PATTERNS IN THE STABLE SL(N) HOMOLOGY OF TORUS KNOTS

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ABSTRACT. Gorsky, Oblomkov, and Rasmussen conjectured that the stable Khovanov homology of $T(n,\infty)$ — which is the limit of the Khovanov homology of the (n,m)-torus link as $m\to\infty$ — is isomorphic to the homology of a certain Koszul complex W_n. In this paper, we define a grading L and conjecture that the L-homogeneous summands of the homology of W_n satisfy a recursive relationship, reminiscent of the inclusion-exclusion principle, which would imply that the homology of W_n is determined by finitely many bidegrees. We present theoretical and computational evidence for this relationship and discuss an analogous conjecture for $\mathfrak{sl}(N)$ and Lee homologies. We also make a conjecture concerning the maximal torsion order appearing in the homology of Koszul complexes corresponding to $\mathfrak{sl}(N)$ analogues of Lee homology.

Keywords: torus knots, Khovanov homology, Lee homology, stable homology, GOR conjecture, torsion order.

1. INTRODUCTION

Khovanov-Rozansky $\mathfrak{sl}(N)$ homology [KR04] is a powerful knot invariant that assigns to each oriented link a bigraded abelian group. When $N=2$, this homology group reduces to Khovanov homology [Kho00]. While torus links are among the simplest links, a computation of their Khovanov-Rozansky $\mathfrak{sl}(N)$ homology remains an outstanding problem in knot theory. For a fixed number of strands n, Stošić [Sto09] showed that the Khovanov homology of $T(n,m)$, after a renormalization, approaches a well-defined limit as $m\to\infty$ — this limit is called the stable Khovanov homology of $T(n,\infty)$. In 2012, Gorsky, Rasmussen, and Oblomkov [GOR13] conjectured that the Khovanov homology of $T(n,\infty)$ is isomorphic to the homology of a Koszul complex with polynomial generators $x_0,\ldots,x_{n-1}$ and exterior generators $\xi_0,\ldots,\xi_{n-1}$. We denote this complex as W_n and refer to it as the GOR complex. It is endowed with a differential d_2 such that $d_2(\xi_k)=\sum_{i=0}^{k}x_ix_{k-i}$ and $d_2(x_k)=0$.

Gorsky and Lewark [GL15] conjectured that the stable $\mathfrak{sl}(N)$ homology of torus knots can be described as a Koszul complex that is similar to the GOR complex but has modified differentials. In this paper, we study the homology of these Koszul complexes. In addition, we consider analogous Koszul complexes corresponding to the stable Lee homology [Lee05] of torus knots, which assigns a bigraded module over univariate polynomials with rational coefficients to each oriented link. We denote these as V_2^N — they are constructed similarly to the GOR complex, except that we have an additional polynomial generator T with $d_2(\xi_0)=x_0^2-T$ (all other differentials remain identical). We also investigate Koszul complexes V_n^N corresponding to $\mathfrak{sl}(N)$ analogues of Lee homology that we refer to as deformed $\mathfrak{sl}(N)$ homology.

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Several authors have investigated the GOR conjecture and analogues thereof. Notably, Hogancamp [Hog14; Hog18] made significant progress towards proving the GOR conjecture. Hogancamp and Mellit determined the Khovanov-Rozansky triply-graded homology of all torus knots in 2019 [HM19], and their analyses could prove useful in proving the GOR conjecture. Despite decades of progress on the GOR conjecture, much work remains in determining the homology of the GOR complex itself — which is precisely the focus of this paper. Despite the algebraic nature of the GOR complex, it is difficult to compute its homologies — though it is simpler than computing the homology of torus knots. The situation is similar for $\mathfrak{sl}(N)$ and deformed $\mathfrak{sl}(N)$ homologies. Better understanding the homology of the GOR complex and its analogues would not only shed light on their fundamental structure but may help prove the GOR conjecture itself.

The GOR complex W_n^N (in $\mathfrak{sl}(N)$ homology) decomposes into a sequence of chain complexes $C_0, C_1, \dots$ where C_i consists of all elements $c \in W_n^N$ with L-degree i (L-degree is the sum of the subscripts of the x_i and ξ_i). Computational evidence suggests the homology of C_L, for L beyond $L_{crit} = \binom{n}{2}$, can be determined from the homologies of C_i with $i < L$. We also make available a computer program that computes the homology of any particular C_i.

1.1. Conjecture on the L-homogeneous summands of W_n.

Formally, our conjecture — which we call the PIE conjecture for its relation to the inclusion-exclusion principle — states the following for regular $\mathfrak{sl}(N)$ homology.

