Toric Ricci solitons in four dimensions

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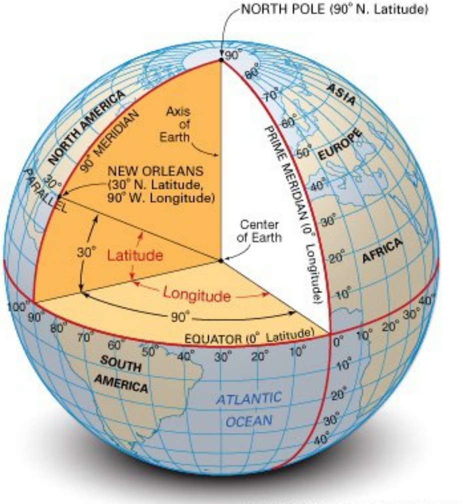
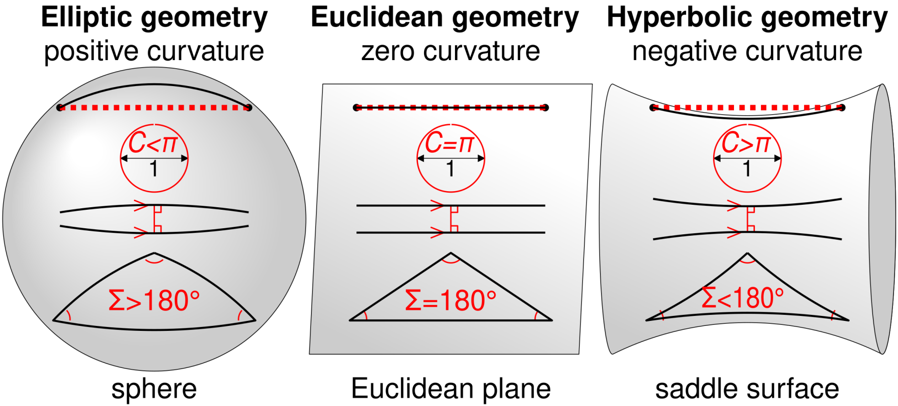
Contents

Contents

Spherical geometry



Comparison of geometries



Riemannian geometry

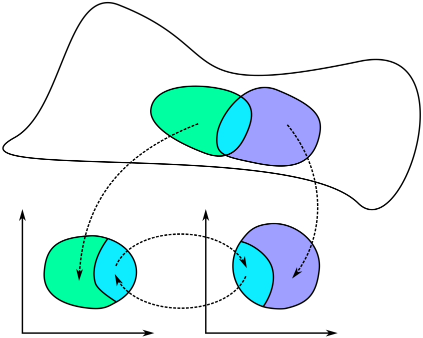
Riemannian metric: local notions of length, area, angle, and curvature

Why Riemannian geometry?

- Generalization of the differential geometry of surfaces embedded in Euclidean space
- Every smooth manifold admits a Riemannian metric
- General relativity: pseudo-Riemannian geometry

Smooth manifolds

Smooth manifold: topological space M with charts assigning coordinates to open subsets of M and smooth transition maps between charts



Smooth manifolds

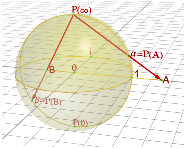
Example (stereographic projection)

The 2-sphere $\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ is a smooth manifold with charts

$$\phi_1(u, v) = \left(\frac{2u}{1 + u^2 + v^2}, \frac{2v}{1 + u^2 + v^2}, \frac{-1 + u^2 + v^2}{1 + u^2 + v^2}\right),$$

$$\phi_2(u, v) = \left(\frac{2u}{1 + u^2 + v^2}, \frac{2v}{1 + u^2 + v^2}, \frac{1 - u^2 - v^2}{1 + u^2 + v^2}\right),$$

which exclude the north pole $(0, 0, 1)$ and the south pole $(0, 0, -1)$ respectively.



