REPRESENTATION THEORY OF THE TWISTED YANGIANS IN COMPLEX RANK

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ABSTRACT. In 2016, Etingof defined the notion of a Yangian in a symmetric tensor category and posed the problem to study them in the context of Deligne categories. This problem was studied by Kalinov in 2020 for the Yangian $Y(\mathfrak{gl}_t)$ of the general linear Lie algebra $\mathfrak{gl}_t$ in complex rank using the techniques of ultraproducts. In particular, Kalinov classified the simple finite-length modules over $Y(\mathfrak{gl}_t)$. In this paper, we define the notion of a *twisted Yangian* in Deligne's categories we extend these techniques to classify finite-length simple modules over the twisted Yangians $Y(\mathfrak{o}_t)$ and $Y(\mathfrak{sp}_t)$ of the orthogonal and symplectic Lie algebras $\mathfrak{o}_t, \mathfrak{sp}_t$ in complex rank.

CONTENTS

1. INTRODUCTION

A *symmetric tensor category* (STC) is an abstraction of the representation category of a group, or more generally, an affine supergroup scheme. However, not all symmetric tensor categories arise as such representation categories; in other words, some do not *fiber* over the category of complex super vector spaces sVec$_\mathbb{C}$. Working with these exotic categories yields new forms of commutative algebra, algebraic geometry, and representation theory beyond the realm of (super) vector spaces. Unless stated otherwise, all symmetric tensor categories considered in this paper are defined over $\mathbb{C}$.

The most prominent examples of STCs that do not fiber over sVec$_\mathbb{C}$ are the *Deligne categories*, introduced in [DMOS82, Del90, Del02, Del07]. These categories generalize classical representation categories such as $\mathrm{Rep}(GL_n)$, $\mathrm{Rep}(O_n)$, and $\mathrm{Rep}(Sp_n)$ to the setting where the rank parameter n is replaced by a complex parameter $t \in \mathbb{C}$. An analogous construction exists for symmetric groups. In recent years, Deligne categories have been the subject of extensive study from multiple perspectives.

The present work concerns Yangians in symmetric tensor categories. The notion was introduced by Etingof in [Eti16], who showed that to any quadratic Lie algebra $\mathfrak{g}$ in a symmetric tensor category $\mathcal{C}$, one can associate an algebra $Y(\mathfrak{g})$, called its *Yangian*. Although this is a categorical rephrasing of Drinfeld's original definition [Dri85], Etingof provided a streamlined construction via R-matrices in the case of $Y(\mathfrak{gl}_t)$, where $\mathfrak{gl}_t = V \otimes V^*$ and V is the tautological object in $\mathrm{Rep}(GL_t)$. Etingof further posed the problem of classifying the irreducible representations of $Y(\mathfrak{gl}_t)$ that have finite-length when viewed as objects in $\mathrm{Rep}(GL_t)$. This question was solved by Kalinov [Kal20], who used ultrafilter and ultraproduct techniques to bypass the lack of a triangular decomposition, following earlier applications of these methods by Harman [Har16] and Deligne [Del07] to Deligne categories. Kalinov showed that simple finite-length representations of $Y(\mathfrak{gl}_t)$ are parametrized by pairs of sequences of Drinfeld polynomials subject to a certain stabilization condition. Independently, Utiralova [Uti19] proved a version of Olshanski's centralizer construction in complex rank for $Y(\mathfrak{gl}_t)$.

In this paper, we introduce the *twisted Yangians* $Y(\mathfrak{sp}_t)$ and $Y(\mathfrak{o}_t)$, corresponding to the categories $\mathrm{Rep}(Sp_t)$ and $\mathrm{Rep}(O_t)$, respectively. Extending Kalinov's methods, we classify their irreducible representations. As expected, irreducible modules are parametrized by Drinfeld polynomials, analogous to those appearing in the integer-rank theory of $Y(\mathfrak{sp}_{2n})$ and $Y(\mathfrak{o}_{2n})$. Specifically, these are highest weight representations, where the highest weight $\mu(u) = (\mu_1(u), \mu_2(u), \ldots)$ is an infinite sequence of $\mu_i(u) \in 1 + \mathbb{C}[\![u^{-1}]\!]$ satisfying

$$\frac{\mu_{i+1}(u)}{\mu_i(u)} = \frac{P_i(u+1)}{P_i(u)}$$

for monic polynomials $P_i(u)$, and the $\mu_i(u)$ eventually stabilize and equal some even series $\mu^m(u) \in 1 + u^{-2}\mathbb{C}[\![u^{-2}]\!]$. This implies that the $P_i(u)$ stabilize and equal 1. As in the integer-rank case, the polynomials $P_i(u)$ are called *Drinfeld polynomials*. Interestingly, the classification has the exact same form for $Y(\mathfrak{sp}_t)$ and $Y(\mathfrak{o}_t)$, even though $Y(\mathfrak{sp}_{2n})$ and $Y(\mathfrak{o}_{2n})$ have different classification theorems. This is perhaps a consequence of the fact that $\mathrm{Rep}(O_t) = \mathrm{Rep}(Sp_{-t})$ as tensor categories.

The organization of this paper is as follows. In Section 2.1, we introduce the representations of the groups GL_N, O_N, Sp_N for an even integer $N = 2n$, discuss the simple objects and the representations of the Lie algebras. We also discuss the positive-characteristic analogs of these categories, as they will be important later on.

In Section 3, we first define the Yangians and twisted Yangians. We then prove positive-characteristic versions of the classification theorems of finite-dimensional irreducible representations of the twisted Yangians, similarly to how Kalinov in [Kal20, Section 3] proves a positive-characteristic analog of the classification theorem of finite-dimensional irreducible representations of $Y(\mathfrak{gl}_n)$. In Section 4, we define the Deligne categories $\mathrm{Rep}\, O_t$ and $\mathrm{Rep}\, Sp_t$ and give their construction using ultraproducts. Finally, in Section 5, we assemble the results of Sections 3 and 4 together using ultraproducts to establish the final classification theorem, Theorem 5.9. Finally, Appendix A proves the necessary results about the Sklyanin determinant used for proving Theorem 3.4, an important tensor product decomposition that relates representations of $Y(\mathfrak{g}_n)$ to those of $SY(\mathfrak{g}_n)$ (where $\mathfrak{g}_n = \mathfrak{sp}_{2n}$ or $\mathfrak{o}_{2n}$), the *special twisted Yangian*, analogous to the Yangian $Y(\mathfrak{sl}_n)$.

Several natural directions remain open. First, one may seek to describe the simple objects of $\mathrm{Rep}(GL_t)$, $\mathrm{Rep}(O_t)$, and $\mathrm{Rep}(Sp_t)$ appearing in irreducible representations of $Y(\mathfrak{gl}_t)$, $Y(\mathfrak{o}_t)$, and $Y(\mathfrak{sp}_t)$. Second, the categorical framework allows one to define a wide class of

Yangians and twisted Yangians beyond the classical types, inviting further classification questions. Finally, it remains to be seen whether an analog of Olshanski's centralizer construction exists for twisted Yangians in complex rank, paralleling [Uti19].

2. The representation categories $\mathbf{Rep}(GL_N)$, $\mathbf{Rep}(SO_N)$, $\mathbf{Rep}(Sp_N)$

In this section, we review some basic properties about representation categories of classical groups and their Lie algebras. We assume all representations are finite-dimensional, and we work over an algebraically closed field $\mathbb{K}$ of characteristic $p \neq 2$. We freely reference [Jan03].

2.1. **Representation categories of classical groups.** Let $\mathcal{G}$ be a reductive algebraic group with fixed choice of split maximal torus H and character lattice $X(H)$. Every module over M over H admits a weight-space decomposition $M = \bigoplus_{\lambda \in X(H)} M_\lambda$. In particular, the Lie algebra $\mathfrak{g}$ of $\mathcal{G}$ admits a weight-space decomposition given by

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha,$$

where R is the set of roots, which are the non-zero weights of H such that $\mathfrak{g}_\alpha \neq 0$. We will let $\mathfrak{h}$ denote the Cartan subalgebra of $\mathfrak{g}$ corresponding to $\mathfrak{g}$. Each $f : H \to \mathbb{G}_m \in X(H)$ gives a Lie algebra map $\mathfrak{h} \mapsto \mathbb{K} \in \mathfrak{h}^*$; in particular, if we fix a basis for $X(H)$, the image of the basis under this map gives a basis of $\mathfrak{h}^*$. We will use the same symbols to describe both bases, identified via this map. This will be convenient to pass from weights of $\mathcal{G}$ to those of $\mathfrak{g}$. Now, let's set up notation for the reductive algebraic groups we will consider.

Let GL_N denote the general linear group scheme, which is the linear algebraic group of $N \times N$ invertible matrices, where $N = 2n$ is an even integer. The Lie algebra $\mathfrak{gl}_N$ of GL_N is the space of all $N \times N$ matrices, for which we will use the usual elementary matrices E_{ij} as basis. However, **rather than using the usual choice of** $1, \ldots, 2n$ **to index rows and columns, we will use** $I_N := \{-n, \ldots, -1, 1, \ldots, n\}$. Consequently, all results we cite from [Kal20] must first be translated to the indices $-n, \ldots, -1, 1, \ldots, n$.

Our choices of split maximal torus H in GL_N and Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{gl}_N$ will be the subgroup and subalgebra of diagonal matrices, respectively. We let $\{\varepsilon_i\}_{i \in I_N}$ denote the basis of $X(H)$ where ε_i is the character $diag(t_{-n}, \ldots, t_n) \mapsto t_i$. Any weight $\lambda \in X(H)$ is therefore of the form $\lambda = \sum_{i \in I_N} \lambda_i \varepsilon_i$ with $\lambda_i \in \mathbb{Z}$. It follows by our convention that the same set of symbols will denote the basis of $\mathfrak{h}^*$ dual to $\{E_{ii}\}_{i \in I_N}$ so that any $\mathfrak{gl}_N$-weight $\lambda \in \mathfrak{h}^*$ can also be expressed $\lambda = \sum_{i \in I_N} \lambda_i \varepsilon_i$ except now with $\lambda_i \in \mathbb{K}$. Finally, we will use the conventional triangular decomposition on $\mathfrak{g}$ where the simple roots are given by $\varepsilon_i - \varepsilon_{i+1}$, corresponding to the root vector $E_{i,i+1}$ for $i \in I_N \setminus \{-1\}$, and $\varepsilon_{-1} - \varepsilon_1$, corresponding to the root vector $E_{-1,1}$.

The main groups in this paper, however, will be subgroup schemes of GL_N preserving a form. In particuar let G be an invertible symmetric (resp. skew-symmetric) matrix valued in

$\mathbb{K}$. The *orthogonal group* $O_N(G)$ (resp. *symplectic group* $Sp_N(G)$) is defined to be subgroup scheme given by the relation

$$\{x \in GL_N \mid x^t G x = G\}.$$

Because $O_N(G)$ is not connected, at times it will be convenient to work instead with the special orthogonal group $SO_N(G)$ of determinant 1 matrices in $O_N(G)$. They both have the same Lie algebra. For the symplectic case we point out that N has to be even for a skew-symmetric form to exist. In fact, from now on we will assume $N = 2n$ is even, as we will not need to work with orthogonal groups for odd N.

Because different choices of G yield isomorphic groups, we choose a particular form for G (defined below) and omit it from the notation. We define the $N \times N$ matrix $G = [g_{ij}]$ by

$$g_{ij} = \begin{cases} \delta_{i,-j} & \text{in the orthogonal case} \\ \delta_{i,-j} \cdot \operatorname{sgn} i & \text{in the symplectic case.} \end{cases}$$

The matrix G has the following forms in the orthogonal and symplectic cases, respectively:

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & -1 & \cdots & 0 & 0 \\ -1 & 0 & \cdots & 0 & 0 \end{bmatrix}. \tag{2.1}$$

This is not the most conventional choice (for instance, see [Hum12]), but it is what Molev uses in [Mol07], which will be useful for our purposes. We let the set of diagonal matrices H in $\mathcal{G} = SO_N, Sp_N$ be our maximal split torus in $\mathcal{G}$ and is given by matrices of the form $diag(t_n^{-1}, \ldots, t_1^{-1}, t_1, \ldots t_n)$. We then define the character $\varepsilon_i \in X(H)$ by sending such a matrix to t_i.

Let's define the Lie algebras of these groups. Define E as the $N \times N$ matrix whose (i, j)-entry is $E_{ij} \in \mathfrak{gl}_N$ (we are using I_N to index the rows and columns of $\mathfrak{gl}_N$ as well). Next, define the matrix F by:

$$F := E - G^{-1}\mathbf{E}^t G,$$

from which a straightforward calculation shows that

$$F_{ij} = E_{ij} - \theta_{ij} E_{-j,-i},$$

where

$$\theta_{ij} := \begin{cases} 1 & \text{in the orthogonal case} \\ \operatorname{sgn} i \cdot \operatorname{sgn} j & \text{in the symplectic case.} \end{cases}$$

Then, the Lie algebras $\mathfrak{o}_N$ and $\mathfrak{sp}$ are the subalgebras of $\mathfrak{gl}_N$ spanned the entries of F in the orthogonal case and symplectic case, respectively. The Cartan subalgebra $\mathfrak{h}$ corresponding to the choice of torus H has basis given by $F_{i,i} = E_{i,i} - \theta_{i,i} E_{-i,-i}$ for $1 \leq i \leq n$; the basis dual to this is precisely the basis induced $\mathfrak{h}^*$ by the basis $\{\varepsilon_i\}_{i=1}^n$ of $X(H)$ given above. Finally, if we let the span of the F_{ij} for $1 \leq i < j \leq N$ denote the span of the positive root spaces, then our choice of positive roots Φ^+ is given by

$$\begin{cases} \{-\varepsilon_i - \varepsilon_j, \varepsilon_i - \varepsilon_j\}_{i<j} & \mathcal{G} = SO_N \\ \{-\varepsilon_i - \varepsilon_j, \varepsilon_i - \varepsilon_j\}_{i<j} \cup \{-2\varepsilon_i\} & \mathcal{G} = Sp_N \end{cases}, \tag{2.2}$$

where i, j range over $1, 2, \ldots, n$.

Now, let's discuss a little about the representation theory. Let $\mathcal{G}$ be any one of GL_N, SO_N or Sp_N with Lie algebra $\mathfrak{g}$. Call any weight $\lambda \in X(H)$ *dominant integral* if it satisfies $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0}$ for all $\alpha \in \Phi^+$, where $\alpha^\vee$ is the coroot associated to α, and let $X(H)^+$ be the set of dominant integral weights. Explicitly, this is given by

$$\begin{cases} \lambda_{-n} \geq \cdots \geq \lambda_n & \mathcal{G} = GL_N \\ -|\lambda_1| \geq \cdots \geq \lambda_{n-1} \geq \lambda_n & \mathcal{G} = SO_N \\ 0 \geq \lambda_1 \geq \cdots \geq \lambda_n & \mathcal{G} = Sp_N \end{cases}. \tag{2.3}$$

with each $\lambda_i \in \mathbb{Z}$.

Impose a partial ordering $\preceq$ on $X(H)$ by $\mu \preceq \lambda$ if $\lambda - \mu$ is the nonnegative linear combination of positive roots. Then, it is well-known that the simple modules all have a unique maximal weight with respect to $\preceq$ called the *highest weight*. Moreover, it must be dominant integral and for each dominant integral weight λ there is a simple module $\mathcal{L}(\lambda)$ with that highest weight. In the case of $\mathcal{G} = GL_N$, we will also denote the simple module of highest weight λ by $L(\lambda)$ or $L(\lambda, p)$ we if we want to emphasize the characteristic. For $\mathcal{G} = SO_N, Sp_N$, we will also denote this simple module by $V(\lambda)$ or $V(\lambda, p)$. This will be so that we are consistent with the notations of Molev in [Mol07] later on when discussing Yangians.

We will also consider another class of modules, called *Weyl modules*. For each dominant integral weight $\lambda \in X(H)^+$, we have a Weyl module by $\Delta(\lambda)$ or $\Delta(\lambda, p)$ defined by the universal property that it admits a surjection to any module generated by its highest-weight vector. In the case the characteristic $p = 0$, we have $\mathcal{L}(\lambda) = \Delta(\lambda)$. For $p > 2$, it can loosely be thought of as a mod p-reduction of the characteristic-zero module $\mathcal{L}(\lambda)$ (so its character is also given by the Weyl character formula). For a more precise definition, see §II.2 in [Jan03].

Let's also state classification simple O_N-modules. Let $\mathcal{G} = SO_N$ and recall that N is even. Then, the simple O_N-modules are indexed by the following:

(1) the collection $\{\lambda \mid \lambda \in X(H)^+, \lambda_1 < 0\}$. In particular, the simple modules are given by the induced modules $\text{ind}_{SO_N}^{O_N} V(\lambda)$ for $\lambda \in X(H)^+$ with $\lambda_1 < 0$. Moreover, replacing λ_1 by $-\lambda_1$ and then inducing gives the same module;

(2) the collection $\{(\lambda, \varepsilon) \mid \lambda \in X(H)^+, \lambda_1 = 0, \varepsilon \in \{-1, 1\}\}$. In this case, the SO_N-module structure on $V(\lambda)$ lifts to the structure of an O_N-module where a distinguished order-2 element acts by ε.

When it is clear we are working with O_N, we will let $X(H)^+$ for O_N be the set of $\lambda \in X(H)$ such that $0 \geq \lambda_1 \geq \cdots \geq \lambda_n$ and denote the simple module corresponding to λ with the notation $V(\lambda)$ or $V(\lambda, p)$. As written, this is ambiguous when $\lambda_1 = 0$, but we deliberately ignore the sign ε for the case $\lambda_n = 0$ because it is not detectable at the level of the Lie algebra $\mathfrak{o}_N$ and basically has no bearing on what is to come. Moreover, it will turn out we won't really be interested in the case where $\lambda_1 \neq 0$ due to stabilization with respect to a certain limit as $N \to \infty$, so in a loose sense we truly only care about the underlying SO_N-module structure.

We will not prove this classification here, but a proof is given in [GW00] (the proof therein assumes characteristic $p = 0$ but only needs $p \neq 2$).

Remark 2.1. For $\mathcal{G} = SO_N$ with $N = 2n+1$ odd, the root system for $\mathcal{G}$ is the same as the even case except with the additional short roots of ε_i and $-\varepsilon_i$. The criteria for dominant integrality for $\lambda \in X(H)$ then becomes the same as that for Sp_N. Since we have $O_N = SO_N \times \mathbb{Z}/2\mathbb{Z}$, it is clear that O_N-representations are indexed by the set $\{(\lambda, \varepsilon) \mid \lambda \in X(H)^+, \varepsilon \in \{-1, 1\}\}$.

2.2. **Representations of classical Lie algebras.** Continue to let $\mathcal{G}$ be one of GL_N, SO_N or Sp_N, and let $\mathfrak{g}$ denote the Lie algebra of $\mathcal{G}$. We are working towards describing the representation theory of the Yangian $Y(\mathfrak{g})$, so it will also be useful to state some basic facts about the representation theory of $\mathfrak{g}$. In characteristic zero, restricting a representation of $\mathcal{G}$ to $\mathfrak{g}$ is essentially the same, so we are mostly concerned with positive characteristic.

