ be the shape $\lambda_1$ with the box at $(hA)^{-1}(n)$ removed, and let $\lambda_2^*$ be the shape $\lambda_2$ with the box at $(hB)^{-1}(n)$ removed. If $\lambda_1^* \neq \lambda_2^*$ and $(\lambda_1^*)' \neq \lambda_2^*$, then we are done, as clearly $hA \setminus n \neq hB \setminus n$ and $(hA \setminus n)' \neq hB \setminus n$. If $\lambda_1^* = \lambda_2^*$ and $(\lambda_1^*)' = \lambda_2^*$, then observe that $\lambda_1$ with the box $(t_{n-1}hA)^{-1}(n)$ taken away cannot be the same or transposed shape as $\lambda_2$ with the box $(t_{n-1}hB)^{-1}(n)$ taken away, so we are done. If $\lambda_1^* = \lambda_2^*$ and $(\lambda_1^*)' \neq \lambda_2^*$, and if $hA \setminus n \neq hB \setminus n$, we are done since $(hA \setminus n)' \neq hB \setminus n$. However, if $hA \setminus n = hB \setminus n$, then observe that $(t_{n-1}hA \setminus n)' \neq t_{n-1}hB \setminus n$ and $t_{n-1}hA \setminus n \neq t_{n-1}hB \setminus n$ because $\lambda_1$ with the box $(t_{n-1}hA)^{-1}(n)$ taken away is not the same shape as $\lambda_2$ with the box $(t_{n-1}hB)^{-1}(n)$ taken away, so we are also done.

Let $h_1, h_2 \in J_n$ such that $h_1A \setminus n \neq h_1B \setminus n, (h_1A \setminus n)' \neq h_1B \setminus n$ and similarly, let $h_2C \setminus n \neq h_2D \setminus n, (h_2C \setminus n)' \neq h_2D \setminus n$. We now select a position to move $n-1$ to in $\lambda_1$ for $h_1A$ and $h_2C$.

If $\lambda_1$ has more than two corners, we choose the corner $(i, j)$ for both $h_1A$ and $h_2C$ such that $(i, j) \neq (h_1A)^{-1}(n)$ and $(i, j) \neq (h_2C)^{-1}(n)$. If $\lambda_1$ has two corners or fewer, then observe that $\lambda_1$ has an extended corner $(k, l)$. Without loss of generality, suppose $(k, l) = (h_1A)^{-1}(n)$. For $h_1A$, we select $(k, l-1)$ or $(k-1, l)$ (whichever it is possible to place $n-1$ in by the definition of an extended corner), and for $h_2C$, we select $(k, l)$.

We do the same process to select a position to move $n-1$ to in $\lambda_2$ for $h_1B$ and $h_2D$. Since $h_1A \setminus n, h_2C \setminus n, h_1B \setminus n, h_2D \setminus n$ are in non-same, non-transposed orbits of partitions of $n-1$, there exist $g_1, g_2 \in J_n$ such that $(g_1h_1A)^{-1}(n-1), (g_1h_1B)^{-1}(n-1), (g_2h_2C)^{-1}(n-1), (g_2h_2D)^{-1}(n-1)$ are in the specified positions as given above, so that $(t_{n-1}g_1h_1A)^{-1}(n) = (t_{n-1}g_2h_2C)^{-1}(n)$ and $(t_{n-1}g_1h_1B)^{-1}(n) = (t_{n-1}g_2h_2D)^{-1}(n)$. Since $t_{n-1}g_1h_1A \setminus n \neq t_{n-1}g_1h_1\ B \setminus n, (t_{n-1}g_1h_1A \setminus n)' \neq t_{n-1}g_1h_1B \setminus n$ and $t_{n-1}g_2h_2C \setminus n \neq t_{n-1}g_2h_2D \setminus n, (t_{n-1}g_2h_2C \setminus n)' \neq t_{n-1}g_2h_2D \setminus n$, there exist $f_1, f_2 \in J_n$ such that $f_1t_{n-1}g_1h_1A = f_2t_{n-1}g_2h_2C$ and $f_1t_{n-1}g_1h_1B = f_2t_{n-1}g_2h_2D$, so the group action $f_2t_{n-1}g_2h_2f_1t_{n-1}g_1h_1$ transforms the pair $(A, B)$ into $(C, D)$, as desired.

