AFFINE SUBREGULAR KAZHDAN-LUSZTIG POLYNOMIALS IN TYPES D AND E

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ABSTRACT. Affine Weyl groups $\widehat{W}$ have a two-sided cell c_{subreg}—the subregular cell—which decomposes into left cells c^j_{subreg}—subregular left cells—indexed by the set $\widehat{S} = \{s_i\}$ of simple reflections. Bezrukavnikov, Kac, and Krylov compute Kazhdan-Lusztig polynomials on the left cell c^0_{subreg} corresponding to the affine reflection $s_0 \in \widehat{S}$ for simply-laced Lie algebras, and find new character formulas for simple modules of affine Lie algebras. We extend their work and provide an explicit description of the left cell module attached to c^j_{subreg} for all $s_j \in \widehat{S}$ in types D and E. As a corollary, we compute the values of all parabolic inverse affine Kazhdan-Lusztig polynomials on the subregular cell. Moreover, while Bezrukavnikov, Kac, and Krylov's methods were geometric, our methods are purely algebraic, so even when $j = 0$ our proof is new.

1. Introduction

Let $\mathfrak{g}$ be a simple Lie algebra, let $\widehat{\mathfrak{g}}$ be the associated affine Lie algebra, and let $Q^\vee$ be the coroot lattice of $\mathfrak{g}$. Let $\widehat{W} = Q^\vee \rtimes W$ be the affine Weyl group, which acts (denoted by $\circ$) on the dual of the Cartan $\widehat{\mathfrak{h}}^*$ by the shifted action by the sum of the fundamental weights $\widehat{\rho}$. For each character $\Lambda \in \widehat{\mathfrak{h}}^*$, we may define two $\widehat{\mathfrak{g}}$-modules: $M(\Lambda)$, the Verma module, and $L(\Lambda)$, the unique irreducible quotient of $M(\Lambda)$. Although by construction the character of $M(\Lambda)$ is easy to describe, the character of $L(\Lambda)$ is much more complicated to describe. For $\kappa > -h^\vee$ where $h^\vee$ is the dual Coxeter number, let $\mathcal{O}_\kappa$ be the category $\mathcal{O}$ of $\widehat{\mathfrak{g}}$-modules of level κ with some finiteness conditions, as in [KT90]. For $\lambda \in \widehat{\mathfrak{h}}^*$ regular (i.e., the stabilizer of the $\widehat{W}$-action is trivial) with integer level $\kappa = \kappa(\lambda) > -h^\vee$ and $v \in \widehat{W}$, there exists an equality in the K-theory of the category $\mathcal{O}_\kappa$ [KT00, Thm 1.1]:

$$[L(v^{-1} \circ \lambda)] = \sum_{w \in \widehat{W}} \epsilon(wv^{-1}) \mathbf{m}^w_v [M(w^{-1} \circ \lambda)],$$

where $\epsilon \colon \widehat{W} \to \{\pm 1\}$ is the sign, and $\mathbf{m}^w_v := \mathbf{m}^w_v(1)$ are special values of the inverse parabolic Kazhdan-Lusztig polynomials. Thus, calculating the character of $L(\Lambda)$ is essentially equivalent to calculating $\mathbf{m}^w_v$.

Computing Kazhdan-Lusztig polynomials is difficult in general, but when v is restricted to a certain subset c^0_{subreg} of $\widehat{W}$, [BKK23] and [KS24] have computed explicit formulas for $\mathbf{m}_v^w$. They use geometric techniques, using Arkhipov-Bezrukavnikov's equivalence and the geometry of Springer fibers. As a corollary, they are able to compute new character formulas for representations of $\widehat{\mathfrak{g}}$. [KS24] applies these character formulas to obtain explicit formulas for flavoured Schur indices of rank one Argyres-Douglas 4d SCFTs (superconfrmal field theories).

Following [BKK23] and [KS24], we define the following:

Definition 1.1. Let $\widehat{W}$ be an affine Weyl group, with simple reflections $\widehat{S}$. Let c_{subreg} be the *subregular cell*, the set of all non-identity elements in $\widehat{W}$ with a unique reduced word. For each $s_i \in \widehat{S}$, let c^i_{subreg} be a *subregular left cell*, the set of elements $w \in c_{\text{subreg}}$ such that $\ell(ws_i) < \ell(w)$.

Remark. The subregular cell c_{subreg} is a two-sided cell, and the subregular lefts c^j_{subreg} is a left cell in the sense of [Lus85]. When $s_i = s_0$ is the affine reflection, the definition matches [KS24, Definition 1.2].

Definition 1.2. For each $s_j \in \widehat{S}$, let $\widehat{W}_j$ be the Coxeter group generated by $\widehat{S}\backslash\{s_j\}$. Let $\widehat{W}^j$ be the set of $v \in \widehat{W}$ such that v is the shortest element in $v\widehat{W}_j$. There is a bijection $\widehat{W}^j \to \widehat{W}/\widehat{W}_j \simeq Q^\vee$, and we let $\nu \mapsto w_\nu$ be the inverse bijection.

Remark. When $j = 0$, the subgroup $\widehat{W}_j$ is the finite Weyl group, and Definition 1.2 recovers [KS24, Definition 1.1].