Let $C_{\mathbb{Z}}$ denote the *fundamental alcove* of $\mathcal{G}$, which is given by

$$C_{\mathbb{Z}} = \{\lambda \in X(H) \mid 0 \leq \langle \lambda + \rho, \alpha^{\vee} \rangle < p', \forall \alpha \in \Phi^+\},$$

where $p' = p$ if $p > 0$ and $p' = \infty$ if $p = 0$ and where ρ is half the sum of the positive roots. For O_N, we define its fundamental alcove to be that for SO_N. Then, we have the following well-known fact (cf. §II.5.6 in [Jan03]):

Proposition 2.2. *For any* $\lambda, \mu \in C_{\mathbb{Z}} \cap X(H)^+$, *we have* $\mathcal{L}(\lambda) = \Delta(\lambda)$, *and moreover there are no non-trivial extensions between* $\mathcal{L}(\lambda)$ *and* $\mathcal{L}(\mu)$.

Corollary 2.3. *For* $\lambda \in C_{\mathbb{Z}} \cap X(H)^+$, *the* $\mathcal{G}$-*module* $\Delta(\lambda)$ *is irreducible upon restriction to* $\mathfrak{g}$.

Proof. If $p = 0$, this is well-known. So suppose $p > 0$. First of all, by Proposition 2.2, $\Delta(\lambda)$ is simple. Then, because $\mathcal{G}$ is reductive, the restriction of a simple $\mathcal{G}$-module to $\mathfrak{g}$ can instead be thought of as module over the first Frobenius kernel over $\mathcal{G}$. But λ is a restricted dominant integral weight because it is both dominant integral and in the fundamental alcove, so the claim follows by the Steinberg tensor product theorem (see §II.3.17 in [Jan03]). $\square$

The following lemma will also be useful (recalling $N = 2n$) :

Lemma 2.4. *Suppose* $p > 2$, *and let* λ *be a dominant integral weight for* $\mathcal{G} = SO_N, Sp_N$. *Then, the condition* $-2\lambda_n + 2n < p$ *implies* $\lambda \in C_{\mathbb{Z}}$.

Proof. We want to show that $-2\lambda_n + 2n < p$ implies

$$0 \leq \langle \lambda + \rho, \alpha^{\vee} \rangle < p$$

for every positive root α. Let us first consider the case of $\mathcal{G} = SO_N$, whose positive roots are $\pm\varepsilon_i - \varepsilon_j$ for $i < j$. In this case, we have

$$\rho = -\varepsilon_2 - 2\varepsilon_3 - \cdots - (n-1)\varepsilon_n.$$

The condition that λ lies in the fundamental alcove is then equivalent to:

$$0 \leq \langle \lambda + \rho, (\pm\varepsilon_i - \varepsilon_j)^{\vee} \rangle = (\pm\lambda_i - \lambda_j) \mp (i-1) + (j-1) < p.$$

Since λ is dominant integral, $i < j$ means $0 \geq \lambda_i \geq \lambda_j$, so the first inequality is satisfied. Additionally, $\pm\lambda_i - \lambda_j \leq -2\lambda_n$ and $\mp(i-1) + (j-1) \leq 2n$, so if $-2\lambda_n + 2n < p$, the second is satisfied, so $-2\lambda_n + 2n < p$ implies λ lies in the fundamental alcove. The same argument

applies, with minor adjustments, to the case of $\mathcal{G} = Sp_N$, whose positive roots are $\pm\varepsilon_i - \varepsilon_j$ for $i < j$ and $-2\varepsilon_i$, and

$$\rho = -\varepsilon_1 - 2\varepsilon_2 \cdots - n\varepsilon_n.$$

This completes the proof. $\qquad\square$

2.3. **Representation categories.** We would like to define some useful notation for later on. Let $\mathcal{G}$ be a reductive algebraic group with Lie algebra $\mathfrak{g}$. If A is some Lie algebra, affine group scheme, or associative algebra algebra (in the category of vector spaces), let $\mathbf{Rep}\, A$ denote the category of finite-dimensional A-modules. For instance, we let $\mathbf{Rep}\, \mathcal{G}$ denote the category of finite-dimensional representations of $\mathcal{G}$, which obviously is a symmetric tensor category. In characteristic 0, it is semisimple.

Now, let A be an associative unital algebra equipped with an algebra homomorphism $U(\mathfrak{g}) \to A$. Let us define $\mathbf{Rep}_{\mathcal{G}}\, A$ to be the category of $\mathcal{G}$-modules M that also admit the structure of an A-module such that the action of $\mathfrak{g}$ on M through $\mathcal{G}$ and through A coincide (we will treat the $\mathcal{G}$-module admitting multiple such structures as separate objects). The morphisms will be $\mathcal{G}$-module homomorphisms. For instance, if $A = U(\mathfrak{g})$ and the equipped algebra homomorphism is the identity map, then this category is just the category $\mathbf{Rep}\, \mathcal{G}$. We will particularly be interested in the case where A is one the (twisted) Yangians $Y(\mathfrak{gl}_N)$, $Y(\mathfrak{o}_N)$, or $Y(\mathfrak{sp}_N)$, defined in §3. We do so because we are interested only in representations that lift to the group, after which we will take ultraproducts to construct representations in complex rank.

3. Twisted Yangians

Before we can develop the theory of twisted Yangians in Deligne's categories, we would like to first discuss twisted Yangians in finite rank. We will review basic results in characteristic zero. It will also be necessary to prove some basic results about twisted Yangians in positive characteristic due to a limit construction later on that passes from fields of positive characteristic to the complex numbers using ultrafilters. [1] We closely follow the presentation in [Mol07]. Retain the notation from Section 2. In particular, we work over the algebraically closed field $\mathbb{K}$ of characteristic $p \neq 2$ and have $N = 2n$ be some even positive integer.

3.1. **The (twisted) Yangians** $Y(\mathfrak{gl}_N)$, $Y(\mathfrak{sp}_N)$, $Y(\mathfrak{o}_N)$. We recall the definitions of the Yangian of $\mathfrak{gl}_N$ and the twisted Yangians of $\mathfrak{o}_N$ and $\mathfrak{sp}_N$, following Chapter 4 of [Mol07], as well as some basic properties.

The Yangian $Y(\mathfrak{gl}_N)$ is the associative algebra generated by the generators generators $t_{ij}^{(r)}$ for $r \geq 1$ and $i, j \in I_N$ subject to the well-known ternary relation (see [Mol07] or (A.1)). It will be useful to consolidate these and define the formal power series

$$t_{ij}(u) := \delta_{ij} + t_{ij}^{(1)}u^{-1} + t_{ij}^{(2)}u^{-2} + \cdots$$

and also to further assemble these into the $N \times N$ matrix $T(u) = [t_{ij}(u)]$, whose coefficients can be viewed as elements of

$$\mathrm{Hom}(V, V \otimes Y(\mathfrak{gl}_N))[\![u^{-1}]\!],$$

where $V = \mathbb{K}^N$. It can be thought of as a generalization of the universal enveloping algebra of $U(\mathfrak{gl}_N)$.

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[1]Discussion of twisted Yangians in positive characteristic appears to be virtually nonexistent (cf. [BT18] for the ordinary Yangian of $\mathfrak{gl}_N$), so we prove what we need here.

Proposition 3.1. *The map*

$$t_{ij}(u) \mapsto \delta_{ij} + E_{ij}u^{-1} \tag{3.1}$$

defines a surjective homomorphism

$$\pi_N : Y(\mathfrak{gl}_N) \to U(\mathfrak{gl}_N).$$

Moreover, we have an embedding $i_n : U(\mathfrak{gl}_N) \hookrightarrow Y(\mathfrak{gl}_N)$ *given by*

$$E_{ij} \mapsto t_{ij}^{(1)}. \tag{3.2}$$

Here E_{ij} denotes the elementary $N \times N$ matrix in $\mathfrak{gl}_N$ which has a 1 in the (i, j) position and zeros elsewhere. Moreover, $Y(\mathfrak{gl}_N)$ is a Hopf algebra and the maps above are compatible with the Hopf algebra structure on $U(\mathfrak{gl}_N)$. A useful power series is the *quantum determinant*, given by

$$\text{qdet}\, T(u) := \sum_{\sigma \in \mathfrak{S}_N} \text{sgn}(\sigma) \cdot t_{1,\sigma(1)}(u - N + 1) \cdots t_{n,\sigma(n)}(u). \tag{3.3}$$

In characteristic zero, the coefficients $d_1, d_2, \ldots$ of the quantum determinant are algebraically independent and generate the center of $Y(\mathfrak{gl}_N)$. In positive characteristic, this is not true. However, as seen in [Kal20], the coefficients are still central and algebraically independent; hence we can define $Z_{HC}(Y(\mathfrak{gl}_N))$ to be the subalgebra generated by the coefficients of $\text{qdet}\, T(u)$. It will also be useful to work with the Yangian $Y(\mathfrak{sl}_N)$ of $\mathfrak{sl}_N$. It is defined as follows. It can be checked that for any power series $f(u) \in 1 + u^{-1}\mathbb{K}[\![u^{-1}]\!]$, the map

$$T(u) \mapsto f(u)T(u) \tag{3.4}$$

defines an automorphism of $Y(\mathfrak{gl}_N)$. The Yangian $Y(\mathfrak{sl}_N)$ of $\mathfrak{sl}_N$ is defined as the subalgebra of $Y(\mathfrak{gl}_N)$ invariant under all such automorphisms. It was shown in [BT18] that

$$Y(\mathfrak{gl}_N) = Z_{HC}(Y(\mathfrak{gl}_N)) \otimes Y(\mathfrak{sl}_N). \tag{3.5}$$

Now, let us define the twisted Yangians $Y(\mathfrak{o}_N)$ and $Y(\mathfrak{sp}_N)$. Since we will be dealing with both cases simultaneously, we let $\mathfrak{g}_n = \mathfrak{o}_N$ or $\mathfrak{sp}_N$. Recall our choice of the matrix G from 2.1, and that we are using $I_N = \{-n, \ldots, -1, 1, \ldots, n\}$ to index the rows and columns. For an $N \times N$ matrix A, we define the transpose operation $'$ with respect to G by

$$A' := G^{-1}A^tG. \tag{3.6}$$

In contrast, [Mol07] defines A' as GA^tG^{-1}. Our choice will make it easier to state definitions categorically later on, as it coincides with the transpose with respect to the bilinear form on V induced by G. However, since our choice of G satisfies $G = \pm G^{-1}$, the two definitions are equivalent. Consequently, all results we cite from [Mol07] remain valid in our setting. If $A = [a_{ij}]$ and $A' = [a'_{ij}]$, we can compute:

$$a'_{ij} = \theta_{ij}a_{-j,-i}.$$

Then $\mathbf{F} = \mathbf{E} - \mathbf{E}'$. We defined $\mathfrak{g}_n$ to be the algebra spanned by the F_{ij}, but we can alternatively describe it as the set of all all $X \in \mathfrak{gl}_N$ satisfying $X' = -X$.

Next, define the R-matrix $R(u)$, an $N \times N$ matrix, by:

$$R(u) := 1 - \frac{P_n}{u} \in R(u) \in (\text{End}(V) \otimes \text{End}(V))[\![u^{-1}]\!],$$

where P_n is the $N \times N$ matrix given by:

$$P_n = \sum_{i,j} E_{ij} \otimes E_{ji}.$$

Note that P_n is the flip map. The transposed R-matrix is then defined as:

$$R'(u) := G_1^{-1} R^{t_1}(u) G_1 = G_2^{-1} R^{t_2}(u) G_2,$$

where G_i and t_i act on the i-th tensor component for $i = 1, 2$.

Similarly, we define the twisted Yangian by introducing a family of generators $\mathbf{s}_{ij}^{(r)}$ and the formal series

$$\mathbf{s}_{ij}(u) := \delta_{ij} + \mathbf{s}_{ij}^{(1)} u^{-1} + \mathbf{s}_{ij}^{(2)} u^{-2} + \cdots .$$

We then assemble these into the $N \times N$ matrix $\mathcal{S}(u)$ with entries $\mathbf{s}_{ij}(u)$. Writing

$$\mathcal{S}(u) = \sum_{i=0}^{\infty} \mathcal{S}_i u^{-i},$$

we note that $\mathcal{S}_0$ is simply the identity matrix.

Definition 3.1. The twisted Yangian $Y(\mathfrak{g}_n)$ is the associative unital algebra generated by the elements $\mathbf{s}_{ij}^{(r)}$, with defining relations given by

$$R(u - v)\mathcal{S}_1(u)R'(-u - v)\mathcal{S}_2(v) = \mathcal{S}_2(v)R'(-u - v)\mathcal{S}_1(u)R(u - v) \tag{3.7}$$

$$\mathcal{S}'(-u) = \mathcal{S}(u) \pm \frac{\mathcal{S}(u) - \mathcal{S}(-u)}{2u}. \tag{3.8}$$

We call the first relation the *quaternary* relation and the second the *symmetry* relation. The following standard results (again, proved in the same way as characteristic zero; see [Mol07, Chapter 2]) will be useful.

Proposition 3.2. *We define the evaluation homomorphism* $Y(\mathfrak{g}_n) \to U(\mathfrak{g}_n)$ *by*

$$\varrho_n \colon \mathbf{s}_{ij}(u) \mapsto \delta_{ij} + F_{ij} \left(u \pm \frac{1}{2} \right)^{-1}. \tag{3.9}$$

Conversely, the embedding $U(\mathfrak{g}_n) \hookrightarrow Y(\mathfrak{g}_n)$ *is given by*

$$i_n \colon F_{ij} \mapsto \mathbf{s}_{ij}^{(1)}. \tag{3.10}$$

Proposition 3.3. *For any series*

$$g(u) = 1 + g_1 u^{-2} + g_2 u^{-4} + \cdots \in 1 + u^{-2}\mathbb{K}[\![u^{-2}]\!],$$

the mapping

$$\mathcal{S}(u) \mapsto g(u)\mathcal{S}(u) \tag{3.11}$$

defines an automorphism of $Y(\mathfrak{g}_n)$.

There is a natural inclusion $Y(\mathfrak{g}_n) \hookrightarrow Y(\mathfrak{gl}_N)$ that aligns with the embedding $\mathfrak{g}_n \hookrightarrow \mathfrak{gl}_N$. This inclusion is given by

$$\mathcal{S}(u) \mapsto T(u)T'(-u). \tag{3.12}$$

We define the *special twisted Yangian* $SY(\mathfrak{g}_n)$ as the subalgebra of $Y(\mathfrak{g}_n)$ that remains invariant under all automorphisms of the form (3.11). Equivalently, it can be characterized as

$$SY(\mathfrak{g}_n) = Y(\mathfrak{sl}_N) \cap Y(\mathfrak{g}_n).$$

We define the scalar $\alpha_q(u)$ as follows:

$$\alpha_q(u) = \begin{cases} 1 & \text{in the orthogonal case} \\ \frac{2u+1}{2u-q+1} & \text{in the symplectic case.} \end{cases} \tag{3.13}$$

Next, define the special map $\varphi_N : \mathfrak{S}_N \to \mathfrak{S}_N$, which acts on permutations as in [Mol07, Section 2.7]. Let $(a_1, \ldots, a_N)$ be a permutation of I_N.

Define

$$\text{sdet}\, \mathcal{S}(u) := (-1)^n \alpha_N(u)$$
$$\times \sum_{\sigma \in \mathfrak{S}_N} \text{sgn}\, \sigma\sigma' \cdot \mathbf{s}'_{-a_{\sigma(1)}, a_{\sigma'(1)}}(-u) \cdots \mathbf{s}'_{-a_{\sigma(n)}, a_{\sigma'(n)}}(-u+n-1) \tag{3.14}$$
$$\times \mathbf{s}_{-a_{\sigma(n+1)}, a_{\sigma'(n+1)}}(u-n) \cdots \mathbf{s}_{-a_{\sigma(N)}, a_{\sigma'(N)}}(u-N+1)$$

to be the *Sklyanin determinant*. In characteristic zero, the coefficients of the Sklyanin determinant generate the center of $Y(\mathfrak{gl}_N)$. In positive characteristic, this is not true. However, as we will see below, they are still central, and hence we can define $Z_{HC}(Y(\mathfrak{gl}_N))$ as the algebra generated by the coefficients of sdet $\mathcal{S}(u)$. Write

$$\text{sdet}\, \mathcal{S}(u) = 1 + c_1 u^{-1} + c_2 u^{-2} + \cdots.$$

Theorem 3.4. *There exists a series* sdet $\mathcal{S}(u) \in 1 + u^{-1} Y(\mathfrak{g}_n)[\![u^{-1}]\!]$ *such that the coefficients of* sdet $\mathcal{S}(u)$ *are central, and the even coefficients* $c_2, c_4, \ldots$ *are algebraically independent and generate* $Z_{HC}(Y(\mathfrak{g}_n))$*; so* $Z_{HC}(Y(\mathfrak{g}_n)) = \mathbb{K}[c_2, c_4, \ldots]$*. We have the following tensor product decomposition:*

$$Y(\mathfrak{g}_n) = Z_{HC}(Y(\mathfrak{g}_n)) \otimes SY(\mathfrak{g}_n). \tag{3.15}$$

See Appendix A for the proof. These results are all well-known in characteristic zero; see [Mol07]. To prove the first statement in positive characteristic, we simply repeat the same arguments as in characteristic zero. To prove the tensor product decomposition, we use (3.5). We do not show that the series sdet $\mathcal{S}(u)$ in Theorem 3.4 matches (3.14) as it is not important for our purposes.

3.2. **Representations of** $Y(\mathfrak{g}_n)$**.** Let $\mathcal{G}_n$ denote the algebraic group O_N or Sp_N such that its Lie algebra is $\mathfrak{g}_n$. Molev derives the classification finite-dimensional simple modules over $Y(\mathfrak{g}_n)$ in [Mol07] in characteristic zero. The goal of this subsection is to extend the results therein to positive characteristic, imposing assumptions as needed. Due to the ultraproduct construction later on, we mainly only care about representations detectable at the level of $\mathcal{G}$, so we will primarily work with representations in the category $\mathbf{Rep}_{\mathcal{G}_n} Y(\mathfrak{g}_n)$ (recall the notation from §2.3).

Finally, we will need some facts about the representation theory of $Y(\mathfrak{gl}_N)$ in positive characteristic. For the definition and theory of highest weight modules for $Y(\mathfrak{gl}_N)$ in positive characteristic, see [Kal20] and [BT18].