□

**Corollary 3.8.** *Let $\lambda$ be a partition of $n$ which is not hook-shaped. Under the group $J_n$, the set of pairs $(A, B)$ for $A, B \in SYT(\lambda)$ has 3 orbits (where $A = B$, $A' = B$, and $A, A' \neq B$) when $\lambda = \lambda'$ and 2 orbits (where $A = B$ and $A \neq B$) otherwise.*

We have provided constructive proofs for Theorem 3.5 and Theorem 3.7, but the path from one pair of standard Young tableaux to another is not straightforward, as the following example illustrates.

**Example 3.9.** *Let $\lambda = 5/2$, an integer partition of 7, and let $A, B, L \in SYT(\lambda)$ as given in Figure 1. The minimal group element which takes the pair $(A, B)$ to $(A, L)$ is 12 Bender–Knuth involutions long: $(A, L) = t_4t_5t_4t_3t_4t_3t_2t_4t_3t_5t_4t_6(A, B)$.*

We observe computationally that when $\lambda$ is not hook-shaped, the image of $J_n$ in $S_{|\mathrm{SYT}(\lambda)|}$ has index at most 2, as demonstrated in Table 1 (Appendix A) for partitions up to 18 boxes.

**Conjecture 3.10.** *Let $\lambda$ be a non-hook-shaped partition of $n$ with $\lambda \neq \lambda'$. Then either $S_N \cong J_n(SYT(\lambda))$ or $A_N \cong J_n(SYT(\lambda))$ (where $N = |SYT(\lambda)|$).*

Given that Conjecture 3.10 holds for a partition $\lambda$, we can ascertain whether $S_N \cong J_n(\mathrm{SYT}(\lambda))$ or $A_N \cong J_n(\mathrm{SYT}(\lambda))$ by verifying whether each $t_i$ is an even permutation. This calculation is less computationally demanding, so we show the distribution of even and odd images $J_n(\mathrm{SYT}(\lambda))$ of partitions from 19 to 52 boxes