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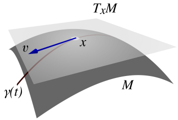
Riemannian metrics

Tangent vector: velocity of curves on a smooth manifold

Tangent space: the vector space $\mathrm{T}_x M$ of tangent vectors at a point x on a smooth manifold M

Riemannian metric: inner products for tangent spaces that vary smoothly from point to point



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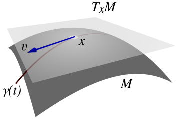
Riemannian metrics

Tangent vector: velocity of curves on a smooth manifold



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Tangent space: the vector space $\mathrm{T}_x M$ of tangent vectors at a point x on a smooth manifold M

Riemannian metric: inner products for tangent spaces that vary smoothly from point to point

For tangent vectors $u, v \in \mathrm{T}_x M$:

$$\|u\| = \sqrt{\langle u, u \rangle}, \qquad \angle(u, v) = \arccos \frac{\langle u, v \rangle}{\|u\| \|v\|}$$

Length of curve $\gamma(t)$:

$$\int_a^b \|\gamma'(t)\| \, dt$$

Riemannian metrics

Coordinates: using charts to describe a Riemannian manifold locally

Riemannian metrics

Coordinates: using charts to describe a Riemannian manifold locally

Standard basis: each tangent space $\mathrm{T}_x M$ has the standard basis

$$\partial_1 = (1, 0, \ldots, 0), \quad \partial_2 = (0, 1, \ldots, 0), \quad \ldots, \quad \partial_n = (0, 0, \ldots, 1)$$

Riemannian metrics

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$$\partial_1 = (1, 0, \dots, 0), \quad \partial_2 = (0, 1, \dots, 0), \quad \dots, \quad \partial_n = (0, 0, \dots, 1)$$

Metric coefficients: entries $g_{ij} = \langle \partial_i, \partial_j \rangle$ of the matrix

$$g = \begin{bmatrix} \langle \partial_1, \partial_1 \rangle & \cdots & \langle \partial_1, \partial_n \rangle \\ \vdots & & \vdots \\ \langle \partial_n, \partial_1 \rangle & \cdots & \langle \partial_n, \partial_n \rangle \end{bmatrix},$$

which uniquely determine the metric.

Riemannian metrics

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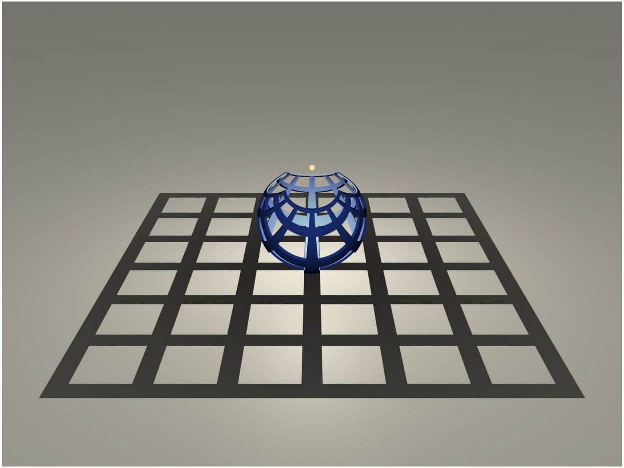
which uniquely determine the metric.

We often use the notation

$$g = \sum_{i,j} g_{ij} dx^i dx^j.$$

Example: stereographic projection



Mathematical Etudes, Steklov Mathematical Institute of Russian Academy of Sciences

Example: stereographic projection

The aforementioned stereographic projection

$$\phi(u,v) = \left(\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{-1+u^2+v^2}{1+u^2+v^2}\right)$$

of the 2-sphere $\mathbb{S}^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1\}$ minus the north pole $(0,0,1)$ has metric

$$g = \frac{4}{(1+u^2+v^2)^2}(du^2+dv^2).$$

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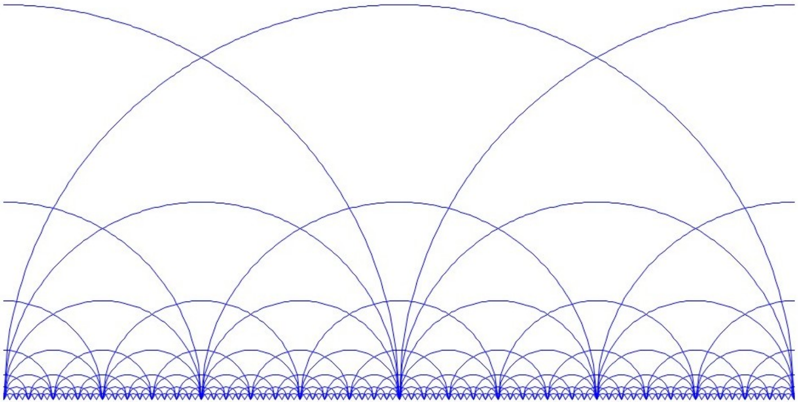
of the 2-sphere $\mathbb{S}^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2=1\}$ minus the north pole $(0,0,1)$ has metric