Definition 3.2. A representation V of the twisted Yangian $Y(\mathfrak{g}_n)$ is called a *highest weight representation* if there exists a nonzero vector $\xi \in V$ such that V is generated by ξ and the following condition holds: for each i, there is a formal power series

$$\mu_i(u) = 1 + \mu_i^{(1)}u^{-1} + \mu_i^{(2)}u^{-2} + \cdots, \quad \mu_i^{(r)} \in \mathbb{K}, \tag{3.16}$$

such that

$$\mathbf{s}_{ij}(u)\xi = 0 \quad \text{for } i < j, \tag{3.17}$$

$$\mathbf{s}_{ii}(u)\xi = \mu_i(u)\xi \quad \text{for } 1 \leq i \leq n. \tag{3.18}$$

The vector ξ is called the *highest weight vector* of V, and the tuple $\mu(u) = (\mu_1(u), \ldots, \mu_n(u))$ is called the *highest weight*.

Notice that the symmetry relation gives

$$\mathbf{s}_{-i,-i} = \mathbf{s}_{ii}(-u) \pm \frac{\mathbf{s}_{ii}(u) - \mathbf{s}_{ii}(-u)}{2u} \tag{3.19}$$

for $i = 1, \ldots, n$. Then (3.19) implies that ξ is an eigenvector for $\mathbf{s}_{-i,-i}$ as well.

Definition 3.3. Let $\mu(u) = (\mu_1(u), \ldots, \mu_n(u))$. The *Verma module* $M(\mu(u))$ over $Y(\mathfrak{g}_n)$ is the quotient of $Y(\mathfrak{g}_n)$ by the left ideal generated by the coefficients of the series

$$\begin{cases} \mathbf{s}_{ij}(u) & i < j \\ \mathbf{s}_{ii} - \mu_i(u) & i = 1, \ldots, n. \end{cases}$$

A standard argument shows that $M(\mu(u))$ has a unique simple quotient, which we will denote by $V(\mu(u))$, and also has the universal property that it surjects onto any highest weight module with the same highest weight.

Recall the notation for simple $\mathcal{G}_n$ and GL_N-modules from §2.1. We have the following proposition:

Proposition 3.5. *Let* $V \in \mathbf{Rep}_{\mathcal{G}_n} Y(\mathfrak{g}_n)$ *be simple and such that* $V = V(\lambda_1, p) \oplus \cdots \oplus V(\lambda_k, p)$ *for* $\lambda_i \in X(H)^+ \cap C_{\mathbb{Z}}$ *with* $(\lambda_i)_1 = 0$. *Then, the module* V *has a unique singular vector (up to scaling), whose* $\mathfrak{g}_n$ *weight is maximal.*

Proof. Let us first contextualize the assumptions. First of all, the condition $(\lambda_i)_1 = 0$ is only there to ensure that each $\mathcal{G}_n$-module appearing above is irreducible when restricting to SO_N and therefore to $\mathfrak{o}_N$, when $\mathcal{G}_n = O_N$. The condition that $\lambda_i \in X(H)^+ \cap C_{\mathbb{Z}}$ is so that the character of $V(\lambda_i, p)$ is given by the Weyl character formula, which loosely speaking is to ensure that for a given λ_i, the module $V(\lambda_i, p)$ "looks the same" as p varies (this will be relevant when working with ultraproducts).

Now, let's prove the statement. The problem here is to rule out the possibility of V not having any singular vectors at all. Since we can regard V as a $\mathfrak{g}_n$-module, we can repeat verbatim the argument given in Theorem 4.2.6 of [Mol07]. $\square$

Hence V must be a highest weight representation $V = V(\mu(u))$ by the universal property of $M(\mu(u))$. We also get the following corollary (Corollary 4.2.7 in [Mol07]).

Corollary 3.6. *If* V *satisfies the conditions of Proposition 3.5, and* $\eta \in V$ *with* $\mathbf{s}_{ij}(u)\eta = 0$ *for all* $i < j$, *then* η *is a scalar multiple of the highest weight vector* ξ *of* V.

Now, we will use what is known about $Y(\mathfrak{gl}_N)$ to say more about representations of $Y(\mathfrak{g}_n)$. As noted at the start of Section 3 in [Kal20], all the statements in Proposition 2.1.2 of [Kal20] except for (d) hold in any characteristic. In particular, $Y(\mathfrak{gl}_N)$ is still a Hopf algebra. Moreover, recall that we can view $Y(\mathfrak{g}_n)$ as a subalgebra of $Y(\mathfrak{gl}_N)$ via the embedding

$$\mathcal{S}(u) = T(u)T'(-u).$$

Letting Δ denote the comultiplication on $Y(\mathfrak{gl}_N)$, the following formula from [Mol07, Section 4.2]

$$\Delta(\mathbf{s}_{ij}(u)) = \sum_{a,b} \theta_{bj} t_{ia}(u) t_{-j,-b}(-u) \otimes \mathbf{s}_{ab}(u) \tag{3.20}$$

holds in any characteristic. This gives us the following proposition:

Proposition 3.7. *There is a functor* $\mathbf{Rep}_{GL_N} Y(\mathfrak{gl}_N) \boxtimes \mathbf{Rep}_{\mathcal{G}_n} Y(\mathfrak{g}_n) \to \mathbf{Rep}_{\mathcal{G}_n} Y(\mathfrak{g}_n)$ *induced by* Δ.

Proof. Notice that Δ is an algebra homomorphism that maps $Y(\mathfrak{g}_n)$ to $Y(\mathfrak{gl}_N) \otimes Y(\mathfrak{g}_n)$. Moreover it sends $U(\mathfrak{g}_n)$ to $U(\mathfrak{g}_n) \otimes U(\mathfrak{g}_n)$. It follows that if $M \in \mathbf{Rep}_{GL_N} Y(\mathfrak{gl}_N)$ and $M' \in M$, then $M \otimes M'$ is a $\mathcal{G}_n$ module where both $\mathfrak{g}_n$ actions are compatible. $\square$

Let ζ be the highest weight vector of the $Y(\mathfrak{gl}_N)$-module $L(\lambda(u))$ (see [Kal20] for the definition of a $Y(\mathfrak{gl}_N)$-highest weight module), where the highest weight is given by $\lambda(u) = (\lambda_1(u), \dots, \lambda_N(u))$. Similarly, let ξ denote the highest weight vector of the $Y(\mathfrak{g}_n)$-module $V(\mu(u))$.

Proposition 3.8. *The submodule* $Y(\mathfrak{g}_n)(\zeta \otimes \xi)$ *of the* $Y(\mathfrak{g}_n)$-*module* $L(\lambda(u)) \otimes V(\mu(u))$ *over* $Y(\mathfrak{g}_n)$ *is a highest weight representation with highest weight vector* $\zeta \otimes \xi$. *Moreover, the highest weight is* $\lambda_i(u)\lambda_{-i}(-u)\mu_i(u)$, *for* $i = 1, \dots, n$.

Proof. The proof is the same as the characteristic zero case; see [Mol07, Proposition 4.2.9]. $\square$

The evaluation homomorphism (3.9) gives a functor $\mathbf{Rep}\,\mathcal{G}_n \to \mathbf{Rep}_{\mathcal{G}_n} Y(\mathfrak{g}_n)$. Similarly, there is a functor $\mathbf{Rep}\,GL_N \to \mathbf{Rep}_{GL_N} Y(\mathfrak{gl}_N)$. Let us call the image of any module under these functors an *evaluation module*. We will use the same notation and not distinguish between a module and its image under this functor.

Let $L(\lambda^{(1)}), \dots, L(\lambda^{(k)})$ be irreducible highest weight GL_N-modules such that each $\lambda^{(i)}$ is dominant integral and in the fundamental alcove for all i, and let $V(\mu)$ be an irreducible $\mathcal{G}_n$-module with μ dominant integral and such that $\mu_1 = 0$ and also in the corresponding fundamental alcove. View these as evaluation modules, and consider their tensor product

$$L(\lambda^{(1)}) \otimes \cdots \otimes L(\lambda^{(k)}) \otimes V(\mu), \tag{3.21}$$

which lies in $\mathbf{Rep}_{\mathcal{G}_n} Y(\mathfrak{g}_n)$ by Proposition 3.7. Define

$$\lambda_i(u) = (1 + \lambda_i^{(1)} u^{-1}) \cdots (1 + \lambda_i^{(k)} u^{-1}). \tag{3.22}$$

Additionally, using the formula (3.9), we can get that the $Y(\mathfrak{g}_n)$-module $V(\mu)$ is a highest-weight representation, with highest weight

$$\mu_i(u) = \frac{1 + (\mu_i \pm \frac{1}{2})u^{-1}}{1 \pm \frac{1}{2}u^{-1}}, \qquad i = 1, \dots, n \tag{3.23}$$

Let ζ be the tensor product of all the highest weight vectors of the modules in (3.21). Then we have the following proposition.

Proposition 3.9. *The submodule $Y(\mathfrak{g}_n)\zeta$ of the $Y(\mathfrak{g}_n)$-module (3.21) is a highest weight representation with the highest weight vector ζ. Moreover, the i-th component of the highest weight is given by $\lambda_i(u)\lambda_{-i}(-u)\mu_i(u)$, where $\lambda_i(u)$ is given by (3.22) and $\mu_i(u)$ is given by (3.23).*

Proof. The proof is the same as the characteristic zero case; see [Mol07, Proposition 4.2.11]
$\square$

We can also consider modules of the form

$$L(\lambda^{(1)}) \otimes \cdots \otimes L(\lambda^{(k)}). \tag{3.24}$$

Let ζ be the tensor product of all the highest weight vectors of modules in (3.24). Then we have the following immediate corollary of Proposition 3.9.

Corollary 3.10. *The submodule $Y(\mathfrak{g}_n)\zeta$ of the $Y(\mathfrak{g}_n)$-module (3.24) is a highest weight representation with the highest weight vector ζ. Moreover, the i-th component of the highest weight is given by $\lambda_i(u)\lambda_{-i}(-u)$.*

Proposition 3.11. *The following holds:*
  (i) *For any element η of the $Y(\mathfrak{g}_n)$-module (3.21) and indices i, j, the expression*

$$\left(1 \pm \frac{1}{2}u^{-1}\right)\mathbf{s}_{ij}(u)\eta$$

  is a polynomial in u^{-1} of degree at most $2k + 1$.
  (ii) *For any element η of the $Y(\mathfrak{g}_n)$-module (3.24), the expression $\mathbf{s}_{ij}(u)\eta$ is a polynomial in u^{-1} of degree at most $2k$.*

Proof. Once again, the proof is the same as the characteristic zero case; see Propositions 4.2.13 and 3.2.11 in [Mol07]. $\square$

The following is a consequence of (3.15).

Proposition 3.12. *Every finite-dimensional irreducible representation of the twisted Yangian $Y(\mathfrak{g}_n)$ remains irreducible when restricted to the special twisted Yangian $SY(\mathfrak{g}_n)$. Moreover, every finite-dimensional irreducible of $SY(\mathfrak{g}_n)$ is of this form.*

We write $\nu(u) \longrightarrow \mu(u)$ if there exists a monic polynomial $P(u)$ such that $\frac{\nu(u)}{\mu(u)} = \frac{P(u+1)}{P(u)}$.

Proposition 3.13. *If $V(\mu(u))$ is finite-dimensional and satisfies the conditions of Proposition 3.5, then*

$$\mu_1(u) \longrightarrow \mu_2(u) \longrightarrow \cdots \longrightarrow \mu_n(u). \tag{3.25}$$

Proof. The proof closely parallels Proposition 4.2.8 in [Mol07]. Let J be the left ideal of $Y(\mathfrak{g}_n)$ generated by the coefficients of $\mathbf{s}_{-i,j}(u)$, for $i, j = 1, \ldots, n$, and consider

$$V^J = \{\eta \in V(\mu(u)) \mid \mathbf{s}_{-i,j}(u)\eta = 0 \text{ for all } i, j \in \{1, \ldots, n\}\}.$$

The highest weight vector $\xi \in V(\mu(u))$ belongs to V^J. The defining relations give

$$[\mathbf{s}_{-i,j}(u), \mathbf{s}_{kl}(v)] = 0 \mod J,$$

for $i, j, k, l > 0$, so the operators $\mathbf{s}_{kl}(u)$ with $k, l > 0$ preserve V^J. Moreover for $i, j, k, l > 0$ the defining relations give

$$(u - v)[\mathbf{s}_{ij}(u), \mathbf{s}_{kl}(v)] = \mathbf{s}_{kj}(u)\mathbf{s}_{il}(v) - \mathbf{s}_{kj}(v)\mathbf{s}_{il}(u) \quad \mathrm{mod}\ J,$$

which is formally identical to that of $Y(\mathfrak{gl}_n)$, so V^J admits a $Y(\mathfrak{gl}_n)$-action by $t_{ij}(u) \mapsto \mathbf{s}_{ij}(u)$. The cyclic span $L := Y(\mathfrak{gl}_n)\xi$ is thus a finite-dimensional highest weight module of $Y(\mathfrak{gl}_n)$ with weight $(\mu_1(u), \ldots, \mu_n(u))$.

Decomposing $V(\mu(u)) = \bigoplus_i V(\mu_i, p)$ as a $\mathfrak{g}_n$-module, and recalling that the Weyl group stabilizes the weights, yields that for any $\mathfrak{gl}_{2n}$-weight η appearing in $V(\mu(u))$,

$$(\eta)_{-n} - (\eta)_n + n \leq 2 \max_i -(\mu_i)_n + n < p.$$

This verifies the condition of Proposition 3.2.1 in [Kal20]. It follows from Theorem 3.2.5 in [Kal20] that (3.25) holds.   □

We now split into the separate cases of $\mathfrak{sp}_{2n}$ and $\mathfrak{o}_{2n}$.

3.2.1. *Finite-dimensional irreducible representations of $Y(\mathfrak{sp}_{2n})$ in positive characteristic.*

Proposition 3.14. *Suppose V is irreducible and finite-dimensional, and as a $\mathfrak{sp}_2$-representation is isomorphic to $V(\mu_1, p) \otimes \cdots \otimes V(\mu_k, p)$ with each μ_i is a non-positive integer and $p > 2 \max -\mu_i + 2$. Then V is a highest representation $V = V(\mu(u), p)$.*

Proof. The condition implies that μ_i is in the fundamental alcove, and hence each $V(\mu_i, p)$ is irreducible and highest weight. Then we can repeat the proof of Theorem 4.2.6 in [Mol07].   □

From now on, assume V satisfies the conditions of (3.14). Write

$$\mu(u) = 1 + \mu^{(1)}u^{-1} + \mu^{(2)}u^{-2} + \cdots, \quad \mu^{(r)} \in \mathbb{K}. \tag{3.26}$$

Proposition 3.15. *There exists an even formal series $g(u) \in 1 + u^{-2}\mathbb{K}[\![u^{-2}]\!]$ such that $g(u)\mu(u)$ is a polynomial in u^{-1}.*

Proof. The proof of Proposition 4.3.1 in [Mol07] carries over verbatim.   □

Then composing the action of $Y(\mathfrak{sp}_2)$ on $V(\mu)$ with the automorphism (3.11) produces an irreducible highest weight representation with polynomial highest weight. Thus, it suffices to consider such representations. In this case,

$$\mu(u) = (1 - \gamma_1 u^{-1}) \cdots (1 - \gamma_{2k} u^{-2k}).$$

As in [Kal20], for $n \in \mathbb{F}_p$, let $[n]$ denote the minimal non-negative representative of n modulo p, inducing a total order $0 < 1 < \cdots < p - 1$ on $\mathbb{F}_p$. If $p = 0$ let $[n] = n$ for a nonnegative integer n. Recall that if (α, β) is a restricted dominant integral weight for GL_2, then $L(\alpha, \beta, p)$ is a simple GL_2-module and remains simple upon restriction to $\mathfrak{gl}_2$.

Proposition 3.16. *Suppose that for every $i = 1, \ldots, k$ the following condition holds: in characteristic $p > 0$ (resp. $p = 0$), if the multiset $\{\gamma_p + \gamma_q \mid 2i - 1 \leq p < q \leq 2k\}$ contains elements of $\mathbb{F}_p$ (resp. elements of $\mathbb{Z}_+$), then $\gamma_{2i-1} + \gamma_{2i}$ is in $\mathbb{F}_p$ (resp. $\mathbb{Z}_+$) is minimal amongst them. Then the representation $V(\mu(u), p)$ of $Y(\mathfrak{sp}_2)$ is isomorphic to the tensor product*

$$L(\gamma_1, -\gamma_2, p) \otimes \cdots \otimes L(\gamma_{2k-1}, -\gamma_{2k}, p), \tag{3.27}$$

regarded as a $Y(\mathfrak{sp}_2)$-module obtained by restriction of the $Y(\mathfrak{gl}_2)$-module.

Proof. We adapt the proof of Proposition 4.3.2 in [Mol07] following the approach of Proposition 3.2.3 in [Kal20]. By Corollary 3.10, if we show the module in question is an irreducible highest weight module, then it has highest weight $\mu(u)$.

Let V be the module (3.27), and consider the embedding (3.12). The (i,j)-entry of $T'(u)$ is $t_{-j,-i}(u)$, so using (3.12)

$$\mathbf{s}_{-1,1}(u) = t_{-1,-1}(u)t_{-1,1}(-u) + t_{-1,1}(u)t_{-1,1}(-u).$$

Since $t_{-1,1}(u)$ acts nilpotentently and $t_{-1,-1}(u)$ acts semisimply (as in [Kal20, Prop. 3.2.3]), it follows that $\mathbf{s}_{-1,1}(u)$ acts nilpotently. Any submodule $N \subset V$ must then contain a singular vector for $\mathbf{s}_{-1,1}(u)$. Thus, irreducibility reduces to showing V has a unique singular vector.

We use induction to prove that any singular vector is proportional to $\zeta := \zeta_1 \otimes \cdots \otimes \zeta_k$, where ζ_i are the highest weight vectors of the $L(\gamma_{2i-1}, -\gamma_{2i}, p)$. The case $k = 1$ is clear. For $k \geq 2$, write

$$\eta = \sum_{r=0}^{q} (E_{1,-1})^r \zeta_1 \otimes \eta_r,$$

with $\eta_r \in L(\gamma_3, -\gamma_4, p) \otimes \cdots \otimes L(\gamma_{2k-1}, -\gamma_{2k}, p)$, $\eta_q \neq 0$, and $q \leq \min\{p-1, [\gamma_1 + \gamma_2]\}$ (if $p = 0$ then $q < \gamma_1 + \gamma_2$). Using (3.20) and the argument of [Mol07, Prop. 4.3.2], η_q is proportional to $\zeta_2 \otimes \cdots \otimes \zeta_k$, so it remains to show $q = 0$.

If $q \geq 1$, then the argument of [Mol07, Prop. 4.3.2] gives

$$q(\gamma_1 + \gamma_2 - q + 1)(\gamma_1 + \gamma_3 - q + 1) \cdots (\gamma_1 + \gamma_{2k} - q + 1) = 0$$

which is impossible given the assumptions on the γ_i, since $\gamma_1 + \gamma_2$ is minimal among $\gamma_1 + \gamma_i$ and $q \leq [\gamma_1 + \gamma_2]$. Thus $q = 0$, so V has a unique singular vector.