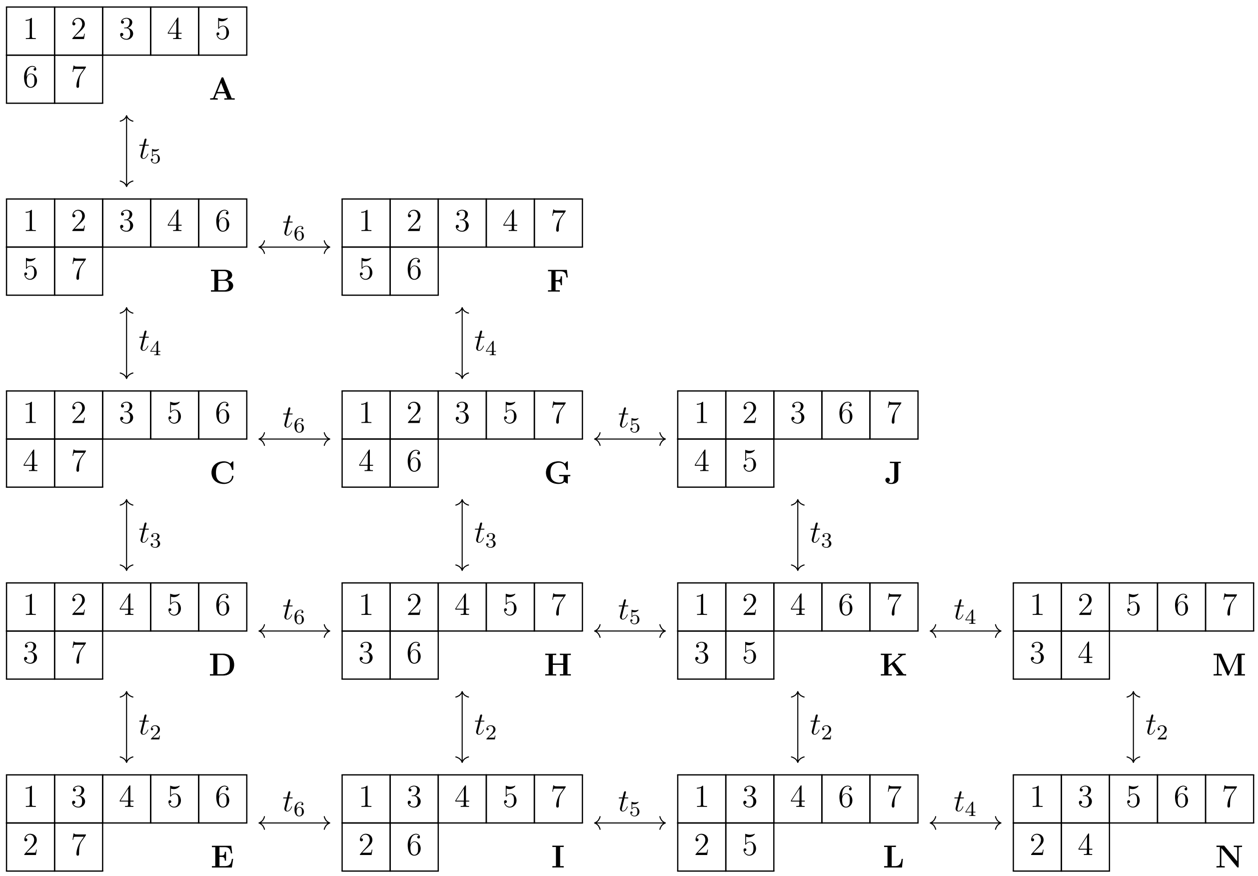



FIGURE 1. All 14 standard Young tableaux of the partition 5/2 and the interactions under nontrivial group operations of $J_7$.

in Table 2 (Appendix A). We include two tables, one which considers all non-hook-shaped partitions $\lambda$ where $\lambda \neq \lambda'$, and the other which only considers "generic" such shapes.

**Definition 3.11.** *Let $\lambda$ be a partition of $n$. We say that $\lambda$ is "generic" if $c_1(\lambda), r_1(\lambda) \leq 2\sqrt{2} \cdot \sqrt{n}$.*

This definition of generic accounts for all randomly chosen partitions of $n$ in the limit according to the Plancherel measure [11]. We conjecture that for generic partitions, the number of even permutations will eventually dominate the number of odd partitions.

**Conjecture 3.12.** *Let $\mathcal{S}_n$ be the set of all generic partitions $\lambda$ of $n$ whose image $J_n(SYT(\lambda))$ is $S_N$, and similarly, let $\mathcal{A}_n$ be the set of all generic partitions $\lambda$ whose image is $A_N$ (where $N = |SYT(\lambda)|$). Then*

$$\lim_{n\to\infty} \frac{|\mathcal{A}_n|}{|\mathcal{S}_n + \mathcal{A}_n|} = 1.$$

When a partition is the same as its transpose, it is easy to see that the image of its standard Young tableaux under $J_n$ must be even.

**Proposition 3.13.** *Let $\lambda$ be a partition of $n > 1$ with $\lambda = \lambda'$. Then $J_n(SYT(\lambda)) \subseteq A_N$ (where $N = |SYT(\lambda)|$).*

*Proof.* This follows readily from the fact that for any $A \in \mathrm{SYT}(\lambda)$, if $t_i A \neq A$, then $t_i A' \neq A'$ and $A \neq A'$. □

## 4. Semistandard Young Tableaux

In this section, we discuss the behavior of semistandard Young tableaux under the cactus group. We prove a condition for 2-row semistandard Young tableaux to be semi-transitive, and we propose a generalization for semistandard Young tableaux with 3 or more rows.

To generalize Bender–Knuth involutions to semistandard Young tableaux as follows, we introduce the following notion of a "free" entry in a semistandard Young tableaux.

**Definition 4.1.** *For an specified index $i$, we say that $i$ is "free" if the box below it does not contain the entry $i+1$, and we say that $i+1$ is "free" if the box above it does not contain the entry $i$.*

The Bender–Knuth involution $t_i$ acts on a semistandard Young tableaux by replacing $a_k$ free entries of $i$ and $b_k$ free entries of $i+1$ with $b_k$ entries of $i$ and $a_k$ entries of $i+1$ for every row $k$ in the Young tableau.

**Example 4.2.** *Consider the following semistandard Young tableaux.*

| | | | | |
|---|---|---|---|---|
| 1 | 1 | 2 | 2 | 3 |
| 2 | 3 | 3 | 3 | |

*The only free boxes containing the entries 2 or 3 are $(1,5), (2,1)$, and $(2,2)$. Since the first row has a single free 3 and no free 2s and the second row has one free 2 and 3, applying $t_2$ yields the following.*

| | | | | |
|---|---|---|---|---|
| 1 | 1 | 2 | 2 | 2 |
| 2 | 3 | 3 | 3 | |

Unlike standard Young tableaux, the set of semistandard Young tableaux of a partition of $n$ is not even transitive under $J_n$.