$$g = \frac{4}{(1+u^2+v^2)^2}(du^2+dv^2).$$

This means that the standard basis $\partial_u = \partial\phi/\partial u$, $\partial_v = \partial\phi/\partial v$ satisfies

- $\|\partial_u\| = \|\partial_v\| = \dfrac{2}{1+u^2+v^2}$; and
- $\langle \partial_u, \partial_v \rangle = 0$, so $\partial_u \perp \partial_v$.

Example: the Poincaré half-plane model



ThatsMaths

Example: the Poincaré half-plane model

The Poincaré half-plane model is a chart for the hyperbolic plane $\mathbb{H}^2$ using the upper half-plane $\{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ with metric

$$g = \frac{dx^2 + dy^2}{y^2}.$$

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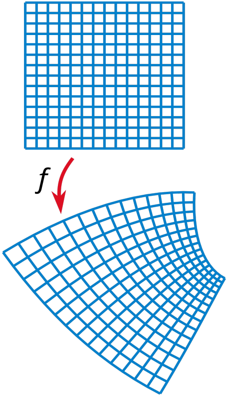
This means that the standard basis ∂_x, ∂_y satisfies

- $\|\partial_x\| = \|\partial_y\| = \dfrac{1}{y}$; and
- $\langle \partial_x, \partial_y \rangle = 0$, so $\partial_x \perp \partial_y$.

Conformal maps



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Conformal map: a map that locally preserves angles

Conformal equivalence: two metrics g and $\tilde{g}$ satisfying

$$\tilde{g} = fg$$

for some scalar function f (conformal factor)

Conformal maps

Examples (conformally Euclidean metrics)

Stereographic projection of the 2-sphere

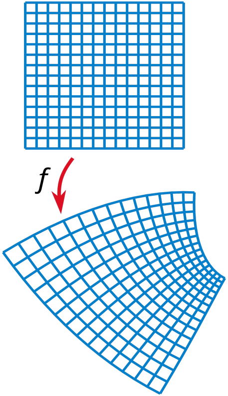
$$g = \frac{4}{(1+x^2+y^2)^2}(dx^2+dy^2)$$

and the Poincaré half-plane model

$$g = \frac{dx^2+dy^2}{y^2}$$

are both conformal to the Euclidean metric

$$g = dx^2+dy^2.$$



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Contents

Curvature

In Euclidean space, we have

$$\frac{\partial^2 f}{\partial x \, \partial y}(x, y) = \frac{\partial^2 f}{\partial y \, \partial x}(x, y)$$

if $f \colon \mathbb{R}^2 \to \mathbb{R}$ has continuous second partial derivatives near (x, y).

Curvature: failure of second (covariant) derivatives to commute

Curvature

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$$\frac{\partial^2 f}{\partial x \partial y}(x, y) = \frac{\partial^2 f}{\partial y \partial x}(x, y)$$

if $f : \mathbb{R}^2 \to \mathbb{R}$ has continuous second partial derivatives near (x, y).

Curvature: failure of second (covariant) derivatives to commute

Curvature tensor: full measure of curvature, analogous to the Hessian

Ricci tensor: trace of the curvature tensor, analogous to the Laplacian

The Ricci flow

Under suitable coordinates, the Ricci tensor $\mathrm{Ric}(g)$ is locally given by

$$\mathrm{Ric}(g) = -\frac{1}{2}\Delta g + \text{(lower-order terms)}$$

where Δ is the Laplace–Beltrami operator.