Finally, this vector generates V by the same argument as in the characteristic zero case, completing the proof. $\square$

Lemma 3.17. *Any $\gamma_1, \ldots, \gamma_{2k} \in \mathbb{K}$ can be rearranged to satisfy the conditions of the previous proposition.*

Proof. First, pick the i, j such that $[\gamma_i + \gamma_j]$ is minimal out of all the pairs i, j for which this is defined. Then we can set γ_i, γ_j to be the new γ_1, γ_2. Repeat for the remaining γ_i. $\square$

Hence, for any V satisfying the conditions of Proposition 3.5, $V = V(\mu(u), p)$ with

$$\mu(u) = (1 - \gamma_1 u^{-1}) \cdots (1 - \gamma_{2k} u^{-2k}),$$

such that $V(\mu(u), p)$ is isomorphic to (3.27). Now we are ready to state the classification theorem for $Y(\mathfrak{sp}_2)$.

Theorem 3.18. *If the irreducible finite-dimensional representation V of $Y(\mathfrak{sp}_2)$ satisfies the conditions of Proposition 3.5, then $V = V(\mu(u), p)$ such that there exists monic polynomial $P(u)$ such that $P(u) = P(-u + 1)$ and*

$$\frac{\mu(-u)}{\mu(u)} = \frac{P(u+1)}{P(u)}. \tag{3.28}$$

Proof. We can simply repeat the first part of the proof of Theorem 4.3.3 in [Mol07]. $\square$

For such $\mu(u)$ as in Theorem 3.18 satisfying equation (3.28) for sutiable monic polynomial $P(u)$, we write $\mu(-u) \implies \mu(u)$. Now we generalize this to the $Y(\mathfrak{sp}_{2n})$ case.

Theorem 3.19. *If V satisfies the conditions of Proposition 3.5 then $V = V(\mu(u))$ with*

$$\mu_1(-u) \Longrightarrow \mu_1(u) \longrightarrow \mu_2(u) \longrightarrow \cdots \longrightarrow \mu_n(u) \qquad (3.29)$$

Proof. Consider the embedding $Y(\mathfrak{sp}_2) \hookrightarrow Y(\mathfrak{sp}_{2n})$ given by $\mathbf{s}_{ij}(u) \mapsto \mathbf{s}_{ij}(u)$ for $i, j \in \{1, -1\}$. If V satisfies Proposition 3.5, the induced $Y(\mathfrak{sp}_2)$-module satisfies Proposition 3.14; its highest weight is $\mu_n(u)$. Since V is finite-dimensional, Theorem 3.18 gives $\mu_1(-u) \Longrightarrow \mu_1(u)$, and Proposition 3.13 finishes the proof. $\qquad\square$

We call the polynomials $P_1(u), \ldots, P_n(u)$ corresponding to each of the arrows the *Drinfeld polynomials* of $V(\mu(u))$.

We also want to go the other way: to construct irreducible representations from Drinfeld polynomials.

Theorem 3.20. *Suppose $p > 2$. Given monic polynomials $P_1(u), \ldots, P_n(u)$ with $P_1(u) = P_1(-u + 1)$, there exists a finite-dimensional representation $V(\mu(u), p)$ of $Y(\mathfrak{sp}_{2n})$ such that*

$$\frac{\mu_i(u)}{\mu_{i+1}(u)} = \frac{P_{i+1}(u+1)}{P_{i+1}(u)}, \qquad i = 1, \ldots, n-1$$

and

$$\frac{\mu_1(-u)}{\mu_1(u)} = \frac{P_1(u+1)}{P_1(u)}.$$

Proof. We may repeat the argument from the second part of the proof of Theorem 4.3.8 in [Mol07]. The existence of an external grading as in the characteristic zero case ensures that $V(\mu(u), p)$ appears as a subquotient in the cyclic $Y(\mathfrak{sp}_{2n})$-span of the highest vector of $L(\lambda(u))$. $\qquad\square$

Using the notation of the second part of the proof of Theorem 4.3.8 in [Mol07], the largest value of $-\mu_n$ in a $\mathfrak{sp}_{2n}$-weight μ appearing in $L(\lambda(u), p)$ coincides with the maximum of $\lambda_{-n} - \lambda_n$ for a $\mathfrak{gl}_{2n}$-weight λ appearing in $L(\lambda(u), p)$. From the remarks following Theorem 3.2.6 in [Kal20], this maximum is given by is the sum of the degrees of the Drinfeld polynomials corresponding to $\lambda(u)$, which are $1, \ldots, 1, Q(u), P_2(u), \ldots, P_n(u)$. So the maximum value of $-\mu_n$ in a $\mathfrak{sp}_{2n}$-weight μ appearing in $L(\lambda(u), p)$ is $\deg Q(u) + \sum_{i=2}^n \deg P_i(u)$. Note that $\deg P_1(u) = 2 \deg Q(u)$. Therefore, if

$$\deg P_1(u) + 2 \sum_{i=2}^n \deg P_i(u) + 2n < p,$$

the corresponding $V(\mu(u))$ satisfies $-2\mu_n + 2n < p$ for all μ appearing in its decomposition into simple $\mathfrak{g}_n$-modules. By Lemma 2.4, V satisfies the conditions of Proposition 3.5 (and vice versa).

As in the $\mathfrak{gl}_n$ case, the Drinfeld polynomials $P_i(u)$ are not unique. If

$$\frac{P_i(u+1)}{P_i(u)} = \frac{Q_i(u+1)}{Q(u)},$$

then $\frac{P_i}{Q_i} = F_i$, where $F_i(u) = F_i(u+1)$. Thus, F_i is a ratio of products of expressions of the form

$$(u + c)(u + 1 + c) \cdots (u + c + p - 1) = (u + c)^p - (u + c)$$

for some $c \in \overline{\mathbb{F}}_p$. Following [Kal20], we define $q_p(u) = u^p - u$, so $P_i(u)$, for $i \geq 2$, is unique up to multiplication or division by polynomials of the form $q_p(u + c)$. However, if we impose that

$$\deg P_1(u) + 2 \sum_{i=2}^{n} \deg P_i(u) + 2n < p,$$

the polynomials $P_i(u)$ are unique since $\deg q_p(u + c) = p$.

Corollary 3.21. *Every finite-dimensional irreducible representations of $SY(\mathfrak{sp}_{2n})$ that satisfies the conditions of* (3.5) *corresponds to a unique tuple $(P_1(u), \ldots, P_n(u))$ of monic polynomials such that $P_1(u) = P_1(-u + 1)$ and $\deg P_1(u) + 2 \sum_{i=2}^{n} \deg P_i(u) + 2n < p$.*

Proof. This follows from the previous two theorems and Proposition 3.12. $\square$

3.2.2. *Finite-dimensional irreducible representations of $Y(\mathfrak{o}_{2n})$ in positive characteristic.* In this section, we look at finite-dimensional irreducible representations of $Y(\mathfrak{o}_{2n})$. We start by looking at finite-dimensional irreducible representations of $Y(\mathfrak{o}_2)$.

Proposition 3.22. *Suppose V is irreducible and finite-dimensional, and as an SO_2 and $\mathfrak{o}_2$-representation is isomorphic to $V(\mu_1, p) \otimes \cdots \otimes V(\mu_k, p)$ with each μ_i is a non-positive integer and $p > 2 \max -\mu_i + 2$. Then V is a highest representation $V = V(\mu(u), p)$.*

Proof. The condition implies that μ_i is dominant integral and in the fundamental alcove (and therefore restricted, so we can freely interchange between the group and the Lie algebra), and hence each $V(\mu_i, p)$ is irreducible and highest weight. Then we can repeat the proof of Theorem 4.2.6 in [Mol07]. $\square$

From now on, assume V satisfies the conditions of (3.22). Write

$$\mu(u) = 1 + \mu^{(1)} u^{-1} + \mu^{(2)} u^{-2} + \cdots, \quad \mu^{(r)} \in \overline{\mathbb{F}}_p. \tag{3.30}$$

Let V be an irreducible finite-dimensional representation of $Y(\mathfrak{o}_2)$ satisfying the conditions of Proposition 3.5, so $V = V(\mu(u))$ with

$$\mu(u) = 1 + \mu^{(1)} u^{-1} + \mu^{(2)} u^{-2} + \cdots, \quad \mu^{(r)} \in \overline{\mathbb{F}}_p.$$

Proposition 3.23. *There exists an even formal series $g(u) \in 1 + u^{-2} \overline{\mathbb{F}}_p[\![u^{-2}]\!]$ such that $(1 + \frac{1}{2} u^{-1}) g(u) \mu(u)$ is a polynomial in u^{-1}.*

Proof. We may repeat the proof of Proposition 4.4.1 in [Mol07] without any changes. $\square$

As in the characteristic zero case, define $\mu'(u) = (1 + \frac{1}{2} u^{-1}) \mu(u)$. Now we may focus on the representations with $\mu'(u)$ a polynomial in u^{-1}. Then write

$$\mu'(u) = (1 - \gamma_1 u^{-1}) \cdots (1 - \gamma_{2k+1} u^{-1})$$

for $\gamma_i \in \overline{\mathbb{F}}_p$. For any $\gamma \in \overline{\mathbb{F}}_p$, we can make the one-dimensional $\mathfrak{o}_2$-representation $V(\gamma)$ into a representation of the twisted Yangian $Y(\mathfrak{o}_2)$ via the evaluation homomorphism, and the corresponding highest weight is given by

$$\frac{1 + (\gamma + \frac{1}{2}) u^{-1}}{1 + \frac{1}{2} u^{-1}},$$

which follows from (3.23).

Proposition 3.24. *Suppose that for every $i = 1, \ldots, k$ the following condition holds: In positive characteristic (resp. characteristic zerp), if the multiset $\{\gamma_p + \gamma_q \mid 2i - 1 \le p < q \le 2k + 1\}$ contains elements of $\mathbb{F}_p$ (resp. $\mathbb{Z}_+$), then $\gamma_{2i-1} + \gamma_{2i}$ is in $\mathbb{F}_p$ (resp. $\mathbb{Z}_+$) and is minimal amongst them. Then the representation $V(\mu(u))$ of $Y(\mathfrak{o}_2)$ is isomorphic to the tensor product*

$$L(\gamma_1, -\gamma_2, p) \otimes L(\gamma_3, -\gamma_4, p) \otimes \cdots \otimes L(\gamma_{2k-1}, -\gamma_{2k}, p) \otimes V(-\gamma_{2k+1} - 1/2, p). \qquad (3.31)$$

Proof. The proof is exactly analogous to the proof of Proposition 3.16. The characteristic zero version is Proposition 4.4.2 in [Mol07]. □

As in the symplectic case, we can always reorder the γ_i to satisfy the conditions of this proposition.

Theorem 3.25. *If the irreducible highest weight representation V of $Y(\mathfrak{o}_2)$ is finite-dimensional and satisfies the conditions of Proposition 3.23 then $V = V(\mu(u), p)$ and there is a pair $(P(u), \gamma)$, where $P(u)$ is a monic polynomial in u with $P(u) = P(-u + 1)$ and $\gamma \in \overline{\mathbb{F}}_p$ with $P(\gamma) \ne 0$ such that*

$$\frac{\mu'(-u)}{\mu'(u)} = \frac{P(u + 1)}{P(u)} \cdot \frac{u - \gamma}{u + \gamma}, \qquad (3.32)$$

where $\mu'(u) = (1 + \frac{1}{2}u^{-1})\mu(u)$.

Proof. We know from above that $V = V(\mu(u), p)$, and it suffices to consider those $\mu(u)$ for which $\mu'(u) = (1 + \gamma_1 u^{-1}) \cdots (1 + \gamma_{2k+1} u^{-1})$ is a polynomial. As a $\mathfrak{o}_2$-module, we know V has the form

$$L(\gamma_1, -\gamma_2) \otimes \cdots \otimes L(\gamma_{2k-1}, -\gamma_{2k}) \otimes V(\gamma_{2k+1} - 1/2).$$

Since this is finite-dimensional, we get that $\gamma_{2i-1} + \gamma_i \in \mathbb{F}_p$ as in [Kal20]. Then set

$$\lambda_1(u) = (1 + \gamma_1 u^{-1})(1 + \gamma_3 u^{-1}) \cdots (1 + \gamma_{2k-1} u^{-1})$$
$$\lambda_2(u) = (1 - \gamma_2 u^{-1})(1 - \gamma_4 u^{-1}) \cdots (1 - \gamma_{2k} u^{-1}).$$

As in the symplectic case, we get a monic polynomial $Q(u)$ in u such that

$$\frac{\lambda_1(u)}{\lambda_2(u)} = \frac{Q(u + 1)}{Q(u)}.$$

Then set

$$P(u) = Q(u)Q(-u + 1)(-1)^{\deg Q},$$

and $\gamma = -\gamma_{2k+1}$; it is easily checked this does the job. The $P(\gamma) \ne 0$ condition follows from the ordering of the γ_i in the same way as in the characteristic zero case. □

Now we look at the general case of finite-dimensional irreducible representations of $Y(\mathfrak{o}_{2n})$, where $n \ge 2$ is a positive integer. We know that the irreducible highest weight representations $V(\mu(u))$ of $Y(\mathfrak{o}_{2n})$ are parametrized by n-tuples

$$\mu(u) = (\mu_1(u), \ldots, \mu_n(u)). \qquad (3.33)$$

We may regard $Y(\mathfrak{o}_2)$ as a subalgebra of $Y(\mathfrak{o}_{2n})$ generated by $\mathbf{s}_{ij}(u)$, $i, j \in \{n, n+1\}$. Let ξ be the highest weight vector of the $Y(\mathfrak{o}_{2n})$-module $V(\mu(u), p)$.

Lemma 3.26. *The $Y(\mathfrak{o}_2)$-module $V = Y(\mathfrak{o}_2)\xi$ is irreducible and isomorphic to the highest weight module $V(\mu_n(u), p)$.*

Proof. We can use the same proof as Lemma 4.11 of [Mol07].   □

Consider the mapping $i \mapsto i'$ for $i \in I_N$ that interchanges indices 1 and -1 while leaving all other indices unchanged. Using the defining relations, it is easily checked that the mapping

$$\mathbf{s}_{ij}(u) \mapsto \mathbf{s}_{i'j'}(u) \tag{3.34}$$

is an automorphism of $Y(\mathfrak{o}_{2n})$.

We now briefly return to the $Y(\mathfrak{o}_2)$ case. For the finite-dimensional representation $V(\mu(u))$ of $Y(\mathfrak{o}_2)$, we know that $V(\mu(u))$ is associated with a unique pair $(P(u), \gamma)$. Let $V(\mu(u))^\sharp$ be the composition of the $Y(\mathfrak{o}_2)$-action on $V(\mu(u))$ with the automorphism (3.34).

Lemma 3.27. *The $Y(\mathfrak{o}_2)$-module $V(\mu(u), p)^\sharp$ is isomorphic to $V(\mu^\sharp(u), p)$, where the series $\mu^\sharp(u)$ is given by*

$$\mu^\sharp(u) = \mu(u) \cdot \frac{u - \gamma + 1}{u + \gamma}.$$

In particular, the $Y(\mathfrak{o}_2)$-module $V(\mu(u), p)^\sharp$ is associated with the pair $(P(u), -\gamma + 1)$.

Proof. We use the same proof as Lemma 4.4.13 in [Mol07].   □

Now, we can state the main classification theorems. Given an *admissible* series $\mu(u)$, we define $\mu^\sharp(u)$ as above.

Theorem 3.28. *If V is a finite-dimensional irreducible $Y(\mathfrak{o}_{2n})$-module, $n \geq 2$, and satisfies the conditions of Proposition 3.5, then $V = V(\mu(u), p)$, the series $\mu_1(u)$ is admissible, and one of the following relations holds:*

$$\mu_1(-u) \Longrightarrow \mu_1(u) \longrightarrow \mu_2(u) \longrightarrow \cdots \longrightarrow \mu_n(u) \tag{3.35}$$

$$\mu_1^\sharp(-u) \Longrightarrow \mu_1^\sharp(u) \longrightarrow \mu_2(u) \longrightarrow \cdots \longrightarrow \mu_n(u). \tag{3.36}$$

Moreover, in positive characteristic, the first one will always hold.

Proof. Consider the module $V(\mu_1(u), p)$ of Lemma 3.26. Examining the $\mathfrak{o}_2$-weights and using that $V(\mu(u), p)$ satisfies the conditions of Proposition 3.5, we see that this module meets the conditions of Proposition 3.22. Suppose $V(\mu_n(u))$ is associated with the number $\gamma \in \overline{\mathbb{F}}_p$. Since V satisfies the conditions of Proposition 3.5, it follows that the $\mathfrak{o}_2$-weights of $V(\mu_n(u), p)$ must all be in $\mathbb{F}_p$. Now $V(\mu_1(u))$ must be given by (3.31). Since the $L(\gamma_{2i-1}, -\gamma_{2i})$ must be finite dimensional, $\gamma_{2i-1} + \gamma_{2i} \in \mathbb{F}_p$. It follows that the $\mathfrak{o}_2$-weights of $L(\gamma_{2i-1}, -\gamma_{2i})$ are all in $\mathbb{F}_p$. Thus, we must have that $-\gamma_{2k+1} - 1/2$ is in $\mathbb{F}_p$, which is equivalent to γ being $1/2$ plus an integer. Proceeding as in the characteristic zero case, it follows that the module $V(\mu(u))^\sharp$ has highest weight

$$\mu^\sharp(u) = (\mu_1(u), \mu_2(u), \dots, \mu_n^\sharp(u))$$

cf. the beginning of the proof of Theorem 4.4.14 in [Mol07]. Moreover, composing the automorphism (3.34) (restricted to $Y(\mathfrak{o}_{2n})$) with the action of $\mathfrak{o}_{2n}$ on $V(\mu(u), p)$ yields another irreducible highest weight representation, so $V(\mu(u))^\sharp$ also satisfies the conditions of Proposition 3.5.

Using (3.32), we obtain

$$\frac{\mu_1(-u)}{\mu_1(u)} = \frac{P(u+1)}{P(u)} \cdot \frac{u + \frac{1}{2}}{u + \gamma} \cdot \frac{u - \gamma}{u - \frac{1}{2}} = \frac{P(u+1)}{P(u)} \cdot \frac{u + \frac{1}{2}}{u - \delta + \frac{1}{2}} \cdot \frac{u + \delta - \frac{1}{2}}{u - \frac{1}{2}}.$$

where $\delta = -\gamma + \frac{1}{2}$ and $P(u) = P(-u+1)$. Note that δ is an integer from our previous discussion. In positive characteristic, or in characteristic zero, and $\gamma \in 1/2 - \mathbb{Z}_+$, δ is a non-negative integer, so we can set

$$Q(u) = P(u)(u - \delta + \tfrac{1}{2}) \cdots (u - \tfrac{1}{2})(u - \tfrac{1}{2}) \cdots (u + (\delta - 1) - \tfrac{1}{2}).$$

Since we have multiplied by an even number of linear factors, we have $Q(u) = Q(-u+1)$. We also have

$$\frac{\mu_1(-u)}{\mu_1(u)} = \frac{Q(u+1)}{Q(u)},$$

so the first condition holds. If $\gamma \in 1/2 + \mathbb{Z}_+$ then δ is a non-positive integer. Consider the $\delta^\sharp, \gamma^\sharp$ associated with $V^\sharp$. By Lemma 3.27, $\gamma^\sharp = -\gamma+1$, so $\delta^\sharp = -\gamma^\sharp+1/2 = -(-\gamma+1)+1/2 = \gamma - 1/2 = -\delta$. Hence $V^\sharp$ satisfies the first condition, so V satisfies the second. $\qquad\square$

As before, we want to go the other way as well.