Conjecture 6 (PIE conjecture)**.** *Fix an $N \geq 2$, $n \geq 1$, and $L > L_{crit} = \binom{n}{2}$. For each subset $K \subset \{0, 1, \dots, n-1\}$, let $\mathcal{C}_K := C_{L - \sum_{i \in K} i} \subset W_n^N$. Then there exists an isomorphism of abelian groups*

$$\bigoplus_{\substack{K \subseteq \{0,1,\dots,n-1\}, \\ |K| \text{ even}}} H_\bullet(\mathcal{C}_K) \cong \bigoplus_{\substack{K \subseteq \{0,1,\dots,n-1\}, \\ |K| \text{ odd}}} H_\bullet(\mathcal{C}_K).$$

The relevance of the inclusion-exclusion principle is the following. Suppose $S_0, \dots, S_{n-1}$ are finite sets. Then the inclusion-exclusion principle states

$$\sum_{|K| \text{ even}} \left| \bigcap_{i \in K} S_i \right| = \sum_{|K| \text{ odd}} \left| \bigcap_{i \in K} S_i \right|,$$

where the sums are over subsets $K \subseteq \{0, \dots, n-1\}$ of sizes of a given parity.

We verify conjecture 6 for W_2^N in $\mathfrak{sl}(N)$ homology and W_3 in Lee homology via propositions 7, 8, and 9. We also follow a more conceptual path towards developing and understanding these conjectures in terms of exact sequences in sections 5 and 6.

1.2. Maximal Torsion Order.

In deformed $\mathfrak{sl}(N)$ homology, the homology of each V_n^N is isomorphic, as a $\mathbb{Q}[x_0]$-module, to the direct sum of $\mathbb{Q}[x_0]$ and some number of copies of $\frac{\mathbb{Q}[x_0]}{x_0^k}$ for various k. In computationally testing the first conjecture, we noticed that there appear to be limits to the order of torsion modules in deformed $\mathfrak{sl}(N)$ homology — in particular, we propose the following conjecture.

Conjecture 10. *If $\frac{\mathbb{Q}[x_0]}{x_0^k}$ is a direct summand of the homology of V_n^N, then K is at most N.*

For deformed $\mathfrak{sl}(N)$ homology, we show that this value of k is obtained by $\frac{\mathbb{Q}[x_0]}{x_0^N} \cdot x_2 \in H_\bullet(V_{n\geq 3})$ (all torus knots with more than 3 strands).

2. Background

2.1. **The GOR Complex.** Gorsky, Oblomkov, and Rasmussen [GOR13] defined a differential graded algebra, which we denote W_n^2, by $W_n^2 := \mathbb{Z}[x_0, \ldots, x_{n-1}] \otimes \Lambda(\xi_0, \xi_1, \ldots, \xi_{n-1})$, where Λ denotes the exterior algebra, with multiplicative bigrading $\deg(x_i) = q^{2i+2}h^{2i}$ and $\deg(\xi_i) = q^{2i+4}h^{2i+1}$ so that the differential d_2 is 0 when applied to terms without any ξ_i and $d_2(\xi_i) := \sum_{k=0}^{i} x_k x_{i-k}$ otherwise. In this paper, the a-degree of a monomial denotes the number of ξ's in that monomial.

2.2. **Extension to $\mathfrak{sl}(N)$ Homology.** In their 2015 paper, Gorsky and Lewark [GL15] defined analogues of the original GOR complex that we denote W_n^N. The complex $W_n^N := \mathbb{Z}[x_0, \ldots, x_{n-1}] \otimes \Lambda(\xi_0, \xi_1, \ldots, \xi_{n-1})$ is endowed with differential d_N defined by $d_N(\xi_i) = \sum_{k_1+k_2+\cdots+k_N=i} x_{k_1} \cdots x_{k_N}$ and $d_N(x_i) = 0$. In W_n^N, ξ_i have bigradings $q^{2i+2N+2}h^{2i+1}$, while x_i have bigradings $q^{2i+2}h^{2i}$. We again define the a-degree of a monomial to be the number of ξ's in that monomial. The complexes W_n^N are conjecturally isomorphic to the stable $\mathfrak{sl}(N)$ homology of $T(n, \infty)$.

Lemma 1. *The complex W_n^N is homotopy equivalent to the complex $\mathbb{Z}[x_0, \ldots, x_{n-1}]/x_0^N \oplus \Lambda(\xi_1, \ldots, \xi_{n-1})$ endowed with differential $d_2(x_i) = 0$ and $d_2(\xi_i) = \sum_{k=0}^{i} x_k x_{i-k}$. In other words, W_n^N is homotopy equivalent to the GOR complex except with ξ_0 omitted and with $\mathbb{Z}[x_0]$ replaced by $\mathbb{Z}[x_0]/x_0^N$.*

Proof. The complex W_n^N may be viewed as the mapping cone

$$A\xi_0 \xrightarrow{x_0^N} A,$$

where A is the chain complex $\mathbb{Z}[x_0, \ldots, x_{n-1}] \otimes \Lambda(\xi_1, \ldots, \xi_{n-1})$ equipped with d_2 defined above. Because multiplication by x_0^N is injective on A, Gaussian elimination [Bar06] shows that W_n^N is homotopy equivalent to the complex $\frac{A}{x_0^N}$ — precisely the one described in the statement of the lemma. □