Bezrukavnikov, Kac, Krylov, and Suzuki [BKK23, KS24] compute Kazhdan-Lusztig polynomials $\mathbf{m}_v^w$ when $v \in c^0_{\text{subreg}}$ and $w \in \widehat{W}^0$. We extend their result in types D and E, and compute $\mathbf{m}_v^w$ when $v \in c^j_{\text{subreg}}$ and $w \in \widehat{W}^j$, for any $s_j \in \widehat{S}$. Furthermore, we prove our formulas purely linear algebraically, so even when $j = 0$, our proof is new.

Theorem 1.3. *Let $\widehat{W}$ be an affine Weyl group of type $\widetilde{D}_n$ or $\widetilde{E}_n$, and let γ be an element of the coroot lattice $Q^\vee$. For any $s_i \in \widehat{S}$, there is a unique element $w_{ij} \in c^j_{\text{subreg}}$ such that $\ell(s_i w_{ij}) < \ell(w_{ij})$ (see Corollary 2.6). Then*

$$\mathbf{m}_{w_{ij}}^{w_\gamma} = \frac{1}{2}\langle K, \Lambda_i\rangle\langle K, \Lambda_j\rangle\langle \gamma, \gamma\rangle + \langle K, \Lambda_i\rangle\langle \gamma, \Lambda_j\rangle - \langle K, \Lambda_j\rangle\langle \gamma, \Lambda_i\rangle$$

is satisfied for all $0 \leq i \leq r$, where Λ_i are defined in §2.1.

Remark. When $j = 0$, this is proved in [BKK23, §5.3]:

$$\mathbf{m}_{w_{i0}}^{w_\gamma} = \langle \Lambda_i, -\gamma + \frac{1}{2}|\gamma|^2 K\rangle. \tag{1.1}$$

To prove (1.1), [BKK23] proves the following result:

Proposition 1.4 ([BKK23, Proposition 5.16])**.** *Let $\mathcal{H}_{\geq_L c^0_{\text{subreg}}}$ denote the left $\widehat{W}$-module associated to the left cell c^0_{subreg} (see §2.4). Let $\widehat{\mathfrak{h}}$ denote the Cartan subalgebra of $\widehat{\mathfrak{g}}$. Then there is an isomorphism of $\widehat{W}$-modules*

$$\mathcal{H}_{\geq_L c^0_{\text{subreg}}} \otimes_{\mathbb{Z}} \mathbb{C} \simeq \widehat{\mathfrak{h}} \otimes_{\mathbb{C}} \mathbb{C}_{\text{sgn}},$$

sending $\overline{C}_1$ to d and $\overline{C}_{w_{i0}}$ to $-\alpha_i^\vee$.

To prove Theorem 1.3, we prove an analogous isomorphism of $\widehat{W}$-modules:

Proposition 1.5. *There is an isomorphism of $\widehat{W}$-modules*

$$\mathcal{H}_{\geq_L c^j_{\text{subreg}}} \simeq \widehat{\mathfrak{h}}^j_{\mathbb{Z}} \otimes \mathbb{Z}_{\text{sgn}},$$

where the module $\widehat{\mathfrak{h}}^j_{\mathbb{Z}}$ is defined in Definition 4.1.

1.1. **Structure of the paper.** In Section 2, we discuss the notation, terminology, and main known results we will use throughout the rest of the paper. We define the notions of Affine Weyl groups, Hecke algebra, and the subregular cell.

In Section 3, we provide an explicit description of the multiplication structure of the canonical basis in the algebra $\mathcal{H}_{\geq_{LR} c_{\text{subreg}}}$. Using this computation, we prove that the $\mathcal{H}$-bimodule $\widetilde{E}_{\text{subreg}}$ introduced in [KS24] is isomorphic to $\mathcal{H}_{\geq_{LR} c_{\text{subreg}}}$.

In Section 4 we prove the isomorphism between two $\widehat{W}$-modules $\widetilde{E}^j_{\text{subreg}}$ and $\widehat{\mathfrak{h}}^j_{\mathbb{Z}} \otimes \mathbb{Z}_{\text{sgn}}$ and compute the inverse Kazhdan-Lusztig polynomials in the subregular cell in types D and E.

2. Preliminaries

2.1. **Affine Weyl groups.** Let $\mathfrak{g}$ be a simple finite dimensional Lie algebra over $\mathbb{C}$ and fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Let Δ be the set of roots of $(\mathfrak{h}, \mathfrak{g})$ and W be the Weyl group of $(\mathfrak{h}, \mathfrak{g})$. Let Q denote the root lattice of $\mathfrak{g}$. Let $\alpha_1, \ldots, \alpha_r \in \Delta$ be a set of simple roots, and $\theta \in \Delta$ be the highest root. Let $\alpha_1^\vee, \ldots, \alpha_r^\vee \in \mathfrak{h}$ be the simple coroots defined by

$$\langle \alpha_j, \alpha_i^\vee \rangle = a_{ij},$$

where $A = (a_{ij})_{i,j=1,\ldots,r}$ is the Cartan matrix of $\mathfrak{g}$. Let $\Delta^\vee$ be the W-orbit of $\{\alpha_1^\vee, \ldots, \alpha_r^\vee\}$. We also denote $\theta^\vee \in \mathfrak{h}$ to be the highest coroot of $\Delta^\vee$.