We now introduce three invariants which help to distinguish these orbits.

**Definition 4.3.** *Call $R$ the set of $r_k(i)$ (the number of occurrences of $i$ in row $k$) for all possible $k$ and $i$.*

**Lemma 4.4.** *The number of locked columns $L_k(i)$ in row $k$ under $t_i$ is a multiple of $\gcd(R)$.*

*Proof.* Observe that $L_k(i) = \sum_{j=1}^{i+1} r_{k+1}(j) - \sum_{j=1}^{i-1} r_k(j)$, a linear combination of elements of $R$. □

We are now in a position to prove our first invariant.

**Proposition 4.5.** *The value* $\gcd(R)$ *is invariant under* $J_N$*.*

*Proof.* Let $A$ be a semistandard Young tableau. It follows from Lemma 4.4 that $\gcd(R)$ is invariant under $t_i$ for an arbitrary row $k$ since the number of occurrences of $i$ and $i+1$ in row $k$ of $t_iA$ are $r_k(i+1)+L_k(i)$ and $r_k(i)-L_k(i)$, respectively, and the number of occurrences of $i$ and $i+1$ in row $k+1$ of $t_iA$ are $r_{k+1}(i+1)-L_k(i)$ and $r_{k+1}(i)+L_k(i)$, respectively. □

Our second invariant only applies to semistandard Young tableaux with two rows.

**Proposition 4.6.** *For* 2*-row semistandard Young tableaux, the value* $\min(R, \{\sum_{i=1}^{j} (r_1(i) - r_2(i+1)) \mid 1 \le j < N\})$ *is invariant under* $J_N$*.*

*Proof.* Let $A$ be a 2-row semistandard Young tableau, and let $w = \min(R, \{\sum_{i=1}^{j} (r_1(i) - r_2(i+1)) \mid 1 \le j < N\})$. Call $r'_1(j)$ the number of occurrences of $j$ in row 1 of $t_iA$. Then $r'_1(i) = r_1(i+1) + L_1(i) \ge w$ and $r'_2(i+1) = r_2(i) + L_1(i) \ge w$. Also,

$$\begin{aligned} r'_1(i+1) &= r_1(i) - L_1(i) \\ &= r_1(i) + \sum_{j=1}^{i-1}(r_1(j) - r_2(j)) - r_2(i) - r_2(i+1) \\ &= r_1(i) + \sum_{j=1}^{i-2}(r_1(j) - r_2(j+1)) + r_1(i-1) - r_2(i) - r_2(i+1) \\ &= \sum_{j=1}^{i}(r_1(j) - r_2(j+1)) \\ &\ge w. \end{aligned}$$

Similarly,

$$\begin{aligned} r'_2(i) &= r_2(i+1) - L_1(i) \\ &= r_2(i+1) + \sum_{j=1}^{i-1}(r_1(j) - r_2(j)) - r_2(i) - r_2(i+1) \\ &= r_2(i+1) + \sum_{j=1}^{i-2}(r_1(j) - r_2(j+1)) + r_1(i-1) - r_2(i) - r_2(i+1) \\ &= \sum_{j=1}^{i-1}(r_1(j) - r_2(j+1)) \\ &\ge w. \end{aligned}$$

To see that $\min(\{\sum_{i=1}^{j}(r_1(i) - r_2(i+1)) \mid 1 \leq j < N\})$ is preserved under $t_i$, observe that

$$\begin{aligned}
\sum_{j=1}^{i}(r_1'(j) - r_2'(j+1)) &= \sum_{j=1}^{i-2}(r_1(j) - r_2(j+1)) + r_1(i-1) \\
&- r_2'(i) + r_1'(i) - r_2'(i+1) \\
&= \sum_{j=1}^{i-2}(r_1(j) - r_2(j+1)) + r_1(i-1) \\
&- (r_2(i+1) - L_1(i)) + (r_1(i+1) + L_1(i)) - (r_2(i) + L_1(i)) \\
&= \sum_{j=1}^{i-2}(r_1(j) - r_2(j+1)) + r_1(i-1) \\
&- r_2(i+1) + r_1(i+1) - r_2(i) + L_1(i) \\
&= \sum_{j=1}^{i-1}(r_1(j) - r_2(j+1)) - r_2(i+1) + r_1(i+1) \\
&+ \sum_{j=1}^{i-2}(r_2(j+1) - r_1(j)) - r_1(i-1) + r_2(i) + r_2(i+1) \\
&= r_1(i-1) - r_2(i) - r_2(i+1) + r_1(i+1) \\
&- r_1(i-1) + r_2(i) + r_2(i+1) \\
&= r_1(i+1) \\
&\geq w.
\end{aligned}$$

□

As arc diagrams are analogous to semistandard Young tableaux of partitions with two rows (under $J_n$) (see [5, 7]), Proposition 4.5 and Proposition 4.6 may be proved with different methods using [3](Lemma 3) and [3](Lemma 2), respectively.