3. Extending the GOR Conjecture to Lee and Deformed $\mathfrak{sl}(N)$ Homologies

The **Lee homology** [Lee05] of an oriented link takes the form of a bigraded module over the polynomial ring $\mathbb{Q}[T]$. It may be viewed as a deformation of Khovanov homology. When bigradings are ignored, the Lee homology of a knot is isomorphic to $\mathbb{Q}[T] \oplus \mathbb{Q}[T] \oplus \bigoplus_{i=1}^{m} \frac{\mathbb{Q}[T]}{T^{n_i}}$ for some nonnegative integers n_i [Lee05]. Of major interest in the study of knot homologies is the value of each n_i: Manolesu and Marengon [MM20] found a knot with one n_i equal to 2, but no higher n_i have been found.

Although the stable Lee homology of torus knots arises naturally as an extension of the GOR conjecture — and while others have likely posited the question before — no authors have previously formally defined a Lee analogue of the GOR conjecture. Therefore, this section is dedicated to making an analogue of the GOR conjecture for both Lee homology and deformed $SL(N)$ homology.

We consider an analogue to GOR conjecture for Lee homology. Let V_n^N be the differential graded algebra $\mathbb{Q}[T, x_0, \ldots, x_{n-1}] \otimes \Lambda(\xi_0, \ldots, \xi_{n-1})$ equipped with the differential d_2 satisfying

$d_2(\xi_0) = x_0^2 - T$ and $d_2(\xi_{k\geq 1}) = \sum_{i=0}^{k} x_i x_{k-i}$, where the q degree of each x_i and ξ_i is $2i$ and the a degree of x_i and ξ_i is 0 and 1 respectively. The following lemma shows that, up to homotopy, we may eliminate ξ_0 and T.

Lemma 2. *The complex V_n^2 is homotopy equivalent to the complex $\mathbb{Q}[x_0,\ldots,x_{n-1}]\otimes\Lambda(\xi_1,\ldots,\xi_{n-1})$ endowed with differential $d_2(x_i)=0$ and $d_2(\xi_i)=\sum_{k=0}^{i} x_k x_{i-k}$. In other words, V_n^2 is homotopy equivalent to the GOR complex except with ξ_0 omitted.*

Proof. The complex V_n^2 may be viewed as the mapping cone

$$A\xi_0 \xrightarrow{x_0^2-T} A,$$

where A is the chain complex $\mathbb{Q}[x_0,\ldots,x_{n-1}]\otimes\Lambda(\xi_1,\ldots,\xi_{n-1})$ equipped with d_2 defined above. The rest of the proof follows reasoning analogous to that of lemma 1. □

We can extend this Lee homology version of the GOR complex to what is the $\mathfrak{sl}(N)$ version of Lee homology — more concisely described as deformed $\mathfrak{sl}(N)$ homology. We denote W_n^{Lee} in deformed $\mathfrak{sl}(N)$ homology as V_n^N, which is the differential graded algebra $\mathbb{Q}[x_0,\ldots,x_{n-1}]\otimes\Lambda(\xi_1,\ldots,\xi_{n-1})$ with differential d_N defined by $d_N(x_i) \coloneqq 0$ and $d_N(\xi_{i\geq 1}) = \sum_{k_1+k_2+\cdots+k_N=i} x_{k_1}\cdots x_{k_N}$. This definition specializes to that of the GOR complex of Lee homology when $N=2$. We define a grading L in Lee and deformed $\mathfrak{sl}(N)$ homology such that the L-degree of x_i and ξ_i are both i.

4. Decomposition of W_n

4.1. **Definition of subcomplexes.** For $\mathfrak{sl}(N)$ homology, the differential d_N reduces a-degree by 1 but does not affect L degree. The same is true for V_n^N. An immediate corollary is that we can write every W_n^N and V_n^N as the direct sum of a countably infinite number of subcomplexes $C_0, C_1, \ldots,$ where C_i contains every $c\in W_n$ with L-degree i. We can perform a similar decomposition for deformed $\mathfrak{sl}(N)$ homology.

4.2. **Elements of C_L.** We can write C_L as a direct sum via a-degree. For concreteness, the following is a basis over $\mathbb{Z}$ of the set of elements of the a-degree k part of C_i in W_n^N:

$$\left\{ \prod_{1\leq j\leq n-1} x_j^{\alpha_j} \prod_{1\leq j\leq n-1} \xi_j^{\beta_j} : \sum_j (j+1)\alpha_j + (j+N-1)\beta_j = i \text{ and } \beta_j = 0,1 \right\}.$$

4.3. **Critical values of L.**

Definition 3 (The critical value of L: L_{crit})**.** We denote by L_{crit} the minimum value of L such that $C_L \subset W_n^N$ or V_n^N has an element that contains the product of all ξ_i's. For all homologies, $L_{crit} = \binom{n}{2}$.