Definition 2.1. Let $\widehat{\mathfrak{g}}$ be the *affine Lie algebra corresponding to* $\mathfrak{g}$,

$$\hat{\mathfrak{g}} := \mathfrak{g}[t^{\pm 1}] \oplus \mathbb{C}K \oplus \mathbb{C}d, \tag{2.1}$$

with Lie bracket defined as follows for $a, b \in \mathfrak{g}$ and $m, n \in \mathbb{Z}$:

$$[at^m, bt^n] = [a, b]t^{m+n} + m(a, b)\delta_{m,-n}K, \quad [d, at^n] = nat^n, \quad [K, \hat{\mathfrak{g}}] = 0.$$

The Lie algebra $\widehat{\mathfrak{g}}$ has a symmetric bilinear form $(,)$, defined by

$$(at^m, bt^n) = \delta_{m,-n}(a,b), \quad (\mathbb{C}K \oplus \mathbb{C}d, \mathfrak{g}[t^{\pm 1}]) = 0,$$
$$(K,K) = (d,d) = 0, \quad (K,d) = 1.$$

This bilinear form restricts to a nondegenerate bilinear form on the Cartan subalgebra of $\widehat{\mathfrak{g}}$:

$$\widehat{\mathfrak{h}} := \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}d.$$

We extend every $\gamma \in \mathfrak{h}^*$ to the linear function on $\widehat{\mathfrak{h}}$ by setting $\langle \gamma, \mathbb{C}K \oplus \mathbb{C}d \rangle = 0$. Let $\delta \in \mathfrak{h}^*$ be the linear function given by $\langle \delta, \mathfrak{h} \oplus \mathbb{C}K \rangle = 0$, $\langle \delta, d \rangle = 1$. Set $\alpha_0 := \delta - \theta \in \mathfrak{h}^*$, $\alpha_0^\vee := K - \theta^\vee \in \mathfrak{h}$. Then $\{\alpha_0, \alpha_1, \ldots, \alpha_r\}$ are simple roots of $\widehat{\mathfrak{g}}$ and $\{\alpha_0^\vee, \alpha_1^\vee, \ldots, \alpha_r^\vee\}$ are simple coroots. Define the fundamental weights $\Lambda_i \in \widehat{\mathfrak{h}}^*$ by

$$\langle \Lambda_i, \alpha_j^\vee \rangle := \delta_{i,j}, \quad i,j = 0,1,\ldots,r.$$

Let $\widehat{W}$ be the Weyl group of $\widehat{\mathfrak{g}}$. The group $\widehat{W}$ is generated by the reflections s_i which are defined by

$$s_i(x) := x - \langle \alpha_i, x \rangle \alpha_i^\vee, \quad x \in \widehat{\mathfrak{h}}, \quad i = 0,1,\ldots,r.$$

The set $\widehat{S} := \{s_0, s_1, \ldots, s_r\}$ is the *set of simple reflections of* $\widehat{W}$.

For $\gamma \in Q^\vee$ define the operator t_γ by

$$t_\gamma(x) = x + \langle \delta, x \rangle \gamma - \big((x,\gamma) + \frac{1}{2}|\gamma|^2 \langle \delta, x \rangle\big) K,$$

where $|\gamma|^2 = (\gamma, \gamma)$.

The group $\widehat{W}$ is generated by s_i where $i = 1, \ldots, r$, and t_γ where $\gamma \in Q^\vee$, so

$$\widehat{W} = Q^\vee \rtimes W. \tag{2.2}$$

2.2. **Iwahori-Hecke algebra.** The Iwahori-Hecke algebra is a deformation of the group algebra

$$\mathbb{Z}[\widehat{W}] = \Big\{ \sum_{w \in \widehat{W}} a_w T_w : a_w = 0 \text{ for all but finitely many } w \in \widehat{W} \Big\},$$

of $\widehat{W}$, which is a ring with multiplication $T_v T_w = T_{vw}$.

Let $\mathcal{A}$ denote the ring $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ of Laurent polynomials over $\mathbb{Z}$ in the indeterminate $q^{\frac{1}{2}}$. For $s_1, s_2 \in \widehat{S}$, let $m(s_1, s_2)$ denote the order of $s_1 s_2$. Define the *Hecke algebra* of the affine Weyl group $\widehat{W}$ to be the unital $\mathcal{A}$-algebra $\mathcal{H}$ generated by the set $\{T_s : s \in \widehat{S}\}$ satisfying the relations

$$\big(T_s - q^{\frac{1}{2}}\big)\big(T_s + q^{-\frac{1}{2}}\big) = 0$$

for all $s \in \widehat{S}$ and

$$T_s T_t T_s \cdots = T_t T_s T_t \cdots$$

for all $s, t \in \widehat{S}$ where both sides have $m(s,t)$ factors. The set $\{T_w : w \in \widehat{W}\}$ forms an $\mathcal{A}$-basis, called the *standard basis* of $\mathcal{H}$.