Our third and final invariant is the most natural invariant on semistandard Young tableaux.

**Definition 4.7.** *Let $A \in SSYT(\lambda, N)$. We denote by $\ell_i(A)$ the number of times $i$ occurs in $A$, and we call $\mathcal{L}(A)$ the "set of counts" of entries in $A$, defined as the multiset $\{\ell_i(A) \neq 0\}$.*

**Proposition 4.8.** *Let $A \in SSYT(\lambda, N)$. Then, $\mathcal{L}(A) = \mathcal{L}(gA)$ for any $g \in J_n$.*

*Proof.* This trivially follows from the observation that the number of occurrences of $i$ and $i+1$ are swapped under the Bender–Knuth operation $t_i$. □

As a consequence of Proposition 4.8, we can extend our definition of the "set of counts" to orbits of semistandard Young tableaux.

**Definition 4.9.** *If $\mathcal{O}$ is an orbit of $SSYT(\lambda, N)/J_N$, we define $\mathcal{L}(\mathcal{O}) := \mathcal{L}(A)$, where $A$ is any semistandard Young tableaux in $\mathcal{O}$.*

Although semistandard Young tableaux are not transitive, we can still analyze when a semistandard Young tableau is transitive under the invariant presented in Proposition 4.8.

**Definition 4.10.** *We say that $A \in SSYT(\lambda, N)$ is "semi-transitive" under $J_n$ if for all $A^* \in SSYT(\lambda, N)$ such that $\mathcal{L}(A) = \mathcal{L}(A^*)$, there exists some $g \in J_n$ such that $gA = A^*$.*

However, in general, semistandard Young tableaux are not even semi-transitive, as the following counterexample shows.

**Example 4.11.** *Let $\lambda = 4/2$. The elements*

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 2 \\ \hline 3 & 3 \\ \cline{1-2}\end{array} \qquad \textit{and} \qquad \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 3 \\ \hline 2 & 3 \\ \cline{1-2}\end{array}$$

*of $SSYT(\lambda, 3)$ are not in the same orbit since the latter semistandard Young tableaux is fixed under all elements of $J_n$ (as it is fixed under both $t_1$ and $t_2$).*

We now present a condition on 2-row semistandard Young tableaux which guarantees semi-transitivity. In order to do this, we first prove the following lemma.

**Lemma 4.12.** *Let $A$ be a 2-row semistandard Young tableau such that $A^{-1}(1) = \{(1, 1)\}$, $S(2, 1) = 2$, and $\ell_2(A) \geq \ell_3(A)$. Then, there exists some $g \in J_N$ such that $gA^{-1}(1) = \{(1, 1)\}$, $gA(2, 1) = 3$, $\ell_2(A) = \ell_2(gA)$, and $\ell_3(A) = \ell_3(gA)$.*

*Proof.* If $\ell_2(A) = \ell_3(A)$, then $t_2$ is our desired group element. Otherwise, $\ell_2(A) > \ell_3(A)$, and we show that $(t_2t_1)^3$ is our desired group element. It is easy to see that $\ell_2(A) = \ell_2((t_2t_1)^3A)$ and $\ell_3(A) = \ell_3((t_2t_1)^3A)$, and since $\ell_1(A) = 1$, it must be that $((t_2t_1)^3A)^{-1}(1) = \{(1, 1)\}$. Now let us show that $((t_2t_1)^3A)(2, 1) = 3$. Let $a = \ell_2(A)$, and let $b_1$ and $b_2$ be the number of 3s in the first and second row, respectively (so that $b_1 + b_2 = \ell_3(A)$). Let $T$ be a function from a semistandard young tableau to a tuple $(w_{1,1}, w_{2,1}, w_{2,2}, w_{3,1}, w_{3,2})$ where $w_{i,j}$ refers to the number of occurrences of $i$ in the row $j$. Observe that $T(A) = (1, a - 1, 1, b_1, b_2)$. So, we have the following sequence:

$$\begin{aligned} T(t_1A) &= (a, 0, 1, b_1, b_2) \\ T(t_2t_1A) &= (a, b_1, b_2, 0, 1) \\ T(t_1t_2t_1A) &= (b, a - b_2, b_2, 0, 1) \\ T(t_2t_1t_2t_1A) &= (b, 0, 1, a - b_2, b_2) \\ T(t_1t_2t_1t_2t_1A) &= (1, b - 1, 1, a - b_2, b_2) \\ T(t_2t_1t_2t_1t_2t_1A) &= (1, a, 0, b_1 - 1, b_2 + 1), \end{aligned}$$

as desired (the last step is because $b - 1 \geq b_2$). Since there are no 2s in the second row, it is indeed true that $((t_2t_1)^3A)(2, 1) = 3$. □

**Theorem 4.13.** *Let $\lambda$ be a 2-row partition of $n$, and let $A \in SSYT(\lambda, N)$. If there is some $1 \leq i \leq N$ such that $\ell_i(A) = 1$, then $A$ is semi-transitive.*

*Proof.* We proceed by induction. Suppose the theorem holds for semistandard young tableaux in $\mathrm{SSYT}(\lambda^*, N)$ where $(r_1(\lambda^*), r_2(\lambda^*)) < (r_1(\lambda), r_2(\lambda))$ under the lexicographic order. Our base case is when $N \leq 2$, which is trivially semi-transitive since if $\ell_1 = 1$, there is only one possible arrangement.

We first set $\ell_1(A) = 1$ (by applying $t_1 \cdots t_i$ where $\ell_i(A) = 1$). We then set $\ell_2(A) = \max \mathcal{L}(A)$ (by applying $t_2 \cdots t_j$ where $\ell_j(A) = \max \mathcal{L}(A)$), which leaves $\ell_1(A) = 1$. Let $k = \min \mathcal{L}(A) \setminus \{1\}$, and set $\ell_3(A) = k$ by the same process. If $A(2, i) = 3$ for all $1 \leq i \leq \min(r_2, k)$ (i.e., there is a maximal number of 3s in row 2), we apply $t_N \cdots t_3$and induct on the resulting semistandard young tableau.

Otherwise, it suffices to show that we can increase the number of 3s in the second row. If $A(2, 1) = 2$, we can do this by Lemma 4.12. Let $a$ be the smallest entry which is not a 3 in the second row. Applying $t_{a-1}, \ldots, t_4$, the smallest entry becomes a 4 (since $A(1, i) \in \{1, 2\}$, which is true since $\ell_2(A) = \max \mathcal{L}(A)$). Suppose $i$ is the smallest value such that $A(2, i) = 4$. We outline a procedure to make each value at $(2, j)$ a 3 for all $1 \leq j \leq i$ while maintaining the counts $\ell_1(A)$, $\ell_2(A)$, and $\ell_3(A)$. Let $v$ be the number of 3s in row 2, and let $w$ be the number of 4s in row 2. We first apply $t_3$, which swaps the values $v$ and $w$. Applying Lemma 4.12, we have one 2, $w - 1$ 3s, and $v$ 4s in the second row. Applying $t_3$ again, we have one 2, $v$ 3s, and $w - 1$ 4s in the second row. Lastly, we apply Lemma 4.12, which gives us $v + 1$ 3s and $w - 1$ 4s in the second row, as desired. It is straightforward to see that the counts $\ell_1(A)$, $\ell_2(A)$, and $\ell_3(A)$ remain unchanged since Lemma 4.12 does not modify the counts.

Since we can always increase the number of 3s in the second row (while keeping the count $\ell_3(A)$ fixed as $k$) until the second row has a maximal number of 3s, we can apply induction on the same shape for $A$, which shows that $\mathrm{SSYT}(\lambda, N)$ is semi-transitive. □

Again, using the translation from arc diagrams to 2-row semistandard Young tableaux given in [5, 7], we can separately prove Theorem 4.13 using [3](Theorem 2).

We conjecture that Theorem 4.13 holds for 3-row semistandard Young tableaux, which we computationally confirmed for all semistandard Young tableaux with $n \leq 48$ and $N \leq 6$.

**Conjecture 4.14.** *Let $\lambda$ be a 3-row partition of $n$, and let $A \in SSYT(\lambda, N)$. If $|\mathcal{L}(A)| > 3$ and if there is some $1 \leq i \leq N$ such $\ell_i(A) = 1$, then $A$ is semi-transitive.*

Theorem 4.13 does not generalize past 3 rows, as the following example illustrates.