Remark 4. Every element e in $C_{L>L_{crit}}$ can be written as Xe', where X is a monomial in x_i's and e' is an element in $C_{L'\leq L_{crit}}$.

5. Chain complexes in C_i

5.1. **Canonical basis elements of the sub-complexes C_L.** The sub-complexes C_L have some intriguing properties that allow us to generate exact sequences in C_i and contemplate inter-degree relationships of homology. In particular, if $L \geq L_{crit}$, then every possible multinomial Ξ in ξ_i's forms part of at least one basis element in C_L. Each of these basis elements also contains a multinomial in X of x_i's — which is equal to 1 if and only if $L = L_{crit}$ and only for the basis element containing the product of all ξ_i's. This implies that for $L > L_{crit}$, every basis element can be expressed as $X\Xi$ where $X \neq 1$ and X, Ξ are multinomials in the even and odd generators.

For any arbitrary $X\Xi \in C_L$, there is some $x_i^{j_i}$ with $j_i > 0$ that appears in X. Therefore, we can construct $X'\Xi = \frac{X\Xi}{x_i}$ with q-degree $2L - 2i$. This construction is a basis element of the complex C_{L-i}.

Now let $\mathcal{B}_L$ denote the canonical basis for $C_L \subset W_n^N$ or V_n^N. We see that $\mathcal{B}_L \subset x_1\mathcal{B}_{L-1} \cup x_2\mathcal{B}_{L-2} \cup \cdots \cup x_{n-1}\mathcal{B}_{L-n+1} = \bigcup_{m=L-n+1}^{L+1} \mathcal{B}_m$ (where multiplying a set by c multiplies all elements of that set by c), but this is not a perfect description of $\mathcal{B}_L$ because it fails to account for multiplicity: a basis element of the form $x_i x_j X \in \mathcal{B}_L$ (where $i \neq j$ and X is a monomial in x_i's) is in both $x_i\mathcal{B}_{L-i}$ and $x_j\mathcal{B}_{L-j}$. To remedy our overcounting, we can direct sum $x_i x_j \mathcal{B}_{L-i-j}$ (for all $1 \leq i \leq j \leq n$) to $\mathcal{B}_L$. However, we must then contend with undercounting basis elements of the form $x_i x_j x_k X$ by direct summing these basis elements with $\bigcup_{m=L-n+1}^{L+1} \mathcal{B}_m$. We continue this pattern of direct summation until we finally add $\prod_{1 \leq i < n} x_i C_{L-1-2-\cdots-(n-1)}$ to one of $\mathcal{B}_L$ or $\bigcup_{m=L-n+1}^{L+1} \mathcal{B}_m$ — this represents our application of the inclusion-exclusion principle to inductively describe the canonical basis $\mathcal{B}_L$ via the canonical bases $\mathcal{B}_{L'<L}$.

For a concrete illustration of this version of the inclusion-exclusion principle, consider C_3 in W_2 of Khovanov ($\mathfrak{sl}(2)$) homology. A basis for this subcomplex — when expressed as $\mathbb{Z}[x_0]$-modules — is

$$\begin{aligned}\mathcal{B}_4 = \{&x_1\xi_1\xi_2,\\ &x_1^3\xi_1, x_1x_2\xi_1, x_1^2\xi_2, x_2\xi_2,\\ &x_1^4, x_1^2x_2, x_2^2\}\end{aligned}$$

We can similarly find bases for C_1, C_2, and C_3:

$$\begin{aligned}\mathcal{B}_1 &= \{\xi_1, x_1\},\\ \mathcal{B}_2 &= \{x_1\xi_1, \xi_2, x_1^2, x_2\}, \text{ and}\\ \mathcal{B}_3 &= \{\xi_1\xi_2, x_1^2\xi_1, x_2\xi_1, x_1\xi_2, x_1^3, x_1x_2\}.\end{aligned}$$

We see that $\mathcal{B}_4 = (x_1\mathcal{B}_3 \cup x_2\mathcal{B}_2) \setminus (x_1x_2\mathcal{B}_1)$. This pattern holds for all $C_{L>L_{crit}}$.

6. Inductive Conjecture of homology of L-homogeneous summands (The PIE Conjecture)

6.1. **Exact sequences of L-homogeneous summands.** Given N, n, and L (representing the L-degree summands of W_n^N or V_n^N) and some set $K \subseteq \{0, 1, \ldots, n-1\}$, define $\mathcal{C}_K \coloneqq C_{L-|K|-\sum_{i\in K} i}$.