2.3. **Inverse Kazhdan-Lusztig polynomials.** Let $\mathcal{H}$ be the Hecke algebra of affine Weyl group $\widehat{W}$. There is a unique ring homomorphism $^{-}\colon \mathcal{H} \to \mathcal{H}$ such that $\overline{q^{1/2}} = q^{-1/2}$ and $\overline{T_w} = T_{w^{-1}}^{-1}$ for any $w \in \widehat{W}$. Let $\mathcal{A}_{>0} = \sum_{n:n>0} \mathbb{Z}q^{\frac{n}{2}}$ and $\mathcal{H}_{>0} = \sum_{w\in\widehat{W}} \mathcal{A}_{>0}T_w$. The *canonical basis* of $\mathcal{H}$ is the set $\{C_w : w \in \widehat{W}\}$ of unique elements satisfying $\overline{C_w} = C_w$ and $C_w \equiv T_w \pmod{\mathcal{H}_{>0}}$. The set $\{C_w : w \in \widehat{W}\}$ forms an $\mathcal{A}$-basis of $\mathcal{H}$. We define the *Kazhdan-Lusztig polynomials* to be the elements $P_{x,w} \in \mathcal{A}_{\leq 0}$ for which

$$C_w = \sum_{x\leq w} P_{x,w}T_x,$$

for all $w \in \widehat{W}$. We let $\mathbf{m}_u^w(q)$ denote the *inverse Kazhdan-Lusztig polynomial*, characterized by:

$$T_w = \sum_{u\leq w} \epsilon(wu^{-1})\mathbf{m}_u^w(q)C_u$$

where $\epsilon\colon \widehat{W} \to \{\pm 1\}$ is the sign. We are mainly interested in $\mathbf{m}_u^w := \mathbf{m}_u^w(1)$ where u is in the subregular cell.

Remark. For a simple reflection $s \in \widehat{S}$, $C_s = T_s - q^{1/2}$.

2.4. **Subregular cell.** We write $x \prec w$ if $x < w$ in the Bruhat ordering and the degree of $P_{x,w}$ is as large as possible: $(\ell(w) - \ell(x) - 1)/2$. Write x—w if either $x \prec w$ or $w \prec x$. Next for each $w \in \widehat{W}$, define

$$L(w) := \{s \in \widehat{S} \mid sw < w\}, \quad R(w) := \{s \in \widehat{S} \mid ws < w\}.$$

Write $x \leq_L w$ if there exists a chain $x = x_0, x_1, \ldots, x_r = w$ such that x_i—x_{i+1} and $L(x_i)$ is not included in $L(x_{i+1})$ for $0 \leq i < r$. We can define an equivalence relation on $\widehat{W}$: $x \sim_L w$ if and only if both $x \leq_L w$ and $w \leq_L x$ hold. The resulting equivalence classes are called the *left cells* of $\widehat{W}$. We can define $x \leq_R w$ and $x \sim_R w$ analogously by replacing R is place of L. The resulting equivalence classes are called the *right cells* of $\widehat{W}$. We can also define $x \leq_{LR}$ to mean that there exists a chain $x = x_0, x_1, \ldots, x_r = w$ such that for each $i < r$, either $x_i \leq_L x_{i+1}$ or $x_i \leq_R x_{i+1}$. This yields an equivalence relation $x \sim_{LR} w$ whose equivalence classes are called the *two-sided cells* of $\widehat{W}$. Note that the subregular two-sided (resp., left and right) cell is a two-sided (resp., left and right) cell.

The two-sided, left, and right cells are important in computations of Kazhdan-Lusztig polynomials because they define sub-$\mathcal{H}$-modules of $\mathcal{H}$:

Proposition 2.2. *Let $v \in \widehat{W}$. Then*

$$\mathcal{H}_{\leq_L v} := \mathcal{A}\{C_w : w \leq_L v\}, \quad \mathcal{H}_{\leq_R v} := \mathcal{A}\{C_w : w \leq_R v\}, \quad \mathcal{H}_{\leq_{LR} v} := \mathcal{A}\{C_w : w \leq_{LR} v\}$$

are sub-left (resp, right and two-sided) $\mathcal{H}$-submodules of $\mathcal{H}$. Moreover,

$$\mathcal{H}_{\geq_L v} := \mathcal{H}/\mathcal{A}\{C_w : w <_L v\}$$

is a left quotient $\mathcal{H}$-module of $\mathcal{H}$ (and similarly for right and two-sided).

In general, two-sided cells are difficult to explicitly characterize, but the subregular two-sided cell $c_{\text{subreg}} \subset \widehat{W}$ admits a concrete combinatorial interpretation. In this paper, we focus on this particular two-sided cell.

A graph-theoretic characterization of the elements of the subregular cell is provided in [KS24].

Proposition 2.3 ([KS24, Lemma 4.1])**.** *In a Coxeter group $\langle s_1, \ldots, s_k \mid (s_i s_j)^{m(i,j)} = 1\rangle$ with $m(i,i) = 1$ and $m(i,j) \geq 2$ for $i \neq j$, if $w = s_{i_1} \cdots s_{i_n}$ is a unique reduced decomposition, then w defines a path $(i_1, \ldots, i_n)$ on the Coxeter-Dynkin diagram such that an edge ij appears less than $m(i,j) - 1$ times consecutively.*

Using this graph-theoretic characterization, it is much easier to express the elements of the subregular cell completely.