**Example 4.15.** *The semistandard Young tableaux*

*are not in the same orbit, as the orbit of $A$ consists of the following semistandard Young tableaux:*

*1-2-2-2-3-4/2-3-3-4/3-4/4-5, 1-2-2-2-3-5/2-3-3-5/3-4/5-5,*

| | | | | | |
|---|---|---|---|---|---|
| 1 | 2 | 2 | 2 | 3 | 4 |
| 2 | 3 | 3 | 4 | | |
| 3 | 4 | | | | |
| 4 | 5 | | | | |

$A =$ 1-2-2-2-3-4/2-3-3-4/3-4/4-5

| | | | | | |
|---|---|---|---|---|---|
| 1 | 2 | 2 | 2 | 4 | 4 |
| 2 | 3 | 3 | 3 | | |
| 3 | 4 | | | | |
| 4 | 5 | | | | |

$B =$ 1-2-2-2-4-4/2-3-3-3/3-4/4-5

*1-2-2-2-4-5/2-3-4-5/4-4/5-5, 1-2-3-3-4-5/3-3-4-5/4-4/5-5,*
*1-1-1-1-4-5/2-3-4-5/4-4/5-5, 1-1-1-1-3-5/2-3-3-5/3-4/5-5,*
*1-1-1-1-3-4/2-3-3-4/3-4/4-5, 1-1-1-1-2-4/2-2-2-4/3-4/4-5,*
*1-1-1-1-2-5/2-2-2-5/3-4/5-5, 1-1-1-1-2-3/2-2-2-3/3-3/4-5,*

*none of which are equal to* $B$.

The orbit of $A$ in the above example is minimal in the sense that there is only one semistandard Young tableaux per assignment of numbers $[\![1, 5]\!]$ to $\mathcal{L}(A)$. In fact, the length of any orbit must be a multiple of the number of possible such assignments.

**Proposition 4.16.** *Let* $\mathcal{O}$ *be an orbit of* $SSYT(\lambda, N)$*, for some partition* $\lambda$ *of* $n$*. Let* $m_k := |\{\ell_i = k \mid \ell_i \in \mathcal{L}(O)\}|$*, for* $1 \leq k \leq \max \ell_i$*. Then*

$$\frac{|\mathcal{O}|}{\binom{N}{m_1, \ldots, m_{\max \ell_i}}}$$

*is an integer.*

*Proof.* Let $\sim$ be the relation defined on $\mathcal{O}$ such that for $A, B \in \mathcal{O}$, we have $A \sim B$ if $\ell_i(A) = \ell_i(B)$ for all $1 \leq i \leq N$. Observe that $\sim$ is an equivalence relation, so it partitions $\mathcal{O}$ into $\binom{N}{m_1, \ldots, m_{\max \ell_i}}$ equivalence classes. Let $S$ be some equivalence class of $\mathcal{O}$. Observe that for any $g \in J_N$, the map $\phi_g : S \to S'$ given by $A \mapsto gA$ is injective and surjective, and the image $S'$ of $\phi_g$ is an equivalence class. Hence, each equivalence class of $\mathcal{O}$ is the same size, and consequently, $\mathcal{O}$ is a multiple of $\binom{N}{m_1, \ldots, m_{\max \ell_i}}$. □

We observe through computations that counterexamples as in Example 4.15 are rare and have small orbits, so we conjecture these become negligible as the number of boxes in a partition tends towards infinity.

**Definition 4.17.** *Let* $\lambda$ *be a partition of* $n$*, and let* $A, B \in SSYT(\lambda, N)$ *with* $\mathcal{L}(A) = \mathcal{L}(B)$ *such that* $|\mathcal{L}(A)|, |\mathcal{L}(B)| > c_1(\lambda)$ *and there are some* $1 \leq i, j \leq N$ *such that* $\ell_i(A) = 1, \ell_j(B) = 1$*. If* $A$ *and* $B$ *are not in the same orbit, then we call the orbits of both* $A$ *and* $B$ *"rigid orbits." We use* $R_\lambda$ *to denote the set of rigid orbits of* $\lambda$*.*