Let α_n be the chain complex

$$\bigoplus_{\substack{K\subseteq\{0,\dots,n-1\},\\|K|=n}} \mathcal{C}_K \xrightarrow{M} \bigoplus_{\substack{K\subseteq\{0,\dots,n-2\},\\|K|=n-1}} \mathcal{C}_K \xrightarrow{M} \dots \xrightarrow{M} \bigoplus_{\substack{K\subseteq\{0,\dots,n-1\},\\|K|=0}} \mathcal{C}_K,$$

where the horizontal map $M_{K,K'} : \mathcal{C}_K \to \mathcal{C}_{K'}$ is defined component-wise as

$$M_{K,K'} \coloneqq \begin{cases} (-1)^{\#\{j\in K\,:\,j<i\}}x_i & \text{if } i \notin K' \text{ and } K = K' \cup \{i\} \\ 0 & \text{otherwise} \end{cases}.$$

For example, α_2 is

$$\begin{tikzcd} & C_{\{1\}} \arrow[dr, "-x_1"] & \\ C_{\{0,1\}} \arrow[ur, "x_0"] \arrow[dr, "x_1"'] & & C_{\emptyset} \\ & C_{\{0\}} \arrow[ur, "x_0"'] & \end{tikzcd}.$$

When necessary for clarity, we let $\alpha_n(L)$ be the chain complex whose rightmost term is C_L. For certain values of L, proposition 5 states that these sequences are exact.

Proposition 5. *All $\alpha_n(L)$ for $L > L_{crit}$ are exact sequences.*

Proof. Set $C_{L<0} \coloneqq 0$ and consider $\bigoplus_{L>0} \alpha_n(L)$. Because of how W_n^N and V_n^N are defined and because $\bigoplus_{i\geq 0} C_L = W_n$, we have

$$\bigoplus_{L\geq 0} \alpha_n(L) = W_n \xrightarrow{\bigoplus_{L>0} M_n(L)} \bigoplus_{i\in S(n-1)} W_n \to \cdots \to \bigoplus_{i\in S(1)} C_{L-i} \xrightarrow{\bigoplus_{L>0} M_1(L)} W_n = K_x \otimes \Lambda[\xi_1, \dots, \xi_n],$$

where K_x denotes the Koszul complex defined by $(\mathbb{Q}[x_0, x_1, \dots, x_n], x \coloneqq x_1, \dots, x_n)$. Because x is a regular sequence, it is well known that the homology of K_x is only supported in the rightmost terms. We see that $H_0(K_x) = \frac{\mathbb{Q}[x_0,x_1,\dots,x_n]}{(x_1,\dots,x_n)} = \mathbb{Q}[x_0]$; therefore, $H_0\left(\bigoplus_{L\geq 0} \alpha_n(L)\right) = \mathbb{Q}[x_0] \otimes \Lambda[\xi_1, \dots, \xi_n]$.

Given an $L \leq L_{crit}$ and a subset $\{i_1, i_2, \dots, i_{max}\} \subseteq \{1, \dots, n-1\}$, we have that $\xi_{i_1}\xi_{i_2}\dots\xi_{i_{\max}} \in C_L$ but $\xi_{i_1}\xi_{i_2}\dots\xi_{i_{\max}} \notin C_{L'<L}$ whenever $\sum i_j = L$, implying $H_0(\alpha_n(L)) = \mathbb{Q}[x_0]\xi_{i_1}\xi_{i_2}\dots\xi_{i_{\max}}$ because the maps M_k only operate by monomials in x_i's. Therefore,

$$\bigoplus_{0\leq L\leq L_{crit}} H_0(\alpha_n(L)) = \mathbb{Q}[x_0] \bigoplus_{0\leq L\leq L_{crit}} \bigoplus_{\sum i_j = L} \xi_{i_1}\xi_{i_2}\dots\xi_{i_{\max}} = \mathbb{Q}[x_0] \otimes \Lambda[\xi_1, \dots, \xi_n].$$

Thus

$$H_\bullet\left(\bigoplus_{L\leq L_{crit}} \alpha_L\right) = H_0\left(\bigoplus_{L\geq 0} \alpha_n(L)\right),$$

as desired. One can consider complexes over $\mathbb{Z}[x_0]/x_0^N$ to recover a similar proof for regular $\mathfrak{sl}(N)$ homology. $\square$

6.2. **The PIE conjecture.** Because α_n are chain complexes and each term of α_n is itself a direct sum of chain complexes, we can take the homology of each term and consider the induced maps M_* on homology. In particular, we define α_n^* to be the sequence of induced maps

$$\bigoplus_{\substack{K\subseteq\{0,\dots,n-1\},\\ |K|=n}} H_\bullet(\mathcal{C}_K) \xrightarrow{M_*} \bigoplus_{\substack{K\subseteq\{0,\dots,n-1\},\\ |K|=n-1}} H_\bullet(\mathcal{C}_K) \xrightarrow{M_*} \dots \xrightarrow{M_*} \bigoplus_{\substack{K\subseteq\{0,\dots,n-1\},\\ |K|=0}} H_\bullet(\mathcal{C}_K).$$

The PIE conjecture proposes a simple way of relating the direct sums of homology terms across L-degrees.