Theorem 3.29. *Suppose $p > 2$. Given monic polynomials $P_1(u), \ldots, P_n(u)$ with $P_1(u) = P_1(-u+1)$, there exists a finite-dimensional representation $V(\mu(u), p)$ of $Y(\mathfrak{o}_{2n})$ such that*

$$\frac{\mu_i(u)}{\mu_{i+1}(u)} = \frac{P_{i+1}(u+1)}{P_{i+1}(u)}, \qquad i = 1, \ldots, n-1$$

and

$$\frac{\mu_1(-u)}{\mu_1(u)} = \frac{P_1(u+1)}{P_1(u)}.$$

Proof. Completely analogous to the proof of Theorem 3.20. $\qquad\square$

In characteristic zero, it follows that the irreducible representations of $Y(\mathfrak{o}_{2n})$ are indexed by Drinfeld polynomials $P_1(u), \ldots, P_n(u)$ with $P_1(u) = P_1(-u+1)$ and a choice of the parameter γ in one of the three subsets

$$\{1/2\}, -1/2 - \mathbb{Z}_+, 3/2 + \mathbb{Z}_+, \tag{3.37}$$

where the first subset corresponds to both conditions (since $\delta = 0 = -\delta = \delta^\sharp$), the second corresponds the first condition, and the third corresponds to the second condition.

In positive characteristic, similarly to the symplectic case, we can get that $V(\mu(u))$ satisfies the conditions of Proposition 3.5 if and only if $\deg P_1(u) + 2\sum_{i=2}^n \deg P_i(u) + 2n < p$.

Corollary 3.30. *Finite-dimensional irreducible representations V of $SY(\mathfrak{o}_{2n})$, $n \geq 2$, satisfying the conditions of Proposition 3.5 are highest weight modules $V = V(\mu(u))$ and are parametrized by tuples $(P_1(u), \ldots, P_n(u))$ of monic polynomials with $P_1(u) = P_1(-u+1)$ and in positive characteristic, $\deg P_1(u) + 2\sum_{i=2}^n \deg P_i(u) + 2n < p$, and in characteristic zero, a choice of one of the subsets in (3.37).*

4. Deligne's categories for classical groups

We will assume in this section that the reader has at least a basic understanding of symmetric tensor categories (STCs) and in particular has some familiarity with the Deligne category $\operatorname{Rep} GL_t$. A comprehensive reference is [EGNO16], and a quick expository reference is [EK23], whose notations and conventions we will mainly follow. Our primary reference for this section will be [Eti16]. We will work over a fixed, algebraically closed field $\mathbb{K}$ of characteristic zero, and we will use the bold-face notation $\mathbf{Rep}\, G$ to denote the ordinary

representation category of finite-dimensional representations for a subgroup G of the general linear group GL_N for a positive integer N.

4.1. **The Deligne categories** $\mathrm{Rep}(O_t)$ **and** $\mathrm{Rep}(Sp_t)$. Let $V = \mathbb{K}^n$ denote the natural representation of GL_N. As an O_N-module, V is faithful and self-dual and therefore generates the category $\mathbf{Rep}\,O_n$, so every simple O_N-module appears in some $V^{\otimes r}$ for a sufficiently large r. By tensor-hom adjunction and self-duality, we see that

$$\mathrm{Hom}_{O_N}(V^{\otimes r}, V^{\otimes s}) = \mathrm{Hom}_{O_N}(V^{\otimes r+s}, \mathbb{1}),$$

where $\mathbb{1}$ is the trivial representation. By the First Fundamental Theorem of invariants for O_N, multilinear O_N-invariant forms on $V^{\otimes r+s}$ are given by the algebra generated by contractions (see [DLZ18] for more details), which means that $r + s$ must be even. By semisimplicity of $\mathbf{Rep}\,O_N$, one can conclude that $\mathbf{Rep}\,O_N$ is the Karoubian closure of the subcategory given by objects of the form $[r] := V^{\otimes r}$ and hom spaces given as above.

It can be show via Brauer duality that a spanning set for $\mathrm{Hom}_{O_N}([r], [s])$ are the perfect matchings on a set of $(r + s)$ dots, which can be visualized as diagrams with a row of r dots below and a row of s dots above, with an arc connecting the two dots which are perfectly matched. Composition is given by first stacking diagrams vertically, then concatenating arcs in the natural way, and finally erasing loops; for each loop we multiply the diagram by n.

As an example, here is the composition of a diagram A in $\mathrm{Hom}_{O_N}([3], [5])$ and a diagram B in $\mathrm{Hom}_{O_N}([5], [3])$ to give a diagram $A \circ B \in \mathrm{Hom}_{O_N}([5], [5])$:

with composition given by
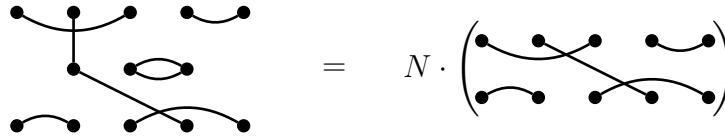
For arbitrary $t \in \mathbb{C}$, we can define the category $\widetilde{\mathrm{Rep}}(O_t)$ with objects $[r]$, with $r \in \mathbb{Z}_+$. The space of morphisms $\mathrm{Hom}([r_1], [r_2])$ is spanned by the Brauer diagrams as above, with the same composition law, where if we remove a loop, we multiply by t. The category $\mathrm{Rep}(O_t)$ is defined to be the Karoubian closure of $\widetilde{\mathrm{Rep}}(O_t)$. We define $\widetilde{\mathrm{Rep}}(Sp_t)$ similarly. In both cases, we let V be the fundamental object $[1]$. For $t \notin \mathbb{Z}$, the categories $\mathrm{Rep}(O_t)$ and $\mathrm{Rep}(Sp_t)$ are semisimple. We will not consider the case of $t \in \mathbb{Z}$ in this paper.[2]

Then V is equipped with a symmetric or alternating isomorphism $\psi : V \to V^*$. If ψ is symmetric, the group $O(V)$ is cut out inside $V \otimes V^*$ by the equations $AA' = A'A = \mathrm{Id}$, where A' is the adjoint of A with respect to ψ. The map $\theta \in \mathrm{End}(V \otimes V^*)$ that sends A to A' is given by

$$V \otimes V^* \xrightarrow{1 \otimes \psi^{-1}} V \otimes V \xrightarrow{\sigma} V \otimes V \xrightarrow{1 \otimes \psi} V \otimes V^*, \tag{4.1}$$

where σ exchanges factors in the tensor product. This can be thought of as taking the adjoint with respect to the symmetric bilinear form on V that is induced by ψ, analogously to (3.6). In the integral rank case, if G is the matrix representation of ψ, then θ sends A to

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[2]In the case $t \notin \mathbb{Z}$, semisimplicity implies that these categories are abelian and therefore are symmetric tensor categories. On the other hand, when $t \in \mathbb{Z}$, they are not abelian, and it is much more difficult to show the abelian envelope exists (see [EAHS18, Cou21]).

$G^{-1}A^t G$. Our choice of θ is the negative of that in [Eti16]. If ψ is alternating, the group $Sp(V)$ is defined in the same way.

This definition works if V is replaced with any object that has a symmetric/alternating form. So when V is the fundamental object of $\mathrm{Rep}(G)$, we write $O_t = O(V)$ or $Sp_t = Sp(V)$ corresponding to the orthogonal and symplectic cases, respectively.

We next define the Lie algebras $\mathfrak{o}_t$ and $\mathfrak{sp}_t$. First, $\mathfrak{gl}(V) = V \otimes V^*$ is naturally an associative algebra, and thus also a Lie algebra, using the commutator. In the orthogonal (resp. symplectic) case, $\mathfrak{o}(V)$ (resp. $\mathfrak{sp}(V)$) is defined as $\ker(-\theta - \mathrm{Id})$. When V is the fundamental object, we get

$$\mathfrak{o}_t = \mathfrak{o}(V) = \mathrm{Lie}O(V) = \mathrm{Lie}O_t \qquad \text{in the orthogonal case}$$
$$\mathfrak{sp}_t = \mathfrak{sp}(V) = \mathrm{Lie}Sp(V) = \mathrm{Lie}Sp_t \qquad \text{in the symplectic case.}$$

Set $\mathfrak{g} = \mathfrak{o}_t$ or $\mathfrak{sp}_t$, corresponding to $G = O_t$ or Sp_t. We define the universal enveloping algebra to be a quotient of the tensor algebra of $\mathfrak{g}$:

$$U(\mathfrak{g}) := T(\mathfrak{g})/(f(\mathfrak{g} \otimes \mathfrak{g})),$$

where $f = i_2 - i_2\sigma - i_1 c$, and i_k is the inclusion of the k-th graded component, and c is the commutator. Note that $\theta \circ \theta = \mathrm{Id}$ implies $(-\theta - \mathrm{Id}) \circ (1 - \theta) = 0$, so the image of $V \otimes V^*$ under $1 - \theta$ lies in $\mathfrak{g}$. Consequently, we have the map

$$V \otimes V^* \xrightarrow{1-\theta} \mathfrak{g} \hookrightarrow U(\mathfrak{g}).$$

Thus, for an algebra, A, with generators given by a set of maps $A_i : V \otimes V^* \to A$, an algebra homomorphism $A \to U(\mathfrak{g})$ is specified by giving a function sending each A_i to an element of $\mathrm{Hom}(V \otimes V^*, U(\mathfrak{g}))$.

4.2. **Ultraproduct construction.** We will employ an alternative presentation of $\mathrm{Rep}(G)$ using ultraproducts. Throughout, let $\mathcal{F}$ be a fixed non-principal ultrafilter. For background on ultrafilters and ultraproducts, see [Kal20, Section 1.2].

Consider the ultraproduct of countably infinite copies of $\overline{\mathbb{Q}}$: $\prod_{\mathcal{F}} \overline{\mathbb{Q}}$. By Łoś's theorem, $\prod_{\mathcal{F}} \overline{\mathbb{Q}}$ is a field. Moreover, it has characteristic 0 since for any $k \in \mathbb{Z}_{\neq 0}$, $k = \prod_{\mathcal{F}} k$. Additionally, it has cardinality of the continuum. By Steinitz's Theorem, it follows that

$$\prod_{\mathcal{F}} \overline{\mathbb{Q}} \cong \mathbb{C}. \tag{4.2}$$

We start with the case of t being a transcendental number. By composing the (non-canonical) isomorphism (4.2) with a suitable automorphism of $\mathbb{C}$ (considered as a vector space over $\overline{\mathbb{Q}}$), we may assume that the isomorphism (4.2) maps $\prod_{\mathcal{F}} 2n$ to t. See [Kal20] for details.

Set $t_n = n$, so that $\prod_{\mathcal{F}} 2t_n = t$ and set $G_n = O_{t_n}$ or Sp_{t_n} correspondingly. Next, let $\mathbf{Rep}(G_n) = \mathbf{Rep}(G_n, \overline{\mathbb{Q}})$, and set $\widehat{\mathcal{C}} = \prod_{\mathcal{F}} \mathbf{Rep}(G_n)$. Let V_{t_n} denote the fundamental representation of G_n, and let $V = \prod_{\mathcal{F}} V_{t_n}$. We have the following theorem (cf. [Eti16, Theorem 2.11 (ii)]).

Theorem 4.1. *If $\mathcal{C}$ is a rigid tensor category, then isomorphism classes of (possibly non-faithful) symmetric tensor functors $\mathrm{Rep}(G) \to \mathcal{C}$ are in bijection with isomorphism classes of objects $X \in \mathbb{C}$ of dimension t with a symmetric/alternating (in the orthogonal/symplectic cases) isomorphism $X \to X^*$. The bijection is given by $F = F(V_t)$.*

Here V_t is the fundamental object of $\mathrm{Rep}(G)$. We now give the ultraproduct construction of $\mathrm{Rep}(G)$.

Theorem 4.2. *The full subcategory of the $\prod_{\mathcal{F}} \overline{\mathbb{Q}}$-linear category $\widehat{\mathcal{C}}$ generated by V under tensor products, direct sums, and direct summands is equivalent to the $\mathbb{C}$-linear category $\mathrm{Rep}(G)$ under the isomorphism $\prod_{\mathcal{F}} \overline{\mathbb{Q}} \cong \mathbb{C}$ with $\prod_{\mathcal{F}} 2t_n = t$.*

Proof. We will use V_t to refer to the tautological object of $\mathrm{Rep}(G)$, and $V = \prod_{\mathcal{F}} V_{t_n}$ to denote the generator of the full subcategory of $\widehat{\mathcal{C}}$ in the statement of the theorem.

Let $\mathcal{C}$ be the full subcategory of $\widehat{\mathcal{C}}$ generated by V under tensor products, direct sums, and direct summands. The object $V = \prod_{\mathcal{F}} V_{t_n}$ has dimension t as the dimension of V_{t_n} is $2t_n$ and $\prod_{\mathcal{F}} 2t_n = t$. Additionally, since $V_{t_n}^* = V_{t_n}$, by Łoś's thoerem we have $V = V^*$. Using Theorem 4.1, we obtain a tensor functor $F : \mathrm{Rep}(G) \to \widehat{\mathcal{C}}$ with $F(V_t) = V$. Since $\mathrm{Rep}(G)$ is generated by V_t under tensor products, direct sums, and direct summands, it follows that the image of $\mathrm{Rep}(G)$ is the full subcategory $\mathcal{C}$ of $\widehat{\mathcal{C}}$. So we know that $F : \mathrm{Rep}(G) \to \widehat{\mathcal{C}}$ is essentially surjective, and we need to show that F is fully faithful. To do this, we show

$$\mathrm{Hom}_{\mathrm{Rep}(G)}(V_t^{\otimes r_1}, V_t^{\otimes r_2}) = \mathrm{Hom}_{\mathcal{C}}(V^{\otimes r_1}, V^{\otimes r_2}) \tag{4.3}$$

and that the composition maps are the same. This implies the result because $\mathrm{Rep}(G)$ and $\mathcal{C}$ can be obtained as the Karoubian envelope of the additive envelope of $V_t^{\otimes r}$ and $V^{\otimes r}$, respectively. Note that by Łoś's theorem we have

$$\mathrm{Hom}_{\mathcal{C}}(V^{\otimes r_1}, V^{\otimes r_2}) = \prod_{\mathcal{F}} \mathrm{Hom}_{G_n}(V_{t_n}^{\otimes r_1}, V_{t_n}^{\otimes r_2})$$

Now if $r_1 + r_2$ is odd, then both sides of Equation (4.3) are empty, as $\mathrm{Hom}_{G_{t_n}}(V_{t_n}^{\otimes r_1}, V_{t_n}^{\otimes r_2})$ is empty. Otherwise, $r_1 + r_2 = 2m$. For sufficiently large n, $\mathrm{Hom}_{G_n}(V_{t_n}^{\otimes r_1}, V_{t_n}^{\otimes r_2})$ has a basis which can be represented by the Brauer diagrams matching r_1 dots in the top row to r_2 dots in the bottom row. In particular, there are $\frac{(2m)!}{m!2^m}$ such diagrams, and $\mathrm{Hom}_{G_n}(V_{t_n}^{\otimes r_1}, V_{t_n}^{\otimes r_2})$ has dimension $\frac{(2m)!}{m!2^m}$, for sufficiently large n. Thus $\prod_{\mathcal{F}} \mathrm{Hom}_{G_n}(V_{t_n}^{\otimes r_1}, V_{t_n}^{\otimes r_2})$ has a basis consisting of these Brauer diagrams. Since by definition, they are a basis of $\mathrm{Hom}_{\mathrm{Rep}(G)}(V_t^{\otimes r_1}, V_t^{\otimes r_2})$, (4.3) follows.

Now we show that the composition maps are the same. In both categories, composition is given by vertical concatenation of diagrams. So we only need to look at what happens when we delete a loop. Each loop deletion can be simplified to $\mathrm{coev} \circ \mathrm{ev}$, which corresponds to multiplication by $2t_n$ in the G_n case. So in $\mathcal{C}$, we multiply by $\prod_{\mathcal{F}} 2t_n$, which is just t, and this matches the rule in $\mathrm{Rep}(G)$. This completes the proof. $\qquad\square$

Next, consider the case where t is algebraic. It follows a similar story as the transcendental case, except instead of taking the ultraproduct of $\mathbf{Rep}(G)$ over copies of $\overline{\mathbb{Q}}$, we take them over copies of $\overline{\mathbb{F}}_{p_n}$, with p_n being an increasing sequence of primes. As shown in [Kal20], we can choose an increasing sequence $p_1, p_2, \dots$ of primes, and an increasing sequence $t_1, t_2, \dots$ of positive integers such that: $\prod_{\mathcal{F}} \overline{\mathbb{F}}_{p_n} \cong \mathbb{C}$, and under this isomorphism, $\prod_{\mathcal{F}} 2t_n = t$. Moreover, the minimal polynomial $q(x)$ of t satisfies $q(2t_i) = 0$ in $\overline{\mathbb{F}}_{p_i}$. Then, we can repeat the argument of the above proof to get the following theorem.

Theorem 4.3. *The full subcategory of the $\prod_{\mathcal{F}} \overline{\mathbb{F}}_{p_n}$-linear category $\widehat{\mathcal{C}}$ generated by V under tensor products, direct sums, and direct summands is equivalent to the $\mathbb{C}$-linear category $\mathrm{Rep}(G)$ under the isomorphism $\prod_{\mathcal{F}} \overline{\mathbb{F}}_{p_n} \cong \mathbb{C}$ with $\prod_{\mathcal{F}} 2t_n = t$.*

We now state the classification of irreducible objects in $\mathrm{Rep}(G)$, see [UH21] and [Eti16].

Theorem 4.4. *The simple objects of $\mathrm{Rep}(G)$ are labeled by tuples $\lambda = (\lambda_1, \ldots, \lambda_k)$ of negative integers with $\lambda_1 \geq \cdots \geq \lambda_k$. Moreover, if V_λ is the simple object corresponding to some negative partition λ, we have*

$$V_\lambda = \prod_{\mathcal{F}} V_\lambda^{(n)},$$

where for sufficiently large n we define $V_\lambda^{(n)}$ to be the simple G_n-module with the highest weight $[\lambda]_{t_n}$ given by

$$[\lambda]_{t_n} = \sum_{i=n-k+1}^{n} \lambda_{n+1-i} \varepsilon_i \in E_n.$$

Note that for sufficiently large n the weight λ is indeed integral and dominant.