Example 2.4. Consider an affine Coxeter group of type $\widetilde{A}_5$. Through the graph-theoretic characterization of the elements of the subregular cell in Proposition 2.3, we know that the subregular elements of type $\widetilde{A}_n$ are in the forms $s_i s_{i+1} \cdots s_j$ or $s_j s_{j-1} \cdots s_i$ where $i < j$ and $s_k = s_{k+n+1}$ for all k. The diagram below shows the corresponding path of subregular element $s_1 s_2 s_3 s_4$.

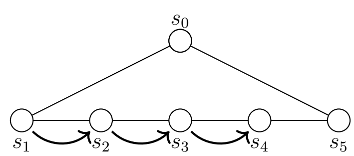

FIGURE 2.1. Path on a Dynkin diagram of type $\widetilde{A}_5$

Example 2.5. Consider an affine Coxeter group of type $\widetilde{B}_5$. From Proposition 2.3, we know subregular elements can be expressed as paths where the edge between s_4 and s_5 can appear twice consecutively, while all other edges appear once consecutively. Therefore we find that there are a finite number of subregular elements. The diagram below shows the corresponding path of subregular element $s_2 s_3 s_4 s_5 s_4$.

Corollary 2.6. *In affine Weyl groups of types D or E, there exists an bijection between $\widehat{S} \times \widehat{S}$ and c_{subreg} for which each $(s_i, s_j) \in \widehat{S} \times \widehat{S}$ is sent to a unique $w_{ij} \in c_{\text{subreg}}$ such that $\ell(s_i w_{ij})$ and $\ell(w_{ij} s_j) < \ell(w_{ij})$.*

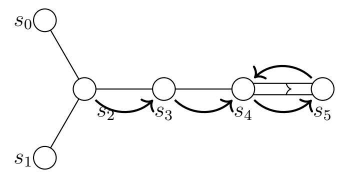

FIGURE 2.2. Path on a Dynkin diagram of type $\widetilde{B}_5$

This corollary follows from Proposition 2.3 since determining two vertices—one starting point and one ending point—in a Coxeter diagram of type D and E determines the path.

3. Lusztig's description of the subregular Hecke algebra

We explicitly describe the multiplication structure of the subregular Hecke algebra $\mathcal{H}_{\geq_{LR} c_{\text{subreg}}}$ in this section, using results from [KS24].

Definition 3.1. For $s_j \in \widehat{S}$, let c^j_{subreg} be the subregular left cell as in §2.4. Then the set c^j_{subreg} inherits a graph structure by drawing an edge between y and w when there exists a $t \in \widehat{S}$ such that $y = tw$. Let Γ_{s_j} be the graph with underlying set c^j_{subreg}, with edges between $y, w \in c_{\text{subreg}}$ if there exists a simple reflection $s \in \widehat{S}$ such that $y = sw$ or $y = ws$. We call Γ_{s_j} the *W-graph* of $\widehat{W}$. We let $\mu\colon c_{\text{subreg}} \to \widehat{S}$ send $w \in \widehat{W}$ to the unique simple reflection $s \in \widehat{S}$ such that $ws < w$.

Remark. The graph Γ_{s_0} is explicitly described in [KS24, §4.1].

[KS24, Definition 3.3] defines the following $\mathcal{H}$-bimodule:

Definition 3.2. Let $\widetilde{E}_{\text{subreg}}$ be the $\mathcal{H}$-bimodule with $\mathbb{Z}[q^{\pm 1/2}]$-basis $\{e_w : w \geq_{LR} c_{\text{subreg}}\} = \{e_1\} \cup \{e_w : w \in c_{\text{subreg}}\}$ such that for $w \in c_{\text{subreg}}$ and $t \in \widehat{S}$,

$$T_t(e_w) = \begin{cases} -q^{-1/2} e_w & \text{if } \ell(tw) < \ell(w) \\ q^{1/2} e_w + \displaystyle\sum_{\substack{y \in c_{\text{subreg}}, ty<y, \\ y=sw \text{ for some } s \in \widehat{S}}} e_y & \text{if } \ell(tw) > \ell(w). \end{cases}$$

$$(e_w)T_t = \begin{cases} -e_w & \text{if } \ell(wt) < \ell(w) \\ q e_w + q^{1/2} \displaystyle\sum_{\substack{y \in c_{\text{subreg}}, yt<y, \\ y=ws \text{ for some } s \in \widehat{S}}} e_y & \text{if } \ell(wt) > \ell(w), \end{cases}$$

and

$$T_t \cdot e_1 = e_1 \cdot T_t = q^{1/2} e_1 + e_t. \tag{3.1}$$

Let E_{subreg} be the sub-bimodule spanned by $\{e_w : w \in c_{\text{subreg}}\}$. For any simple reflection $s_j \in \widehat{S}$, let $\widetilde{E}^j_{\text{subreg}}$ be the sub-$\mathcal{H}$-module spanned by $\{e_w : w \in c^j_{\text{subreg}} \cup \{1\}\}$, and let E^j_{subreg} be the sub-module spanned by $\{e_w : w \in c^j_{\text{subreg}}\}$.