Conjecture 6 (PIE conjecture)**.** *Given a specific $L > L_{crit}$, N, and n, define $\mathcal{C}_K \coloneqq C_{L+|K|+\sum_{i\in K} i}$. Then as $|K|$ ranges between 0 and $n-1$ inclusive, we have*

$$\bigoplus_{\substack{K\subseteq\{0,1,\dots,n-1\},\\ |K| \textit{ even}}} H_\bullet(\mathcal{C}_K) \cong \bigoplus_{\substack{K\subseteq\{0,1,\dots,n-1\},\\ |K| \textit{ odd}}} H_\bullet(\mathcal{C}_K).$$

This isomorphism respects a-gradings and can be made to respect q-gradings if a shift is applied to each $\mathcal{C}_K$.

For example, in W_3^N and V_3^N, the conjecture states

$$H_\bullet(C_L) \oplus H_\bullet(C_{L-3}) \cong H_\bullet(C_{L-2}) \oplus H_\bullet(C_{L-1}).$$

Meanwhile, in W_4^N and V_4^N, we have

$$H_\bullet(C_L) \bigoplus \begin{matrix} H_\bullet(C_{L-3}) \\ \oplus \\ H_\bullet(C_{L-4}) \\ \oplus \\ H_\bullet(C_{L-5}) \end{matrix} \quad\cong\quad \begin{matrix} H_\bullet(C_{L-1}) \\ \oplus \\ H_\bullet(C_{L-2}) \\ \oplus \\ H_\bullet(C_{L-3}) \end{matrix} \bigoplus H_\bullet(C_{L-6}).$$

6.3. **Illustrating the PIE conjecture with α_3.** A quick check reveals that the short exact sequence α_3 induces a long exact sequence on homology when $L > L_{crit}$:

$$\begin{tikzcd}
H_2(C_{L-3}) \arrow[r, "f_2"] & H_2(C_{L-2}) \oplus H_2(C_{L-1}) \arrow[r, "g_2"] & H_2(C_L) \arrow[dll, "\partial_2"'] \\
H_1(C_{L-3}) \arrow[r, "f_1"'] & H_1(C_{L-2}) \oplus H_1(C_{L-1}) \arrow[r, "g_1"] & H_1(C_L) \arrow[dll, "\partial_1"'] \\
H_0(C_{L-3}) \arrow[r, "f_0"'] & H_0(C_{L-2}) \oplus H_0(C_{L-1}) \arrow[r, "g_0"] & H_0(C_L)
\end{tikzcd}.$$

Showing that the connecting homomorphisms ∂_2 and ∂_1 are 0 — and then that each layer is split exact — would suffice to prove our conjecture on homology in the case of $n = 3$. For $n \geq 3$, however, α_n is no longer a short exact sequence, and we need other tools to deduce relationships between $H_\bullet(C_L)$.

7. Automated Computation of Homology

We can automate the procedure shown in Section 4 to systematically generate each sub-complex C_L and find its homology. We represent the complex via a monomial basis, apply Gaussian elimination [Bar06], represent the maps between layers as matrices, and use their Smith normal forms to compute homology. Tables 1(A) and 1(B) detail to what extent the inductive conjecture on C_L was tested. The conjecture withstood all computational checks.

$\mathfrak{sl}(N)$	W_3	W_4	W_5	W_6	W_7	W_8
2	85_3	80_5	75_7	70_9	65_{11}	60_{13}
3	80_6	75_9	70_{12}	65_{15}	60_{18}	
4	75_{10}	70_{14}	65_{18}	60_{22}		
5	70_{15}	65_{20}	60_{25}			
6	65_{21}	60_{27}	55_{33}			
7	60_{28}	55_{35}				
8	55_{36}	50_{44}				

(A) Regular $\mathfrak{sl}(N)$ homology.

$\mathfrak{sl}(N)$	V_3	V_4	V_5	V_6	V_7
2	∞_3	65_6	55_{10}	45_{15}	35_{21}
3	60_3	55_6	45_{10}	35_{15}	30_{21}
4	55_3	45_6	35_{10}	30_{15}	
5	50_3	40_6	30_{10}	25_{15}	
6	45_3	35_6	30_{10}		
7	40_3	30_6	20_{10}		
8	35_3	25_6	20_{10}		
9	30_3	25_6	20_{10}		

(B) Deformed $\mathfrak{sl}(N)$ homology.

TABLE 1. The maximum value of L for which the inductive conjecture was tested for W_n or V_n (columns) in a given $\mathfrak{sl}(N)$ homology (rows), where ∞ denotes complexes whose homology is fully described by propositions 8 and 9. Subscripts denote the value of L_{crit} in a particular W_n^N or V_n^N.