5. TWISTED YANGIANS IN COMPLEX RANK

5.1. The twisted Yangian $Y(\mathfrak{g})$. Let $\mathfrak{g} = \mathfrak{o}_t$ or $\mathfrak{sp}_t$, and let $G = O_t$ or Sp_t, respectively. There is a G-equivariant isomorphism $\psi : V \to V^*$, symmetric in the orthogonal case and alternating in the symplectic case. Set

$$(\pm, \mp) = \begin{cases} (+, -) & \text{if } \mathfrak{g} = \mathfrak{o}_t, \\ (-, +) & \text{if } \mathfrak{g} = \mathfrak{sp}_t. \end{cases}$$

For each $i \in \mathbb{Z}_{>0}$, let $V_i \cong V$ be a copy of V in $\mathrm{Rep}(G)$. Consider the tensor algebra

$$B = T \left(\bigoplus_{i=1}^{\infty} V_i \otimes V_i^* \right),$$

which is an ind-object of $\mathrm{Rep}(G)$. Note that B is generated by images of the maps

$$\mathcal{S}_i : V \otimes V^* \to V_i \otimes V_i^* \subset B.$$

We introduce the formal power series

$$\mathcal{S}(u) = 1 + \sum_{i>0} \mathcal{S}_i u^{-i} \in \mathrm{Hom}(V \otimes V^*, B)[\![u^{-1}]\!],$$

where 1 is regarded as an element of $\mathrm{Hom}(V \otimes V^*, B)$ via the inclusion $\mathbb{1} \hookrightarrow B$. For convenience, set $\mathcal{S}_0 = 1$. Via tensor-hom adjunction, we may also view

$$\mathcal{S}(u) \in \mathrm{Hom}(V, V \otimes B)[\![u^{-1}]\!].$$

Define the R-matrix

$$R(u) = 1 - \frac{\sigma}{u} \in \mathrm{Hom}(V \otimes V, V \otimes V)[\![u^{-1}]\!],$$

where σ is the flip map on $V \otimes V$.

Define

$$\mathcal{S}'(u) = \sum_{i \geq 0} \mathcal{S}'_i u^{-i}, \qquad \mathcal{S}'_i = \mathcal{S}_i \circ \theta,$$

so that $\mathcal{S}'(u) \in \mathrm{Hom}(V \otimes V^{\cdot} B)[\![u^{-1}]\!]$. By tensor-hom adjunction, σ may be regarded as an element of $\mathrm{Hom}(V \otimes V^* \otimes V \otimes V^*, \mathbb{1})$. Define

$$\sigma' = \sigma \circ (1_{V \otimes V^*} \otimes \theta) \in \mathrm{Hom}(V \otimes V^* \otimes V \otimes V^*, \mathbb{1}),$$

and set
$$R'(u) = 1 - \frac{\sigma'}{u}.$$
Let $V_I \cong V_{II} \cong V$ be two copies of V. For $i \in I, II$, let
$$\mathcal{S}^i(u) \in \mathrm{Hom}(V_I \otimes V_{II}, V_I \otimes V_{II} \otimes B)[\![u^{-1}]\!]$$
be the series $\mathcal{S}(u)$ acting on the V_i component. Set
$$Q(u,v) = (u-v)(-u-v)\Big[R(u-v)\mathcal{S}^I(u)R'(-u-v)\mathcal{S}^{II}(v)$$
$$- \mathcal{S}^{II}(v)R'(-u-v)\mathcal{S}^I(u)R(u-v)\Big]$$
as an element of $\mathrm{Hom}(V_I \otimes V_{II}, V_I \otimes V_{II} \otimes B)[\![u^{-1}, v^{-1}]\!]$. Via tensor-hom adjunction, we may also regard $Q(u,v)$ as an element of $\mathrm{Hom}(V_I \otimes V_I^* \otimes V_{II} \otimes V_{II}^*, B)[\![u^{-1}, v^{-1}]\!]$.

Finally, define
$$K(u) = \mathcal{S}'(-u) \mp \mathcal{S}(u) - \frac{\mathcal{S}(u) - \mathcal{S}(-u)}{2u} \in \mathrm{Hom}(V, V \otimes B)[\![u^{-1}]\!],$$
which, again by adjunction, may be regarded as an element of $\mathrm{Hom}(V \otimes V^*, B)[\![u^{-1}]\!]$.

Write
$$Q(u,v) = \sum_{i,j} Q_{i,j} u^{-i} v^{-j} \quad \text{and} \quad K(u) = \sum_i K_i u^{-i}.$$

Then, the definition of the Yangian is as follows.

Definition 5.1. The twisted Yangian of $\mathfrak{g}$, denoted $Y(\mathfrak{g})$, is the algebra which is the quotient of B by the quadratic relations given by the Q_{ij} and the K_{ij}.

We now will use ultraproducts and Łoś's theorem to generalize Proposition 3.2. Since our definition reduces to Section 3.1 in integer rank, we can use Łoś's theorem to get
$$Y(\mathfrak{g}) = \prod_{\mathcal{F}} Y(\mathfrak{g}_n).$$

Moreover, θ and θ_{t_n} satisfy the same relations in their corresponding categories, so again, by Łoś's theorem, we get
$$\theta = \prod_{\mathcal{F}} \theta_{t_n}.$$

Proposition 5.1. *We have the embedding* $i : U(\mathfrak{g}) \hookrightarrow Y(\mathfrak{g})$ *given by*
$$(1 - \theta) \mapsto \mathcal{S}_1. \tag{5.1}$$
Here $(1-\theta)$ *is a map* $V \otimes V^* \to U(\mathfrak{g})$, *which generates* $U(\mathfrak{g})$, *and* $\mathcal{S}_1$ *is a map* $V \otimes V^* \to Y(\mathfrak{g})$. *Furthermore, we have the evaluation map* $\varrho\colon Y(\mathfrak{g}) \to U(\mathfrak{g})$ *given by*
$$\mathcal{S}(u) \mapsto 1 + (1 - \theta) \cdot \left(u + \frac{1}{2}\right)^{-1}. \tag{5.2}$$

Here the coefficients of $\mathcal{S}(u)$ *are maps from* $V \otimes V^* \to Y(\mathfrak{g})$, *which generate* $Y(\mathfrak{g})$, *and the coefficients of the right hand side are maps from* $V \otimes V^*$ *to* $U(\mathfrak{g})$.

Proof. First of all, consider the map (5.1). In integer rank, this map factors through $U(\mathfrak{g}_n)$. Since factoring through is a first order property, we can conclude from Łoś's theorem that this map factors through for the ultraproduct $Y(\mathfrak{g})$ as well.

Next, consider the map (5.2), which immediately gives an algebra homomorphism from B to $U(\mathfrak{g}_t)$. This map factors through in integer rank when we pass from B to $Y(\mathfrak{g}_t)$, and since factoring through by the Q_{ij} and the K_i is a first-order property, it follows from Łoś's theorem that we have a map from $Y(\mathfrak{g}) \to U(\mathfrak{g})$ that extends the original map. $\square$

Proposition 5.2. *If $g(u) = 1 + g_1 u^{-2} + g_2 u^{-4} + \cdots \in \mathbb{C}[\![u^{-2}]\!]$ is an even power series, then*

$$\mathcal{S}(u) \mapsto g(u)\mathcal{S}(u) \tag{5.3}$$

defines an automorphism of $Y(\mathfrak{g})$.

Proof. Replacing $\mathcal{S}(u)$ with $g(u)\mathcal{S}(u)$ in the defining relations $Q(u,v)$ and $K(u,v)$ just multiplies by some power of $g(u)$, so the defining relations continue to hold. $\square$

The twisted Yangian $Y(\mathfrak{g})$ embeds naturally into the Yangian $Y(\mathfrak{gl}_t)$. The Yangian $Y(\mathfrak{gl}_t)$ can be defined within both $\mathrm{Rep}(O_t)$ and $\mathrm{Rep}(Sp_t)$ exactly as in [Kal20]. Similarly, we define the series $T'(u)$ analogously to $\mathcal{S}'(u)$. We record the following useful fact.

Proposition 5.3. *We have the embedding $Y(\mathfrak{g}) \hookrightarrow Y(\mathfrak{gl}_t)$ given by*

$$\mathcal{S}(u) \mapsto T(u)T'(-u). \tag{5.4}$$

Proof. Consider the map in the integral rank case, which is just (3.12). This defines an embedding $Y(\mathfrak{g}_n) \hookrightarrow Y(\mathfrak{gl}_{t_n})$. Being an embedding is a first-order property, so it follows from Łoś's theorem, that (5.4) defines an embedding. $\square$

This map also embeds the special twisted Yangian $SY(\mathfrak{g})$ into $Y(\mathfrak{sl}_t)$.

5.2. **Finite-length representations of $Y(\mathfrak{g})$.** Here again we set t to be transcendental and $\mathfrak{g} = \mathfrak{o}_t$ or $\mathfrak{sp}_t$. Recall $\mathfrak{g}_n = \mathfrak{o}_{2t_n}$ or $\mathfrak{sp}_{2t_n}$ and $G_n = O_{2t_n}$ or Sp_{2t_n} correspondingly. Also, when t is transcendental, set $p_n = 0$.

First we define the category of $Y(\mathfrak{g})$-modules.

Definition 5.2. Denote by $\mathrm{Rep}_0(Y(\mathfrak{g}))$ the category with objects being objects $M \in \mathrm{Rep}(G)$ together with an element $\mu_M \in \mathrm{Hom}(Y(\mathfrak{g}) \otimes M, M)$ such that

    (1) M is a representation of $Y(\mathfrak{g})$, i.e. if μ is the product map of the Yangian, then
$$\mu_M \circ (1 \otimes \mu_M) = \mu_M \circ (\mu \otimes 1) \text{ as elements of } \mathrm{Hom}(Y(\mathfrak{g}) \otimes Y(\mathfrak{g}) \otimes M, M).$$
    (2) The map $\mu_M \circ (i \otimes 1)$ gives the standard structure of a $\mathfrak{g}$ representation on M.

The morphisms in $\mathrm{Rep}_0(Y(\mathfrak{g}))$ are morphisms of $\mathrm{Rep}(G)$, which commute with the representation structure. Note that we consider only honest objects of $\mathrm{Rep}(G)$ and not ind-objects.

Similarly to Theorems 4.2 and 4.3, we can give an ultraproduct construction of $\mathrm{Rep}_0(Y(\mathfrak{g}))$.

Lemma 5.4. *The category $\mathrm{Rep}_0(Y(\mathfrak{g}))$ is defined as the full subcategory of*

$$\prod_{\mathcal{F}} \mathbf{Rep}_0(Y(\mathfrak{g}_n), p_n)$$

consisting of those objects whose image in $\prod_{\mathcal{F}} \mathbf{Rep}_{p_n}(G_n)$ lies in $\mathrm{Rep}(G)$. Moreover, the irreducible representations of $Y(\mathfrak{g})$ correspond to collections of representations of $Y(\mathfrak{g}_n)$ such

that almost all are irreducible. In other words, $\mathrm{Irr}(\mathrm{Rep}_0(Y(\mathfrak{g})))$ *is the full subcategory of*

$$\prod_{\mathcal{F}} \mathrm{Irr}(\mathbf{Rep}_0(Y(\mathfrak{g}_n), p_n)),$$

whose image is contained in $\mathrm{Rep}(G)$.

Proof. The proof is exactly the same as [Kal20, Lemma 4.2.2]. □

The same statement of course holds for representations of the special twisted Yangian.

It follows that for $M \in \mathrm{Rep}_0(Y(\mathfrak{g}))$, we can write $M = \prod_{\mathcal{F}} M_n$, where M_n is a representation of $Y(\mathfrak{g}_{t_n})$. If M is irreducible, by Łoś's theorem, almost all of the M_n are irreducible. The analogous statement holds for the special twisted Yangian.

Consider the case where t is transcendental and $\mathfrak{g} = \mathfrak{o}_t$. Suppose that $M \in \mathrm{Rep}_0(SY(\mathfrak{g}))$. Then, almost all of the $Y(\mathfrak{g}_n)$-modules M_n corresponding to M must be irreducible. Consequently, for almost all n, M_n corresponds to a choice of the parameter γ belonging to one of the three sets in (3.37). We refer to them as types A, B, and C, respectively.

Recall the $\sharp$-automorphism of $Y(\mathfrak{o}_{2n})$ defined in (3.34). We also denote by $\sharp$ its ultraproduct, which, by Łoś's theorem, defines an automorphism of $Y(\mathfrak{o}_t)$. If M is of type C, let $M^\sharp$ be the $Y(\mathfrak{o}_t)$-module obtained by composing the $\sharp$-automorphism with the action on M. Then $M^\sharp$ is of type B. Finally, if t is algebraic, we regard all irreducible representations of $Y(\mathfrak{o}_t)$ as being of type A.

Lemma 5.5. *Fix* $c \in \mathbb{Z}$. *For algebraic* t *and for almost all* n, *we have* $p_n - 2t_n > c$.

Proof. Suppose to the contrary that $p_n - 2t_n \leq c$ for almost all n. Since there are finitely many possibilities, $p_n - 2t_n = d$ for some d. This means

$$q(2t_n) = q(-d) \pmod{p_n},$$

so $q(-d)$ has an infinite number of prime divisors, implying $q(-d) = 0$. However, $q(x)$, being the minimal polynomial of t, cannot have integral roots, a contradiction. □

Proposition 5.6. *Suppose* $M \in \mathrm{Rep}_0(SY(\mathfrak{g}))$ *is irreducible. Then,* $M \sqsubset L(\lambda(u))$ *unless* $\mathfrak{g} = \mathfrak{o}_t$ *and* M *is type* C, *in which case* $M^\sharp \sqsubset L(\lambda(u))$. *Here* $L(\lambda(u))$ *is a irreducible* $Y(\mathfrak{sl}_t)$-*module, considered as a* $SY(\mathfrak{g})$-*module via the embedding* (5.4) *and* $\lambda(u)$ *is a highest weight for* $Y(\mathfrak{gl}_t)$, *which is a pair of sequences* $(\lambda_i^\bullet(u), \lambda_i^\circ(u))$.

Proof. Since every irreducible module of the special twisted Yangian arises as the restriction of an irreducible $Y(\mathfrak{g})$-module (a consequence of Łoś's theorem, as the claim holds in integral rank), consider the irreducible $Y(\mathfrak{g})$-module corresponding to M, which we also denote by M. Let $\{M_n\}$ be the sequence of $Y(\mathfrak{g}_{t_n})$-modules associated with M. Since M is irreducible, Łoś's theorem implies that M_n is irreducible for almost all n.

As an object of $\mathrm{Rep}(G)$, M decomposes as a finite direct sum of highest-weight irreducible representations, i.e.,

$$M = \bigoplus_{j=1}^{K} V(\mu_j),$$

for some negative partitions μ_j. Then, for almost all n, we have

$$M_n = \bigoplus_{j=1}^{K} V([\mu_j]_{t_n}, p_n). \tag{5.5}$$

Symplectic Case: Suppose M_n is irreducible as a $Y(\mathfrak{g}_{t_n}) = Y(\mathfrak{sp}_{2t_n})$-module. Since $\max_i -([\mu_i]_{t_n})_{t_n}$ stabilizes for large n, it follows from Lemma 5.5 that M_n satisfies the conditions of Proposition 3.5 for almost all n.

By Theorem 3.19 and the proof of Theorem 3.20 (cf. [Mol07, the second part of Theorem 4.3.8]), M_n is obtained by composing an automorphism $\mathcal{S}(u) \mapsto g_n(u)\mathcal{S}(u)$, where

$$g_n(u) = 1 + g_1^{(n)} u^{-2} + g_2^{(n)} u^{-4} + \cdots$$

is an even series, with the action on a subquotient M_n' of the finite-dimensional irreducible $Y(\mathfrak{gl}_{2t_n})$-module $L(\lambda^{(n)}(u))$, regarded as a $Y(\mathfrak{sp}_{2t_n})$-module via the embedding (3.12).

Since $SY(\mathfrak{sp}_{2t_n})$ is invariant under the automorphism $\mathcal{S}(u) \mapsto g_n(u)\mathcal{S}(u)$, we may replace M_n with M_n', obtaining isomorphic $SY(\mathfrak{sp}_t)$-modules via ultraproducts:

$$M = \prod_{\mathcal{F}} M_n \cong \prod_{\mathcal{F}} M_n' = M'.$$

Furthermore, the embedding sends $SY(\mathfrak{sp}_{2t_n})$ into $Y(\mathfrak{sl}_{2t_n})$, allowing us to view $L(\lambda^{(n)}(u))$ as a $Y(\mathfrak{sl}_{2t_n})$-module.

By construction, $\lambda^{(n)}(u)$ has a sequence of Drinfeld polynomials. Moreover, if another weight $\mu^{(n)}(u)$ has the same Drinfeld polynomials, then $\lambda^{(n)}(u)$ and $\mu^{(n)}(u)$ differ by multiplication by some even power series, so $L(\lambda^{(n)}(u)) \cong L(\mu^{(n)}(u))$ as $Y(\mathfrak{sl}_{2t_n})$-modules.

By [Kal20, Theorem 3.2.6], $L(\mu^{(n)}(u))$ decomposes as a tensor product of evaluation modules:

$$L(\mu^{(n)}(u)) \cong L(\lambda_1^{(n)}, p_n) \otimes \cdots \otimes L(\lambda_{k_n}^{(n)}, p_n).$$

It follows that:

$$M_n \sqsubset L(\lambda^{(n)}(u)) \sqsubset L(\lambda_1^{(n)}, p_n) \otimes \cdots \otimes L(\lambda_{k_n}^{(n)}, p_n),$$

where the second subquotient is to be interpreted as $Y(\mathfrak{sl}_{2t_n})$-modules, and the $\lambda_i^{(n)}$ are dominant integral $\mathfrak{gl}_{2t_n}$-weights.

The highest weight of M with respect to the $\mathfrak{sp}_{2t_n}$-action is $\widetilde{\lambda}_1^{(n)} + \cdots + \widetilde{\lambda}_{k_n}^{(n)}$, where

$$(\widetilde{\lambda}_j^{(n)})_i = (\lambda_j^{(n)})_i - (\lambda_j^{(n)})_{-i}.$$

We can restrict to n with $2t_n > \max(l(\mu_j))$, since all cofinite subsets are in $\mathcal{F}$.

Since M_n has a unique highest weight vector, this highest weight must be $[\mu_j]_{2t_n}$ for some j. Moreover, for all larger n, this remains the highest weight, as it depends only on the support of $[\mu_j]_{2t_n}$. Let μ be the corresponding negative partition, so that:

$$[\mu]_{t_n} = \widetilde{\lambda}_1^{(n)} + \cdots + \widetilde{\lambda}_{k_n}^{(n)}.$$

Since $[\mu]_{t_n}, \widetilde{\lambda}_1^{(n)}, \ldots, \widetilde{\lambda}_{k_n}^{(n)}$ are all negative partitions, and the negative partition associated to $[\mu]_{t_n}$ is constant for large n, there are finitely many choices for the $\widetilde{\lambda}_j^{(n)}$, so one such choice must occur for almost all n. Denote these limiting negative partitions by $\widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_k$.