By [KS24, Proposition 3.4], we have:

Theorem 3.3. *There is an isomorphism of $\mathcal{H}$-bimodules*

$$\widetilde{E}_{\text{subreg}} \simeq \mathcal{H}_{\geq_{LR} c_{\text{subreg}}}$$

sending e_w to C_w for all $w \in c_{\text{subreg}}$. In particular, there is an isomorphism of $\mathcal{H}$-modules

$$\widetilde{E}^j_{\text{subreg}} \simeq \mathcal{H}_{\geq_L c^j_{\text{subreg}}}.$$

4. Main Results

4.1. **Defining the $\widehat{W}$-module $\widehat{\mathfrak{h}}^j_{\mathbb{Z}}$.** Recall that in types D and E [BKK23, Proposition 5.16] gives an isomorphism of $\widehat{W}$-modules

$$\mathcal{H}_{\geq_L c^0_{\text{subreg}}} \simeq \widehat{\mathfrak{h}}_{\mathbb{Z}} \otimes \mathbb{Z}_{\text{sgn}},$$

where $\widehat{\mathfrak{h}}_{\mathbb{Z}}$ is the integral form of the Cartan algebra $\widehat{\mathfrak{h}}$. It sits in a short exact sequence

$$0 \to \mathfrak{h}_{\mathbb{Z}} \oplus \mathbb{Z}K \to \widehat{\mathfrak{h}}_{\mathbb{Z}} \to \mathbb{Z}d \to 0.$$

For each $0 \leq j \leq r$ we define another $\widehat{W}$-module $\widehat{\mathfrak{h}}^j_{\mathbb{Z}} = \mathfrak{h}_{\mathbb{Z}} \oplus \mathbb{Z}K \oplus \mathbb{Z}d^j$ again as a rank one extension of $\mathfrak{h}_{\mathbb{Z}} \oplus \mathbb{Z}K$, sitting in a short exact sequence

$$0 \to \mathfrak{h}_{\mathbb{Z}} \oplus \mathbb{Z}K \to \widehat{\mathfrak{h}}^j_{\mathbb{Z}} \to \mathbb{Z}d^j \to 0.$$

Definition 4.1. For any $0 \leq j \leq r$, let $\widehat{\mathfrak{h}}^j_{\mathbb{Z}} = \mathfrak{h}_{\mathbb{Z}} \oplus \mathbb{Z}K \oplus \mathbb{Z}d^j$ be the $\widehat{W}$-module with the usual $\widehat{W}$-action on $\mathfrak{h}_{\mathbb{Z}} \oplus \mathbb{Z}K$ and for any $\gamma \in Q^\vee$,

$$t_\gamma(d^j) = d^j + \langle K, \Lambda_j \rangle \gamma - \left(\langle \gamma, \Lambda_j \rangle + \frac{1}{2}|\gamma|^2 \langle K, \Lambda_j \rangle \right) K \tag{4.1}$$

and for any $0 \leq i \leq r$,

$$s_i(d^j) = \begin{cases} -d^j & \text{if } i \neq j \\ -d^j + \alpha_j^\vee & \text{if } i = j. \end{cases} \tag{4.2}$$

Remark. When $j = 0$, the module $\widehat{\mathfrak{h}}^j_{\mathbb{Z}}$ is the integral form of the Cartan $\widehat{\mathfrak{h}}$ of $\widehat{\mathfrak{g}}$.

We now check our formulas define a well-defined action of $\widehat{W}$ on $\mathfrak{h}_{\mathbb{Z}} \oplus \mathbb{Z}K \oplus \mathbb{Z}d^j$.

Proposition 4.2. *For any $j \neq 0$, the formulas (4.1) and (4.2) make $\widehat{\mathfrak{h}}^j_{\mathbb{Z}}$ a $\widehat{W}$-module.*

Proof. We need to check that for any $x \in \widehat{\mathfrak{h}}^j_{\mathbb{Z}}$, any $\gamma, \gamma' \in Q^\vee$, and any $w \in W$, the following equalities hold:

$$t_\gamma t_{\gamma'}(x) = t_{\gamma+\gamma'}(x) \tag{4.3}$$

$$w t_\gamma w^{-1}(x) = t_{w(\gamma)}(x). \tag{4.4}$$

When $x \in \mathfrak{h}_{\mathbb{Z}} \oplus \mathbb{Z}K$, the relation is clearly satisfied, so it suffices to check for $x = d^j$. Equation (4.3) holds since

$$\begin{aligned} t_\gamma(t_{\gamma'}(d^j)) &= t_\gamma\Big(d^j + \langle K, \Lambda_j\rangle\gamma' - \big(\langle \gamma', \Lambda_j\rangle + \frac{1}{2}\langle K, \Lambda_j\rangle|\gamma'|^2\big)K\Big) \\ &= d^j + \langle K, \Lambda_j\rangle(\gamma+\gamma') - \big(\langle \gamma+\gamma', \Lambda_j\rangle + \frac{1}{2}\langle K, \Lambda_j\rangle(|\gamma|^2 + 2(\gamma,\gamma') + |\gamma'|^2)\big)K \\ &= d^j + \langle K, \Lambda_j\rangle(\gamma+\gamma') - \big(\langle \gamma+\gamma', \Lambda_j\rangle + \frac{1}{2}\langle K, \Lambda_j\rangle|\gamma+\gamma'|^2\big)K \\ &= t_{\gamma+\gamma'}(d^j). \end{aligned}$$