**Conjecture 4.18.** *Let* $\lambda$ *be a* $k$*-row partition of* $n$ *where* $k > 3$*, and let* $N$ *be some nonnegative integer. Then*

$$\lim_{n \to \infty} \frac{|R_\lambda|}{|SSYT(\lambda, N)/J_N|} = 0,$$

*and if* $\mathcal{R} \in R_\lambda$*, then*

$$\lim_{n\to\infty} \frac{|\mathcal{R}|}{|SSYT(\lambda, N)|} = 0.$$

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## Appendix A. Computational data on the image $J_n(\mathrm{SYT}(\lambda))$

| $\lvert\lambda\rvert$ | # $S_N$ | # $A_N$ | # other |
|---|---|---|---|
| 2 | 0 | 0 | 0 |
| 3 | 0 | 0 | 0 |
| 4 | 0 | 0 | 0 |
| 5 | 2 | 0 | 0 |
| 6 | 4 | 0 | 0 |
| 7 | 8 | 0 | 0 |
| 8 | 10 | 2 | 0 |
| 9 | 16 | 4 | 0 |
| 10 | 30 | 0 | 0 |
| 11 | 38 | 6 | 0 |
| 12 | 48 | 14 | 0 |
| 13 | 72 | 14 | 0 |
| 14 | 100 | 18 | 0 |
| 15 | 148 | 10 | 0 |
| 16 | 186 | 24 | 0 |
| 17 | 244 | 32 | 0 |
| 18 | 318 | 44 | 0 |

Table 1. The number of images which are $A_N, S_N$, or other of $J_n$ on $\mathrm{SYT}(\lambda)$ for all non-hook-shaped $\lambda$ where $\lambda \neq \lambda'$.

| $\|\lambda\|$ | # odd | # even |
|---|---|---|
| 19 | 400 | 66 |
| 20 | 486 | 114 |
| 21 | 620 | 144 |
| 22 | 756 | 216 |
| 23 | 970 | 254 |
| 24 | 1208 | 332 |
| 25 | 1536 | 386 |
| 26 | 1834 | 564 |
| 27 | 2174 | 796 |
| 28 | 2666 | 1008 |
| 29 | 3044 | 1476 |
| 30 | 3636 | 1920 |
| 31 | 3758 | 3034 |
| 32 | 4740 | 3554 |
| 33 | 5942 | 4144 |
| 34 | 6976 | 5274 |
| 35 | 9164 | 5656 |
| 36 | 10928 | 6980 |
| 37 | 13878 | 7688 |
| 38 | 15480 | 10460 |
| 39 | 18514 | 12592 |
| 40 | 22650 | 14602 |
| 41 | 28492 | 16002 |
| 42 | 32416 | 20664 |
| 43 | 40348 | 22814 |
| 44 | 47580 | 27488 |
| 45 | 53206 | 35816 |
| 46 | 54832 | 50608 |
| 47 | 41542 | 83088 |
| 48 | 51610 | 95528 |
| 49 | 64016 | 109368 |
| 50 | 71824 | 132254 |
| 51 | 96470 | 143316 |
| 52 | 114716 | 166704 |

| $\|\lambda\|$ | # odd | # even |
|---|---|---|
| 19 | 358 | 62 |
| 20 | 420 | 106 |
| 21 | 516 | 132 |
| 22 | 654 | 202 |
| 23 | 806 | 244 |
| 24 | 980 | 304 |
| 25 | 1298 | 368 |
| 26 | 1504 | 528 |
| 27 | 1720 | 732 |
| 28 | 2056 | 900 |
| 29 | 2424 | 1378 |
| 30 | 2830 | 1740 |
| 31 | 2638 | 2818 |
| 32 | 3592 | 3366 |
| 33 | 4426 | 3864 |
| 34 | 5026 | 4838 |
| 35 | 6488 | 5182 |
| 36 | 7556 | 6234 |
| 37 | 10430 | 7008 |
| 38 | 11404 | 9166 |
| 39 | 13430 | 10740 |
| 40 | 16120 | 12222 |
| 41 | 21542 | 14014 |
| 42 | 23956 | 17706 |
| 43 | 29580 | 19060 |
| 44 | 34112 | 22588 |
| 45 | 36846 | 29042 |
| 46 | 40120 | 42082 |
| 47 | 28400 | 67078 |
| 48 | 34666 | 76036 |
| 49 | 42074 | 85966 |
| 50 | 49826 | 108688 |
| 51 | 66464 | 116772 |
| 52 | 77248 | 134248 |

TABLE 2. Number of images which are odd and even $J_n$ on $\mathrm{SYT}(\lambda)$ for all non-hook-shaped $\lambda$ where $\lambda \neq \lambda'$ (left) and all generic non-hook-shaped $\lambda$ where $\lambda \neq \lambda'$ (right).