As an example of a non-trivial check performed by this program, consider $C_{39} \subset V_5^4$ in deformed $SL(4)$ homology. For V_5^4, our conjecture states that the direct sum of eight particular homology groups is isomorphic to the direct sum of eight other homology groups. But some homology groups appear on both sides, so conjecture 6 reduces to

$$H_\bullet(C_{39}) \oplus H_\bullet(C_{34}) \oplus H_\bullet(C_{34}) \oplus H_\bullet(C_{29}) \cong H_\bullet(C_{38}) \oplus H_\bullet(C_{37}) \oplus H_\bullet(C_{31}) \oplus H_\bullet(C_{30}).$$

for every a-degree of V_5^4. Table 2 shows that this holds true for elements with a-degree 0 (all elements with no ξ_i's) in $C_{39} \subset V_5^4$. (We omit calculated values for other a-degrees due to space).

$H_\bullet(C_L)$	Copies of $\frac{\mathbb{Q}[x_0]}{x_0}$	Copies of $\frac{\mathbb{Q}[x_0]}{x_0^2}$	Copies of $\frac{\mathbb{Q}[x_0]}{x_0^3}$	Copies of $\frac{\mathbb{Q}[x_0]}{x_0^4}$
$H_\bullet(C_{39})$	37	33	40	37
$H_\bullet(C_{38})$	30	37	33	40
$H_\bullet(C_{37})$	33	30	37	33
$H_\bullet(C_{34})$	24	30	27	33
$H_\bullet(C_{31})$	24	21	27	24
$H_\bullet(C_{30})$	19	24	21	27
$H_\bullet(C_{29})$	21	19	24	21

TABLE 2. An easily parsable representation of the homology of the bottom layer (elements with no ξ_i) of $C_{39} \subset V_5^4$. The table shows that conjecture 6 is true in this specific case.

8. THE CASE OF W_2^N IN $\mathfrak{sl}(N)$ HOMOLOGY

The homology of W_2^N is well-known to experts, but we recompute it in this section for the reader's convenience.

Proposition 7. *The homology of $C_L \subset W_2^N$ is given by*

(1) 0 *in a-degree 2,*
(2) *a $\mathbb{Z}^{N-1}$ in a-degree 1,*
(3) *and a $\mathbb{Z}^{N-1} \oplus \mathbb{Z}/N$ in a-degree 0.*

Proof. The subcomplex $C_L \in W_2^N$ over $\mathbb{Z}[x_0]$ is as follows:

$$\begin{tikzcd} x_1^L \arrow[dr, "x_0^N"] & x_1^{L-1}\xi_1 \arrow[d, "Nx_0^{N-1}"] & x_1^{L-1}\xi_1\xi_1 \arrow[dl, "x_0^N"] \\ & x_1^L & \end{tikzcd}$$

By direct computation, we see that homology is 0 in a-degree 2, a $\mathbb{Z}^{N-1}$ in a-degree 1, and a $\mathbb{Z}^{N-1} \oplus \mathbb{Z}/N$ in a-degree 0. □

9. THE CASE OF W_3 IN LEE HOMOLOGY

We now extend the split exactness of α_2 to Lee homology.

Proposition 8. *Given $C_L \subset V_3^2$, if $L > 0$ is odd, then $H_\bullet(C_L)$ is isomorphic to $\frac{\mathbb{Q}[x_0]}{x_0}$ supported in a-degree 0.*

Proof. We have the following bases for a-degree 2, 1, and 0:

$$x_1^{2m-2}\xi_1\xi_2, x_1^{2m-4}x_2\xi_1\xi_2, \dots, x_2^{m-1}\xi_1\xi_2 \text{ in } a\text{-degree } 2,$$
$$x_1^{2m}\xi_1, \dots, x_2^{m}\xi_1, x_1^{2m-1}\xi_2, \dots, x_1x_2^{m-1}\xi_2 \text{ in } a\text{-degree } 1,$$
$$\text{and } x_1^{2m+1}, \dots, x_1x_2^{m} \text{ in } a\text{-degree } 0.$$

Because the differential satisfies $d_2(\xi_1) = 2x_0x_1$ and $d_2(\xi_2) = 2x_0x_2 + x_1^2$, Gaussian elimination allows us to cancel all m basis elements with a-degree 2 with the m basis elements $x_1^{2m}\xi_1, \ldots, x_1^2x_2^{m-1}\xi_1$. We can similarly cancel $x_1^{2m-1}\xi_2, \ldots, x_1x_2^{m-1}\xi_2$ with $x_1^{2m+1}, \ldots, x_1^3x_2^{m-1}$. We are left with $x_2^m\xi_1 \xrightarrow{2x_0} x_1x_2^m$. Hence the homology is $\frac{\mathbb{Q}[x_0]}{x_0}$ in a-degree 0. □

Proposition 9. *Given $C_L \subset V_3^2$, if $L > 0$ is even, then $H_\bullet(C_L)$ is isomorphic to $\frac{\mathbb{Q}[x_0]}{x_0^2}$ supported in a-degree 0.*

Proof. By a similar computation to that in the proof of proposition 8, C_L deformation retracts onto the subcomplex with four generators: $x_1x_2^m\xi_1, x_2^m\xi_2, x_1^2x_2^m, x_2^{m+1}$. By Gaussian elimination along $x_2^m\xi_2 \to x_1^2x_2^m$, we obtain a complex with a single generator in a-degree 1, a single generator in a-degree 0, and the differential is multiplication by $-4x_0^2$. Thus the homology is $\frac{\mathbb{Q}[x_0]}{x_0^2}$ supported in a-degree 0. □

10. Bounding Orders in $\mathfrak{sl}(N)$ homology

In computationally verifying Conjecture 6, we found that torsion orders in both deformed $\mathfrak{sl}(N)$ homology appear to take only specific values.