To check (4.2), it suffices to let $w = s_i$ for some $1 \le i \le r$ since the simple reflections generate W. Recall that from Section 2, we have

$$s_i(\lambda) = -\lambda + \langle \lambda, \alpha_i\rangle \alpha_i^\vee$$

for $\lambda \in \widehat{\mathfrak{h}}_{\mathbb{Z}} \otimes \mathbb{Z}_{\mathrm{sgn}}$. We proceed by case work. If $i \neq j$,

$$\begin{aligned} s_i(t_\gamma(s_i(d^j))) &= s_i(t_\gamma(-d^j)) \\ &= s_i\left(-d^j - \langle K, \Lambda_j\rangle\gamma + \left(\langle \gamma, \Lambda_j\rangle + \frac{1}{2}\langle K, \Lambda_j\rangle|\gamma|^2\right)K\right) \\ &= d^j + \langle K, \Lambda_j\rangle(\gamma - \langle \gamma, \alpha\rangle\alpha^\vee) - \left(\langle \gamma, \Lambda_j\rangle + \frac{1}{2}\langle K, \Lambda_j\rangle|\gamma|^2\right)K \\ &= d^j + \langle K, \Lambda_j\rangle s_i(\gamma) - \left(\langle s_i(\gamma), \Lambda_j\rangle + \frac{1}{2}\langle K, \Lambda_j\rangle|s_i(\gamma)|^2\right)K \\ &= t_{s_i(\gamma)}(d^j) \end{aligned}$$

since $\langle \alpha^\vee, \Lambda_j\rangle = 0$ and $|\gamma| = |s_i(\gamma)|$. If $i = j$,

$$\begin{aligned} s_i(t_\gamma(s_i(d^j))) &= s_i(t_\gamma(-d^j + \alpha_j^\vee)) \\ &= s_i\left(-d^j - \langle K, \Lambda_j\rangle\gamma + \left(\langle \gamma, \Lambda_j\rangle - \langle \alpha_j^\vee, \gamma\rangle + \frac{1}{2}\langle K, \Lambda_j\rangle|\gamma|^2\right)K + \alpha_j^\vee\right) \\ &= d^j + \langle K, \Lambda_j\rangle(\gamma - \langle \gamma, \alpha\rangle\alpha^\vee) - \left(\langle \gamma, \Lambda_j\rangle - \langle \alpha_j^\vee, \gamma\rangle + \frac{1}{2}\langle K, \Lambda_j\rangle|\gamma|^2\right)K \\ &= d^j + \langle K, \Lambda_j\rangle s_i(\gamma) - \left(\langle s_i(\gamma), \Lambda_j\rangle + \frac{1}{2}\langle K, \Lambda_j\rangle|s_i(\gamma)|^2\right)K = t_{s_i(\gamma)}(d^j) \end{aligned}$$

since $\langle s_i(\gamma), \Lambda_j\rangle = \langle \gamma, \Lambda_j\rangle - \langle \alpha_j^\vee, \gamma\rangle$. ∎

4.2. **Proving the isomorphism of $\widehat{W}$-modules.** By Theorem 3.3, to prove Proposition 1.5, it suffices to check:

Proposition 4.3. *For any $0 \leq j \leq r$ there is an isomorphism of $\widehat{W}$-modules*

$$\widetilde{E}^j_{\mathrm{subreg}} \simeq \widehat{\mathfrak{h}}^j_{\mathbb{Z}} \otimes \mathbb{Z}_{\mathrm{sgn}}$$

sending e_1 to d^j and $e_{w_{ij}}$ to $-\alpha_j^\vee$.

First, we check the isomorphism on the submodule E^j_{subreg}.

Lemma 4.4. *For any $0 \leq j \leq r$ there is an isomorphism of $\widehat{W}$-modules*

$$E^j_{\mathrm{subreg}} \simeq (\mathfrak{h}_{\mathbb{Z}} \oplus \mathbb{Z}K) \otimes \mathbb{Z}_{\mathrm{sgn}}$$

sending $e_{w_{ij}}$ to $-\alpha_j^\vee$.

Proof. Observe that there is an isomorphism of $\widehat{W}$-modules $\widetilde{E}^j_{\mathrm{subreg}} \simeq \widetilde{E}^0_{\mathrm{subreg}}$ sending $e_{w_{ij}}$ to $e_{w_{i0}}$ since the action of s_i on both $e_{w_{ij}}$ and $e_{w_{i0}}$ intertwines with the $\widehat{W}$ action. Then from [BKK23, Proposition 5.16] we know that $\widetilde{E}^0_{\mathrm{subreg}} \simeq (\mathfrak{h}_{\mathbb{Z}} \oplus \mathbb{Z}K) \otimes \mathbb{Z}_{\mathrm{sgn}}$ so the isomorphism $E^j_{\mathrm{subreg}} \simeq (\mathfrak{h}_{\mathbb{Z}} \oplus \mathbb{Z}K) \otimes \mathbb{Z}_{\mathrm{sgn}}$ follows. □

Using this Lemma, we proceed to prove that $\widetilde{E}^j_{\mathrm{subreg}} \simeq \widehat{\mathfrak{h}}^j_{\mathbb{Z}} \otimes \mathbb{Z}_{\mathrm{sgn}}$.