For each $\lambda_i^{(n)}$, we have the decomposition

$$\lambda_i^{(n)} = \chi_i^{(n)} + d_i^{(n)},$$

where $\chi_i^{(n)}$ is a $\mathfrak{gl}_{2s_n}$-weight with $(\chi_i^{(n)})_1 = 0$ for large enough n, and $d_i^{(n)} \in \overline{\mathbb{F}}_{p_n}$ is constant along all components. Then, $\widetilde{\lambda}_i = \widetilde{\chi}_i^{(n)}$.

For large enough n, we must have $(\widetilde{\chi}_i^{(n)})_1 = 0$ so $(\chi_i^{(n)})_1 = 0$ implies $(\chi_i^{(n)})_{-1} = 0$. Similarly, we get $(\chi_i^{(n)})_k = 0$ for $k = -t_n + l(\mu), \ldots, t_n - l(\mu)$; so the entries of $\chi_i^{(n)}$ have at most $l(\mu)$ positive and $l(\mu)$ negative parts. Thus there are finitely many possibilities for $\chi_i^{(n)}$, so it is eventually fixed for almost all n. Hence there exists a bipartition η_i such that $[\eta_i]_{2t_n} = \chi_i^{(n)}$ for almost all n. Thus,

$$M_n \sqsubset L([\eta_1]_{2t_n} + c_1^{(n)}, p_n) \otimes \cdots \otimes L([\eta_k]_{2t_n} + c_k^{(n)}, p_n)$$

for some constants $c_i^{(n)}$. Setting $c_i = \prod_{\mathcal{F}} c_i^{(n)}$, and using Łoś's theorem, we get:

$$M \sqsubset L(\eta_1 + c_1) \otimes \cdots \otimes L(\eta_k + c_k).$$

Additionally, we have the following:

$$L(\lambda^{(n)}(u)) \sqsubset L([\eta_1]_{2t_n} + c_1^{(n)}, p_n) \otimes \cdots \otimes L([\eta_k]_{2t_n} + c_k^{(n)}, p_n),$$

so the ultraproduct $L = \prod_{\mathcal{F}} L(\lambda^{(n)}(u))$ lies in $\mathrm{Rep}(GL_t)$. By [Kal20, Definition 4.2.9], L is isomorphic to $L(\lambda(u))$ for some $Y(\mathfrak{gl}_t)$-highest weight $\lambda(u)$, where $\lambda(u)$ is a pair of sequences $(\lambda_i^{\bullet}(u), \lambda_i^{\circ}(u))$. Then, by Łoś's theorem:

$$M \sqsubset L(\lambda(u)).$$

Orthogonal Case: The argument is analogous. For algebraic t (i.e., in positive characteristic), the proof proceeds identically. In characteristic zero, it applies for types A and B modules since, if M_n is type A or B, the second part of the proof of [Mol07, Theorem 4.4.14] implies that M_n is a subquotient of $L(\lambda^{(n)}(u))$, which in turn is a subquotient of $L(\lambda_1^{(n)}) \otimes \cdots \otimes L(\lambda_{k_n}^{(n)})$. If M_n is type C, then $M_n^{\sharp}$ is of type B, and $M_n^{\sharp} \sqsubset L(\lambda(u))$. $\qquad\square$

Also, note that the $Y(\mathfrak{sl}_t)$-module $L(\lambda(u))$ completely determines M, by the same argument as in [Kal20, Corollary 4.2.8]. Let $M^{\natural} = M$, $M_n^{\natural} = M_n$ unless M is type C in which case set $M^{\natural} = M^{\sharp}$ and $M_n^{\natural} = M_n^{\sharp}$. Then we have

$$M^{\natural} \sqsubset L(\lambda(u)) \sqsubset L(\eta_1 + c_1) \otimes \cdots \otimes L(\eta_k + c_k),$$

where $L(\lambda(u))$ is the finite-dimensional irreducible highest-weight representation of $Y(\mathfrak{gl}_t)$, and $\lambda(u)$ is a pair of sequences $(\lambda_i^{\bullet}(u), \lambda_i^{\circ}(u))$ in $\mathbb{C}[\![u^{-1}]\!]$ given by

$$\lambda_i^{\bullet}(u) = \prod_{j=1}^{k}(1 + [(\eta_j^{\bullet})_i + c_j]u^{-1}) \quad \text{and} \quad \lambda_i^{\circ}(u) = \prod_{j=1}^{k}(1 + [(\eta_j^{\circ})_i - c_j]u^{-1})$$

along with an element

$$\lambda^m = \prod_k (1 + c_k u^{-1}).$$

These are defined up to simultaneous multiplication by any $f(u) \in \mathbb{C}[\![u^{-1}]\!]$. We define the highest weight of $M^{\natural}$, as a $Y(\mathfrak{g})$-module, as a sequence $\mathbb{C}[\![u^{-1}]\!]$ equal to

$$\mu_i(u) = \lambda_i^{\bullet}(-u)\lambda_i^{\circ}(-u)$$

along with an element

$$\mu^m(u) = \lambda^m(-u)\lambda^m(u).$$

Set $l(\mu) = l(\lambda)$. Note that multiplying λ by $f(u) \in \mathbb{C}[\![u^{-1}]\!]$ multiplies μ by $f(u)f(-u) \in \mathbb{C}[\![u^{-2}]\!]$. Thus μ is defined up to simultaneous multiplication by $g(u) \in \mathbb{C}[\![u^{-2}]\!]$. Since $\mu^m(u)$ is an even sequence, we may assume that $\mu^m(u) = 1$. So with this assumption, $\mu(u)$ is uniquely determined.

We can relate the highest weight of $M^\natural$ to the highest weights of $M_n^\natural$ in the following way. For $n > l(\mu)$, define

$$([\mu]_{t_n})_{t_n+1-i} = \begin{cases} \mu_i(u) & \text{for } i \leq l(\mu), \\ \mu^m(u) & \text{otherwise}, \end{cases}$$

where the coefficients are also reduced to $\overline{\mathbb{F}}_{p_n}$, i.e. c_j is replaced with $c_j^{(n)}$, etc.

By Łoś's theorem, for almost all n, we have

$$M_n' \sqsubset L([\lambda]_{2t_n}(u)) \sqsubset L([\eta_1]_{2t_n} + c_1^{(n)}, p_n) \otimes \cdots \otimes L([\eta_k]_{2t_n} + c_k^{(n)}, p_n).$$

From [Mol07, Corollary 4.2.12], the $t_n + 1 - i$-th component of the highest weight of $M_n^\natural$ is simply

$$([\lambda]_{2t_n})_{-t_n-1+i}(-u)([\lambda]_{2t_n})_{t_n+1-i}(u),$$

which is easily checked to equal $([\mu]_{t_n})_{t_n+1-i}$.

We can now prove the main classification theorems.

Theorem 5.7. *For every irreducible $M \in \text{Rep}_0(SY(\mathfrak{sp}_t))$, there exists a sequence of monic polynomials, denoted by $P(u) = P_1(u), P_2(u), \ldots$ such that*

- *the corresponding highest weight satisfies*

$$\frac{\mu_{i+1}(u)}{\mu_i(u)} = \frac{P_i(u+1)}{P_i(u)}. \tag{5.6}$$

- *for sufficiently large i, the $\mu_i(u)$ stabilizes and equals $\mu^m(u) = 1$, and the $P_i(u)$ stabilize and equal 1.*

Moreover, for any such $P(u)$, there is M with highest weight satisfying (5.6). This also gives a one-to-one corresponence between irreducible modules and sequences of monic polynomials $P(u)$.

The polynomials in the sequence $P(u)$ are referred to as the *Drinfeld polynomials* corresponding to M.

Proof. For the first part, consider the corresponding sequence of modules $M_n \in \text{Rep}_0(SY(\mathfrak{sp}_t))$, almost all of which are irreducible with highest weight equal to $[\mu]_{t_n}(u)$, where $\mu(u)$ is the highest weight of M. Let the Drinfeld polynomials corresponding to $[\mu]_{t_n}(u)$ be

$$Q^{(n)}(u), P_{t_n-1}(u), \ldots, P_1^{(n)}(u).$$

Then, we have

$$\frac{([\mu]_{t_n})_{t_n+1-(i+1)}(u)}{([\mu]_{t_n})_{t_n+1-(i)}(u)} = \frac{P_i^{(n)}(u+1)}{P_i^{(n)}(u)}.$$

Since

$$\deg Q^{(n)}(u) + 2 \sum_{i=1}^{t_n-1} \deg P_i^{(n)}(u) + 2t_n < p,$$

the same argument as in the proof of [Kal20, Theorem 4.2.11] applies to show that there exist polynomials $P_1(u), P_2(u), \ldots$ such that (5.6) holds. Moreover, as $\mu_i(u)$ stabilizes and equals $\mu^m(u) = 1$ for sufficiently large i, it follows that $P_i(u) = 1$ for all sufficiently large i.

For the second statement, we may simply use an analogous argument to the second part of the proof of [Kal20, Theorem 4.2.11].

Finally, note that the highest weight of M is unique subject to the condition $\mu^m(u) = 1$, so the Drinfeld polynomials are uniquely determined. $\qquad\square$

Theorem 5.8. *For every irreducible $M \in \mathrm{Rep}_0(SY(\mathfrak{o}_t))$, M is type A and there exists a sequence of monic polynomials, denoted by $P(u) = P_1(u), P_2(u), \ldots$, such that*

- *the corresponding highest weight satisfies*

$$\frac{\mu_{i+1}(u)}{\mu_i(u)} = \frac{P_i(u+1)}{P_i(u)}. \tag{5.7}$$

- *for sufficiently large i, the $\mu_i(u)$ stabilizes and equals $\mu^m(u) = 1$, and the $P_i(u)$ stabilize and equal 1.*

Moreover, for any such $P(u)$, there is M with highest weight satisfying (5.7). This also gives a one-to-one correspondence between irreducible modules and sequences of monic polynomials $P(u)$.

Proof. If t is algebraic, the proof is identical to that for the symplectic case. Now suppose t is transcendental. We know that $\mu^m(u) = 1$. Thus, for sufficiently large n, the highest weight $([\mu]_n)(u)$ of $M_n^\natural$ satisfies $([\mu]_n)_1(u) = 1$. It also must satisfy (3.32). The γ corresponding to it is in one of the first two subsets in (3.37), since $M_n^\natural$ is type A or B. However, for γ in the second subset, (3.32) impossible for $\mu(u) = 1$, i.e. $\mu'(u) = 1 + 1/2u^{-1}$. So $\gamma = -1/2$, and $M_n^\natural$ is type A. Thus $M_n^\natural$ is type A for almost all n, so M is type A. The rest of the proof is exactly as in the symplectic case. $\qquad\square$

Finally, let us discuss the classification of $Y(\mathfrak{g})$-modules. We know that

$$Y(\mathfrak{g}) = SY(\mathfrak{g}) \otimes Z_{HC}(Y(\mathfrak{g})).$$

Hence a representation of $Y(\mathfrak{g})$ is given by a tensor product of a representation of $SY(\mathfrak{g})$ with a representation of $\mathbb{C}[\![z_2, z_4, \ldots]\!]$. This is simply a sequence of complex numbers, represented as an even series $g(u) \in \mathbb{C}[\![u^{-2}]\!]$. Two such representations corresponding to the same irreducible representations of $SY(\mathfrak{g})$ differ by multiplication by a one-dimensional representation of $Y(\mathfrak{g})$. Thus, a specific choice of $g(u) \in 1 + u^{-2}\mathbb{C}[\![u^{-2}]\!]$, determines a unique representation of $Y(\mathfrak{g})$. Thus we have the following.

Theorem 5.9. *Irreducible objects of $\mathrm{Rep}_0(Y(\mathfrak{g}))$ are in one-to-one correspondence with tuples $(P(u), g(u))$, where $P(u) = P_1(u), P_2(u), \ldots$ is a sequence of Drinfeld polynomials and $g(u) \in 1 + u^{-2}\mathbb{C}[\![u^{-2}]\!]$.*

It is interesting to note that the classification does not depend on whether $\mathfrak{g} = \mathfrak{o}_t$ or $\mathfrak{sp}_t$. One may expect this given that $\mathrm{Rep}(O_t) = \mathrm{Rep}(Sp_{-t})$ as tensor categories, and only a slight modification needs to be made to the braiding for them to be equivalent as symmetric tensor categories.

References

[BT18]   Jonathan Brundan and Lewis Topley, *The p-centre of Yangians and shifted Yangians*, Mosc. Math. J. **18** (2018), no. 4, 617–657.

[Cou21] Kevin Coulembier, *Monoidal abelian envelopes*, Compositio Mathematica **157** (2021), no. 7, 1584–1609.

[Del90] Pierre Deligne, *Catégories tannakiennes*, The Grothendieck Festschrift, Vol. II, Progr. Math., vol. 87, Birkhäuser Boston, Boston, MA, 1990, pp. 111–195.

[Del02]   _____, *Catégories tensorielles*, Mosc. Math. J. **2** (2002), no. 2, 227–248, Dedicated to Yuri I. Manin on the occasion of his 65th birthday.

[Del07]   _____, *La catégorie des représentations du groupe symétrique S_t, lorsque t n'est pas un entier naturel*, Algebraic groups and homogeneous spaces, Tata Inst. Fund. Res. Stud. Math., vol. 19, Tata Inst. Fund. Res., Mumbai, 2007, pp. 209–273.

[DLZ18]   P. Deligne, G.I. Lehrer, and R.B. Zhang, *The first fundamental theorem of invariant theory for the orthosymplectic super group*, Advances in Mathematics **327** (2018), 4–24, Special volume honoring David Kazhdan.

[DMOS82] Pierre Deligne, James S. Milne, Arthur Ogus, and Kuang-yen Shih, *Hodge cycles, motives, and Shimura varieties*, Lecture Notes in Mathematics, vol. 900, Springer-Verlag, Berlin-New York, 1982. MR 654325

[Dri85]   Vladimir G. Drinfeld, *Hopf algebras and the quantum Yang-Baxter equation*, Dokl. Akad. Nauk SSSR **283** (1985), no. 5, 1060–1064.

[EAHS18]   Inna Entova-Aizenbud, Vladimir Hinich, and Vera Serganova, *Deligne categories and the limit of categories* Rep($GL(m|n)$), International Mathematics Research Notices **2020** (2018), no. 15, 4602–4666.

[EGNO16]   Pavel Etingof, Shlomo Gelaki, Dmitri Nikshych, and Victor Ostrik, *Tensor categories*, vol. 205, American Mathematical Soc., 2016.

[EK23]   Pavel Etingof and Arun S. Kannan, *Lectures on symmetric tensor categories*, Representations of Algebras and Related Structures: Proceedings of the International Conference on Representations of Algebras, ICRA 2020 (Trondheim, Norway; Bielefeld, Germany) (Aslak Bakke Buan, Henning Krause, and Øyvind Solberg, eds.), EMS Press, 11 2023, pp. 191–234.

[Eti16]   Pavel Etingof, *Representation theory in complex rank, II*, Adv. Math. **300** (2016), 473–504.

[GW00]   Roe Goodman and Nolan R Wallach, *Representations and invariants of the classical groups*, Cambridge University Press, 2000.

[Har16]   Nate Harman, *Deligne categories as limits in rank and characteristic*, 2016.

[Hum12]   James E Humphreys, *Introduction to lie algebras and representation theory*, vol. 9, Springer Science & Business Media, 2012.

[Jan03]   Jens Carsten Jantzen, *Representations of algebraic groups*, vol. 107, American Mathematical Soc., 2003.

[Kal20]   Daniil Kalinov, *Finite-dimensional representations of Yangians in complex rank*, International Mathematics Research Notices (2020), no. 20, 6967–6998.

[Mol07]   Alexander Molev, *Yangians and classical Lie algebras*, Mathematical Surveys and Monographs, vol. 143, American Mathematical Society, Providence, RI, 2007.

[UH21]   Alexandra Utiralova and Serina Hu, *Harish-Chandra bimodules of finite K-type in Deligne categories*, 2021.

[Uti19]   Alexandra Utiralova, *Olshanski's centralizer construction and Deligne tensor categories*, 2019.

Appendix A. The Sklyanin determinant in positive characteristic

In this appendix, we define the Sklyanin determinant in positive characteristic and prove Theorem 3.4. Our approach follows Sections 2.5–2.9 of [Mol07]. Since many proofs mirror those in the positive characteristic setting, we will only provide brief outlines where appropriate.