Conjecture 10. *If $\frac{\mathbb{Q}[x_0]}{x_0^m}$ is a direct summand of $H_\bullet(V_n^N)$, then m is less than or equal to N.*

Proposition 11. *If $n \geq 3$, then x_2 generates a copy of $\frac{\mathbb{Q}[x_0]}{x_0^N}$ in $H_\bullet(V_n^N)$.*

Proof. Note that $d_N\xi_1 = Nx_0^{N-1}x_1$ and $d_N\xi_2 = Nx_0^{N-1}x_2 + \binom{N}{2}x_0^{N-2}x_1^2$. Therefore, in L-degree 2, there are four generators over $\mathbb{Q}[x_0]$: $x_1^2, x_2, x_1\xi_1$, and ξ_2. By direct computation, x_2 generates a copy of $\frac{\mathbb{Q}[x_0]}{x_0^N}$ in homology. □

10.1. Proof of Maximal Order Conjecture for $a = 0$.

We prove conjecture 10 for terms with no ξ_i (i.e. terms with a-degree 0).

Lemma 12. *Consider the ideal $(d_N\xi_1, \ldots, d_N\xi_n)$ generated by the polynomials $d_N\xi_1, \ldots, d_n\xi_n$ within the polynomial ring $\mathbb{Q}[x_0, \ldots, x_n]$. Then $x_0^Nx_n$ is a member of this ideal.*

Proof. For each n-tuple of rational numbers $(a_0, \ldots, a_{n-1})$, let $P(a_0, \ldots, a_{n-1})$ be the polynomial

$$P(a_0, \ldots, a_{n-1}) \coloneqq a_0x_0d_N\xi_n + \cdots + a_{n-1}x_{n-1}d_N\xi_1.$$

Let the vector subspace V be the set of n-tuples $(a_0, \ldots, a_{n-1})$ such that if $v \in V$, then $P(v)$ is a scalar multiple of $x_0^Nx_n$.

Note that

$$d_n\xi_i = \sum_{j_1+j_2+\cdots+j_N=i} x_{j_1}\cdots x_{j_N} = \sum_{\substack{k_0+\cdots+k_{n-1}=N,\\ 0k_0+k_1+\cdots+Nk_{n-1}=i}} x_0^{k_0}\cdots x_{n-1}^{k_{n-1}}.$$

Therefore, $(a_0, \ldots, a_{n-1}) \in V$ if and only if for every monomial $X = x_0^{k_0}x_1^{k_1}\cdots x_{n-1}^{k_{n-1}} \neq x_0^Nx_n$ with $\sum ik_i = n$ and $\sum k_i = N+1$, the coefficient on X in $P(a_0, \ldots, a_{n-1})$ is 0.

If we set $a_i = \frac{1-Ni}{n}$ for $i = 0, \ldots, n-1$, then

$$\sum_{i=0}^{n-1} \frac{1-Ni}{n} b_0 = \sum_{i=0}^{n-1} b_i - \frac{N}{n} \sum_{i=0}^{n-1} i b_i = 0.$$

Therefore, $\left(1, 1-\frac{N}{n}, \ldots, 1-\frac{N(n-1)}{n}\right) \in A$, so $P\left(1, 1-\frac{N}{n}, \ldots, 1-\frac{N(n-1)}{n}\right)$ equals $N x_0^N x_n$. Thus $x_0^N x_n$ lies in the ideal.

□

Theorem 13. *In a-degree 0 of every $H_\bullet(V_n)$ for $n \geq 1$ in every deformed $\mathfrak{sl}(N)$ homology, the maximal possible torsion is x_0^N.*

Proof. Notice that, in the bottom layer, homology is given by $\frac{\mathbb{Q}[x_0, \ldots, x_{n-1}]}{(d_N \xi_1, d_N \xi_2, \ldots, d_N \xi_{n-1})}$. By Lemma 12, $x_0^N x_i = 0$ in this quotient for $i = 1, \ldots, n-1$, but 1 generates a copy of $\mathbb{Q}[x_0]$. Therefore, if $r \in \mathbb{Q}[x_0, \ldots, x_{n-1}]$ is a polynomial where the coefficient of x_0^k in r is 0 for all k, then $x_0^N r = 0$ in this quotient. □

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