Proof of Proposition 4.3. Let φ be the linear map $\widetilde{E}^j_{\mathrm{subreg}} \to \widehat{\mathfrak{h}}^j_{\mathbb{Z}} \otimes \mathbb{Z}_{\mathrm{sgn}}$ sending e_1 to d^j and $e_{w_{ij}}$ to $-\alpha_j^\vee$. This sends basis elements to basis elements, so φ is an isomorphism of abelian groups. It suffices to check that φ intertwines the $\widehat{W}$-action. Moreover, by Lemma 4.4 we know φ is an isomorphism on the submodule E^j_{subreg}, it suffices to check that for any $w \in \widehat{W}$,

$$\varphi(w(e_1)) = w(d^j).$$

In fact, since $\widehat{W}$ is generated by simple reflections, it suffices to check the equality on $w \in \widehat{S}$. We prove this for $j \neq 0$, and the argument for $j = 0$ is similar.

When $w = s_i$ for $i \neq 0$, then $\varphi(s_i(e_1)) = s_i(d^j)$ since by (4.2)

$$s_i(d^j) = \begin{cases} d^j & \text{if } i \neq j \\ d^j - \alpha_j^\vee & \text{if } i = j. \end{cases}$$

and by (3.1)

$$s_i(e_1) = \begin{cases} e_1 & t \neq s_j \\ e_1 + e_{s_j} & i = j. \end{cases}$$

When $w = s_0$, we have $s_0(e_1) = e_1$ while since $s_0 = t_{\theta^\vee} s_\theta$ where θ is the highest root,

$$\begin{aligned} s_0(d^j) &= t_{\theta^\vee}(s_\theta(d^j)) = t_{\theta^\vee}(d^j - \langle \Lambda_j, \theta \rangle \theta^\vee) \\ &= d^j + \langle K, \Lambda_j \rangle \theta - \langle \Lambda_j, \theta \rangle \theta^\vee - \langle K - \theta, \Lambda_j \rangle K = d^j - \langle \alpha_0, \Lambda_j \rangle K = d^j. \end{aligned}$$ □

4.3. **Computation of inverse Kazhdan-Lusztig polynomials.** The inverse Kazhdan-Lusztig polynomials $\mathbf{m}_{w_i}^{w_\gamma}$ were defined in §2.3 by the relation

$$T_\gamma = T_\gamma(C_1) = C_1 + \sum_{i=0}^{r} \mathbf{m}_{w_i}^{w_\gamma} C_{w_i}.$$

By Proposition 1.5 the computation of inverse Kazhdan-Lusztig polynomials reduces to calculating the integers $\mathbf{m}_{w_k}^{w_\gamma}$ for which

$$t_\gamma(e_1) = e_1 + \sum_{i=0}^{r} \mathbf{m}_{w_{ij}}^{w_\gamma} e_{w_{ij}}.$$

Theorem 4.5. *Let $\widehat{W}$ be an affine Weyl group of type $\widetilde{D}_n$ or $\widetilde{E}_n$, and let γ be an element of $Q^\vee$. For any $s_i \in \widehat{S}$, there is a unique element $w_{ij} \in c^j_{\text{subreg}}$ such that $\ell(s_i w_{ij}) < \ell(w_{ij})$ (see Corollary 2.6). Then*

$$\mathbf{m}_{w_{ij}}^{w_\gamma} = \frac{1}{2}\langle K, \Lambda_i\rangle\langle K, \Lambda_j\rangle\langle \gamma, \gamma\rangle + \langle K, \Lambda_i\rangle\langle \gamma, \Lambda_j\rangle - \langle K, \Lambda_j\rangle\langle \gamma, \Lambda_i\rangle$$

is satisfied for all $0 \le i \le r$.

Proof. Recall that for all $\gamma \in Q^\vee$,

$$t_\gamma(e_1) = e_1 + \sum_{k=0}^{r} \mathbf{m}_{w_{kj}}^{w_\gamma} e_{w_{kj}}.$$

By definition, we have $\langle \Lambda_i, \alpha_j^\vee\rangle = \delta_{ij}$, so

$$\begin{aligned}
\mathbf{m}_{w_i}^{w_\gamma} &= \left\langle \Lambda_i, \sum_{k=0}^{r} \mathbf{m}_{w_{kj}}^{w_\gamma} e_{w_{kj}} \right\rangle = \langle \Lambda_i, t_\gamma(e_1) - e_1\rangle \\
&= \left\langle \Lambda_i, \langle K, \Lambda_j\rangle\gamma - \big(\langle \gamma, \Lambda_j\rangle + \frac{1}{2}|\gamma|^2\langle K, \Lambda_j\rangle\big)K \right\rangle \\
&= \frac{1}{2}\langle K, \Lambda_i\rangle\langle K, \Lambda_j\rangle\langle \gamma, \gamma\rangle + \langle K, \Lambda_i\rangle\langle \gamma, \Lambda_j\rangle - \langle K, \Lambda_j\rangle\langle \gamma, \Lambda_i\rangle. \qquad \square
\end{aligned}$$

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