Consider the R-matrix $R(u)$, which takes values in $(\operatorname{End}\mathbb{C}^N)^{\otimes 2}$. We define the R-matrices $R_{ij}(u)$, acting on the ith and jth components of $(\mathbb{C}^N)^{\otimes m}$. Similarly, for an $N \times N$ matrix

$$M \in (\operatorname{End}\mathbb{C}^N) \otimes Y(\mathfrak{g}_n)[\![u^{-1}]\!],$$

we define

$$M_i \in (\operatorname{End}\mathbb{C}^N)^{\otimes m} \otimes Y(\mathfrak{g}_n)[\![u^{-1}]\!]$$

as acting on the ith component of $(\mathbb{C}^N)^{\otimes m}$. We will refer to the following equation as the *ternary relation*

$$R(u-v)T_1(u)T_2(u) = T_2(u)T_1(u)R(u-v), \tag{A.1}$$

where $T(u)$ is from section 3.1 and the subscript indicates which components it acts on. We now introduce the rational function $R(u_1, \ldots, u_m)$ with values in $(\operatorname{End}\mathbb{C}^N)^{\otimes m}$, defined as

$$R(u_1, \ldots, u_m) = (R_{m-1,m})(R_{m-2,m}R_{m-2,m-1}) \cdots (R_{1m} \cdots R_{12}), \tag{A.2}$$

where $u_1, \ldots, u_m$ are independent variables, and we abbreviate $R_{ij} = R_{ij}(u_i - u_j)$. Here, $R_{ij}(u)$ denotes the R-matrix acting on the ith and jth components of $(\operatorname{End}\mathbb{C}^N)^{\otimes m}$. Using the Yang-Baxter equation (see (A.8)) and the fact that we may commute R_{ij} and R_{kl} for pairwise distinct i, j, k, l, a simple induction shows

$$R(u_1, \ldots, u_m) = (R_{12} \cdots R_{1m}) \cdots (R_{m-2,m-1}R_{m-2,m})(R_{m-1,m}). \tag{A.3}$$

Define

$$T_i = T_i(u_i), \quad \mathcal{S}_i = \mathcal{S}_i(u_i), \qquad 1 \le i \le m$$

and

$$R'_{ij} = R'_{ji} = R'_{ij}(-u_i - u_j).$$

For an arbitrary permutation $(p_1, \ldots, p_m)$ of the indices $1, \ldots, m$, we abbreviate

$$\langle \mathcal{S}_{p_1}, \ldots, \mathcal{S}_{p_m} \rangle = \mathcal{S}_{p_1}(R'_{p_1 p_2} \cdots R'_{p_1 p_m})\mathcal{S}_{p_2}(R'_{p_2 p_3} \cdots R'_{p_2 p_m}) \cdots \mathcal{S}_{p_m}. \tag{A.4}$$

Proposition A.1. *We have the identity*

$$R(u_1, \ldots, u_m)T_1 \cdots T_m = T_m \cdots T_1 R(u_1, \ldots, u_m).$$

Proof. First, by repeated application of the ternary relation (A.1) and the fact that R_{ij} and T_k are permutable for i, j, k distinct, we get

$$(R_{1m} \cdots R_{12})T_1(T_2 \cdots T_m) = (T_2 \cdots T_m)T_1(R_{1m} \cdots R_{12}).$$

Since

$$R(u_1, \ldots, u_m) = R(u_2, \ldots, u_m)(R_{1m} \cdots R_{12}),$$

by induction on m we are done. $\qquad\square$

Proposition A.2. *We have the identity*

$$R(u_1, \ldots, u_m) \langle \mathcal{S}_1, \ldots, \mathcal{S}_m \rangle = \langle \mathcal{S}_m, \ldots, \mathcal{S}_1 \rangle R(u_1, \ldots u_m). \tag{A.5}$$

Proof. Let i, j, k be distinct indices in $\{1, \ldots, m\}$. Note that we have

$$R_{ij}\mathcal{S}_i R'_{ij}\mathcal{S}_j = \mathcal{S}_j R'_{ji}\mathcal{S}_i R_{ij} \tag{A.6}$$

$$R_{ij}R'_{ik}R'_{jk} = R'_{jk}R'_{ik}R_{ij}. \tag{A.7}$$

Indeed, (A.6) is simply the quaternary relations, and for (A.7), we start with the Yang-Baxter equation

$$R_{ij}R_{ik}R_{jk} = R_{jk}R_{ik}R_{ij}. \tag{A.8}$$

Then apply the transposition with respect to the matrix G on the k'th component to get

$$R_{ij}R'_{jk}R'_{ik} = R'_{ik}R'_{jk}R_{ij}. \tag{A.9}$$

Now we have $P_{ij}R^t_{jk}P_{ij} = R^t_{ik}$, where P is the permutation operator and

$$R^t_{jk} = R^t_{jk}(-u_j - u_k).$$

Then we get that

$$(G_k^{-1}P_{ij}G_k)R'_{jk}(G_k^{-1}P_{ij}G_k) = (G_k^{-1}P_{ij}G_k)(G_k^{-1}R^t_{jk}G_k)(G_k^{-1}P_{ij}G_k) = G_k^{-1}R^t_{ik}G_k$$

which is simply R'_{ik}. So we conjugate both sides of (A.9) by $(G_k^{-1}P_{ij}G_k)$, and since conjugating R_{ij} gives us R_{ij} again, since R_{ij} and G_k commute, and $P_{ij}R_{ij}P_{ij} = R_{ij}$, we get (A.7). From here, we can proceed in exactly the same way as the proof of Proposition 2.5.1 in [Mol07], replacing "A^t" transpose symbols with the "A'" transpose with respect to G. $\square$

Consider the symmetric group $\mathfrak{S}_m$ acting on $\{1, \ldots, m\}$. The anti-symmetrizer in the group algebra of $\mathfrak{S}_m$ is given by

$$\sum_{\sigma \in \mathfrak{S}_m} \mathrm{sgn}(\sigma) \cdot \sigma \in \mathbb{C}[\mathfrak{S}_m].$$

Let A_m be the image of this anti-symmetrizer under the natural action of $\mathfrak{S}_m$ on $(\mathbb{C}^N)^{\otimes m}$. We use $e_1, \ldots, e_N$ to be the standard basis vectors of $\mathbb{C}^N$, we get

$$A_m(e_{i_1} \otimes \cdots \otimes e_{i_m}) = \sum_{\sigma \in \mathfrak{S}_m} \mathrm{sgn}(\sigma) \cdot e_{i_{\sigma(1)}} \otimes \cdots \otimes e_{i_{\sigma(m)}}. \tag{A.10}$$

Note that we have $A_m^2 = m!A_m$. For $m \leq N$ the right hand side is nonzero because we assume $N < p$, see Proposition 3.5.

Proposition A.3. *For $m \leq N$, if $u_i - u_{i+1} = 1$ for all $i = 1, \ldots, m-1$, then*

$$R(u_1, \ldots, u_m) = A_m.$$

Proof. We can use the same proof as Proposition 1.6.2 in [Mol07]. There are no division by zero problems because all denominators are $\leq m \leq N < p$. $\square$

Now set $m = N$ and

$$u_i = u - i + 1, \quad i = 1, \ldots, N.$$

By the previous propositions,

$$A_N T_1 \cdots T_m = T_m \cdots T_1 A_N \tag{A.11}$$

and

$$A_N \langle S_1, \ldots, S_N \rangle = \langle S_N, \ldots, S_1 \rangle A_N. \tag{A.12}$$

The image of A_N on $(\mathbb{C}^N)^{\otimes N}$ is one-dimensional. Indeed, for any basis element $e_{i_1} \otimes \cdots \otimes e_{i_N}$, if $i_k = i_\ell$ for some $k \neq \ell$, then by (A.10),

$$A_N(e_{i_1} \otimes \cdots \otimes e_{i_N}) = 0.$$

Thus, A_N is nonzero only on pure tensors indexed by permutations of $(1, \ldots, N)$, where it evaluates to

$$\sum_{\sigma \in \mathfrak{S}_N} \mathrm{sgn}(\sigma) \cdot e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(N)}.$$

It follows that (A.11) and (A.12) reduce to A_N multiplied by a scalar series with coefficients in $Y(\mathfrak{gl}_N)$ or $Y(\mathfrak{g}_n)$, respectively.

Definition A.1. The *quantum determinant* (resp. *Skylanin determinant*) of the matrix $T(u)$ (resp. $\mathcal{S}(u)$) is the formal series with coefficients in $Y(\mathfrak{gl}_N)$ (resp. $Y(\mathfrak{g}_n)$) given by

$$\mathrm{qdet}\, T(u) = d_0 + d_1 u^{-1} + d_2 u^{-2} + \cdots ,$$

respectively

$$\mathrm{sdet}\, \mathcal{S}(u) = c_0 + c_1 u^{-1} + c_2 u^{-2} + \cdots ,$$

such that the element (A.11) (resp (A.12)) equals $A_N \mathrm{qdet}\, T(u)$ (resp. $A_N \mathrm{sdet}\, \mathcal{S}(u)$).

Proposition A.4. *We have for any permutation $\pi \in \mathfrak{S}_N$*

$$\mathrm{qdet}\, T(u) = \mathrm{sgn}(\pi) \sum_{\sigma \in \mathfrak{S}_N} \mathrm{sgn}(\sigma) \cdot t_{\sigma(1),\pi(1)}(u) \otimes \cdots \otimes t_{\sigma(N),\pi(N)}(u - N + 1)$$

$$= \mathrm{sgn}(\pi) \sum_{\sigma \in \mathfrak{S}_N} \mathrm{sgn}(\sigma) \cdot t_{\pi(1),\sigma(1)}(u - N + 1) \otimes \cdots \otimes t_{\pi(N),\sigma(N)}(u).$$

Proof. The characteristic zero proof works just as well in positive characteristic, see the proof of Proposition 1.6.6 in [Mol07]. $\square$

Due to this, we see that this definition of the quantum determinant matches the one in [Kal20], so we know that its coefficients are central and algebraically independent.

Theorem A.5. *Under the embedding (3.12),*

$$\mathrm{sdet}\, \mathcal{S}(u) = \alpha_N(u)\, \mathrm{qdet}\, T(u)\, \mathrm{qdet}\, T(-u + N - 1),$$

where $\alpha_N(u)$ is given by (3.13).

Proof. Consider the mappings

$$T(u) \mapsto G^{-1} T(u) G,$$
$$T(u) \mapsto T^t(u),$$
$$T(u) \mapsto T(-u).$$

The first is an automorphism, while the second and third are anti-automorphisms by Propositions 1.3.1 and 1.3.3 in [Mol07] ((1.22), (1.26), (1.25)). The proof follows from the characteristic-zero case. Their composition yields

$$T(u) \mapsto G^{-1} T^t(-u) G = T'(-u), \tag{A.13}$$

an automorphism denoted τ_N.

From (3.12), we obtain $S_i = T_i T_i'$, where $T_i = T_i(u - i + 1)$ and $T_i^\sigma = T_i'(-u + i - 1)$. Thus, the left-hand side of (A.12) rewrites as

$$A_N T_1 T_1^\sigma R_{12}' \cdots R_{1N}' T_2 T_2^\sigma R_{23}' \cdots R_{2N}' \cdots T_{N-1} T_{N-1}^\sigma R_{N-1,N}' T_N T_N^\sigma, \tag{A.14}$$

where $R_{ij}' = R_{ij}(-2u + i + j - 2)$. Applying the ternary relation (A.1) and partial transpose with respect to G on the first component,

$$T_1'(u) R'(u - v) T_2(v) = T_2(v) R'(u - v) T_1'(u).$$

Thus,

$$T_i^\sigma R_{ij}' T_j = T_j R_{ji}' T_i^\sigma. \tag{A.15}$$

For $i \neq j, k$, both T_i and T_i^σ commute with R_{jk}', allowing us to rewrite (A.14) as

$$A_N T_1 (T_1^\sigma R_{12}' T_2) R_{13}' \cdots R_{1N}' (T_2^\sigma R_{23}' T_3) \cdots (T_{N-1}^\sigma R_{N-1,N}' T_N) T_N^\sigma. \tag{A.16}$$

Applying (A.15) repeatedly, we obtain

$$A_N T_1 \cdots T_N R_{12}' \cdots R_{1N}' R_{23}' \cdots R_{2N}' \cdots R_{N-1,N}' T_1^\sigma \cdots T_N^\sigma.$$

This simplifies to

$$\text{qdet}\, T(u) A_N \langle I_1, I_2, \ldots, I_N \rangle T_1^\sigma \cdots T_N^\sigma,$$

where I_i is the identity operator. Since the mapping $\mathcal{S}(u) \mapsto I$ defines a representation of $T(\mathfrak{g}_n)$, we obtain from (A.12)

$$A_N \langle I_1, \ldots, I_n \rangle = \langle I_N, \ldots, I_1 \rangle A_N = A_N \beta_N(u), \tag{A.17}$$

for some scalar function $\beta_N(u)$. Applying τ_N to (A.11),

$$A_N T_1^\sigma \cdots T_N^\sigma = A_N \tau_N(\text{qdet}\, T(u)).$$

Since

$$\text{qdet}\, T(u) = \sum_{\sigma \in \mathfrak{S}_N} \text{sgn}(\sigma) \cdot t_{\sigma(1),1}(u) \cdots t_{\sigma(N),N}(u - N + 1),$$

and τ_N maps $T(u) \mapsto T'(-u)$, we find

$$\tau_N(\text{qdet}\, T(u)) = \sum_{\sigma \in \mathfrak{S}_N} \text{sgn}(\sigma) \cdot t_{N,N+1-\sigma(1)}(-u) \cdots t_{1,N+1-\sigma(N)}(-u + N - 1).$$

Introducing $\pi(i) = N + 1 - i$, this rewrites as

$$\sum_{\sigma \in \mathfrak{S}_N} \text{sgn}(\sigma) \cdot t_{\pi(1),(\pi \circ \sigma)(1)}(-u) \cdots t_{\pi(N),(\pi \circ \sigma)(N)}(-u + N - 1)$$

$$= \text{sgn}(\pi) \sum_{\sigma' \in \mathfrak{S}_N} \text{sgn}(\sigma') \cdot t_{\pi(1),\sigma'(1)}(-u) \cdots t_{\pi(N),\sigma'(N)}(-u + N - 1).$$

Thus, $\tau_N(\text{qdet}\, T(u)) = \text{qdet}\, T(-u + N - 1)$, leading to

$$A_N \text{sdet}\, \mathcal{S}(u) = A_N \beta_N(u) \text{qdet}\, T(u) \text{qdet}\, T(-u + N - 1).$$

Applying this to $e_1 \otimes \cdots \otimes e_N$,

$$\text{sdet}\, \mathcal{S}(u) \sum_{\sigma \in \mathfrak{S}_N} \text{sgn}(\sigma) e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(N)}$$

$$= \beta_N(u) \text{qdet}\, T(u) \text{qdet}\, T(-u + N - 1) \sum_{\sigma \in \mathfrak{S}_N} \text{sgn}(\sigma) e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(N)}.$$

It follows that
$$\mathrm{sdet}\,\mathcal{S}(u) = \beta_N(u)\,\mathrm{qdet}\,T(u)\,\mathrm{qdet}\,T(-u+N-1).$$
To show $\beta_N(u) = \alpha_N(u)$, consider first the orthogonal case. With
$$R(u) = 1 - \frac{P}{u}, \quad P = \sum_{i,j} e_{ij} \otimes e_{ji},$$
we obtain
$$R'(u) = 1 - \frac{P'}{u}, \quad P' = \sum_{i,j} e_{ij} \otimes e_{N+1-i,N+1-j}.$$
By direct computation, $PP' = P'$, so
$$(1-P)R'(u) = (1-P).$$
For $i < j$, this implies
$$A_N R'_{ij} = A_N.$$
Thus, $\alpha_N(u) = 1$, agreeing with (3.13). In the symplectic case, applying R'_{jk} to the basis vector
$$v = e_1 \otimes e_2 \otimes \cdots \otimes e_n \otimes e_{n+1} \otimes \cdots \otimes e_{2n}$$
yields
$$\frac{2u - 2n + 3}{2u - 2n + 1} A_{2n} v.$$
Continuing this process leads to
$$\frac{2u + 1}{2u - 2n + 1} A_{2n} v,$$
matching (3.13) and proving $\beta_N(u) = \alpha_N(u)$. $\qquad\square$

As a consequence of this theorem, the coefficients of $\mathrm{sdet}\,\mathcal{S}(u)$ are central, allowing us to define $Z_{HC}(\mathfrak{g}_n)$ as the algebra generated by these coefficients.

Applying Theorem A.5, we immediately obtain
$$\alpha_N(-u+N-1) \cdot \mathrm{sdet}\,\mathcal{S}(u) = \alpha_N(u) \cdot \mathrm{sdet}\,\mathcal{S}(-u+N-1).$$
From this, it follows that all the odd coefficients can be expressed in terms of the even ones. Consequently, $Z_{HC}(Y(\mathfrak{gl}_n))$ is generated by the even coefficients. Finally, the fact these coefficients are algebraically independent follows in the same way as the characteristic zero case; see the proof of Theorem 2.8.2 in [Mol07].

Next, we construct a series $\widetilde{d}(u) \in 1 + u^{-1}Z_{HC}(Y(\mathfrak{gl}_n))[\![u^{-1}]\!]$ satisfying
$$\widetilde{d}(u)\widetilde{d}(u-1)\cdots\widetilde{d}(u-N+1) = \mathrm{qdet}\,T(u) = 1 + d_1 u^{-1} + d_2 u^{-2} + \cdots.$$
Writing
$$\widetilde{d}(u) = 1 + \widetilde{d}_1 u^{-1} + \widetilde{d}_2 u^{-2} + \cdots,$$
expanding the power series reveals that the u^{-k} coefficient of $\widetilde{d}(u-m)$ takes the form $\widetilde{d}_k$ plus terms depending only on $\widetilde{d}_i$ for $i < k$. Thus, the u^{-k} coefficient of $\widetilde{d}(u)\widetilde{d}(u-1)\cdots\widetilde{d}(u-N+1)$ is $N\widetilde{d}_k$ plus terms involving only $\widetilde{d}_i$ for $i < k$. Since $N < p$, we can recursively determine the d_i starting from $k = 1$.

By Proposition A.4, automorphisms of the form (3.4) map
$$\widetilde{d}(u) \mapsto f(u)\widetilde{d}(u).$$

Finally, we define
$$\widetilde{t}_{ij}(u) = \widetilde{d}(u)^{-1} t_{ij}(u).$$
It follows that $\widetilde{t}_{ij}(u)$ remains invariant under automorphisms of the form (3.4), implying that $\widetilde{t}_{ij}(u) \in Y(\mathfrak{sl}_N)$.

Now we can finish the proof of Theorem 3.4.

First, we establish the decomposition
$$Y(\mathfrak{g}_n) = Z_{HC}(Y(\mathfrak{g}_n))SY(\mathfrak{g}_n). \tag{A.18}$$
We regard $Y(\mathfrak{g}_n)$ as a subalgebra of $Y(\mathfrak{gl}_N)$ via the embedding (3.12). Under this embedding, we have
$$\mathbf{s}_{ij}(u) = \sum_k \theta_{kj} t_{ik}(u) t_{N+1-j,N+1-k}(-u).$$

Define the series
$$\widetilde{\mathbf{s}}_{ij}(u) = (\widetilde{d}(u)\widetilde{d}(-u))^{-1} \mathbf{s}_{ij}(u).$$
Since $\widetilde{d}(u)$ has central coefficients, it follows that
$$\widetilde{\mathbf{s}}_{ij}(u) = \sum_k \theta_{kj} \widetilde{t}_{ik}(u) \widetilde{t}_{N+1-j,N+1-k}(-u).$$

Now, we compute
$$\alpha_N(u)^{-1} \operatorname{sdet} \mathcal{S}(u) = \operatorname{qdet} T(u) \operatorname{qdet} T(-u + N - 1)$$
$$= (\widetilde{d}(u)\widetilde{d}(-u))(\widetilde{d}(u-1)\widetilde{d}(-u+1))\cdots(\widetilde{d}(u-N+1)\widetilde{d}(-u+N-1)).$$

From this, it follows that all coefficients of $\widetilde{d}(u)\widetilde{d}(-u)$ can be expressed as polynomials in the coefficients of $\operatorname{sdet}\mathcal{S}(u)$, implying that $\widetilde{d}(u)\widetilde{d}(-u) \in Z_{HC}(\mathfrak{g}_n)$.

Similarly to how $\widetilde{t}_{ij}(u)$ belongs to $Y(\mathfrak{sl}_N)$, we deduce that $\widetilde{\mathbf{s}}_{ij}(u) \in SY(\mathfrak{g}_n)$. Thus, the decomposition
$$\mathbf{s}_{ij}(u) = \widetilde{d}(u)\widetilde{d}(-u)\widetilde{\mathbf{s}}_{ij}(u)$$
yields the desired result (A.18).

Finally, the tensor product decomposition follows from the relation
$$Y(\mathfrak{gl}_N) = Z_{HC}(Y(\mathfrak{gl}_N)) \otimes Y(\mathfrak{sl}_N),$$
along with the inclusions $Z_{HC}(Y(\mathfrak{g}_n)) \subset Z_{HC}(Y(\mathfrak{gl}_N))$ (which follows from Theorem A.5) and $SY(\mathfrak{g}_n) \subset Y(\mathfrak{sl}_N)$ (by the same argument as in characteristic zero).

This completes the proof of (3.4).

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