The Ricci flow

Under suitable coordinates, the Ricci tensor $\mathrm{Ric}(g)$ is locally given by

$$\mathrm{Ric}(g) = -\frac{1}{2}\Delta g + \text{(lower-order terms)}$$

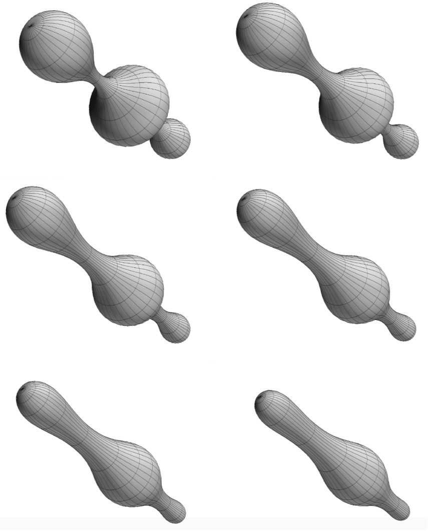
where Δ is the Laplace–Beltrami operator.

Ricci flow:

$$\frac{\partial g}{\partial t} = -2\,\mathrm{Ric}(g),$$

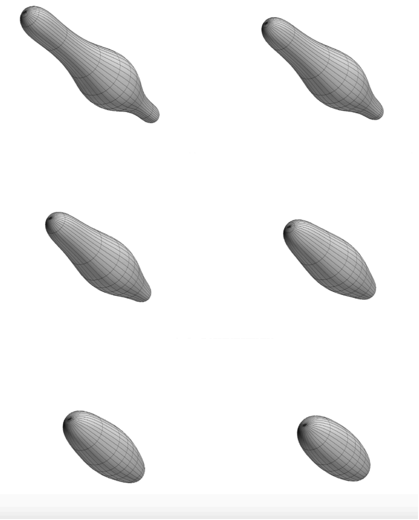
widely used to find and classify canonical metrics in differential geometry and general relativity.

The Ricci flow



J. Hyam Rubinstein, Robert Sinclair, Experimental Mathematics

The Ricci flow



J. Hyam Rubinstein, Robert Sinclair, Experimental Mathematics

The Poincaré conjecture

Theorem (Poincaré conjecture)

Let M be a compact three-dimensional topological manifold. If every simple closed curve in M can be deformed continuously to a point, then M is homeomorphic to the 3-sphere.

The Poincaré conjecture

Theorem (Poincaré conjecture)

Let M be a compact three-dimensional topological manifold. If every simple closed curve in M can be deformed continuously to a point, then M is homeomorphic to the 3-sphere.

- Conjectured by Henri Poincaré in 1904
- One of the Clay Mathematics Institute's seven Millennium Prize Problems
- Resolved by Grigori Perelman in 2002–2003, who proved the stronger Thurston's geometrization conjecture based on Richard S. Hamilton's work on the Ricci flow

Ricci solitons

Ricci solitons: self-similar solutions to the Ricci flow

Ricci solitons model the formation of singularities in the Ricci flow.

- Classified in ≤ 3 dimensions
- Complicated singularity behaviors in ≥ 4 dimensions

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Einstein metrics: $\mathrm{Ric}(g) = \lambda g$ for some constant λ

Einstein metrics are the vacuum solutions to the Einstein field equations in the theory of general relativity.

Examples (Einstein metrics)

- Euclidean space, the n-sphere, hyperbolic space
- Every two-dimensional manifold admits an Einstein metric

Contents

Toric metrics

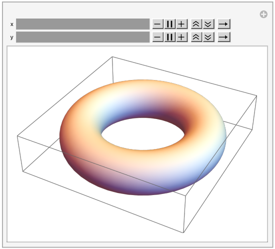
Toric metric: metric that admits an isometric torus action

Under suitable coordinates, a four-dimensional toric metric takes the form

$$g = g_{\text{base}} \oplus g_{\text{torus}}$$

where

- g_{base} is a metric in the base directions x and y;
- g_{torus} is a metric in the toric directions s and t; and
- the coefficients of g depend on x and y but not s or t.



Mathematica

Toric Ricci solitons

I studied toric Ricci solitons of the form

$$g = \frac{1}{q(x,y)^2}\left(\frac{dx^2}{A(x)} + \frac{dy^2}{B(y)} + A(x)\,ds^2 + B(y)\,dt^2 + 2\sqrt{A(x)B(y)}\cos\theta(x,y)\,ds\,dt\right). \quad (\star)$$

This is a generalization of a class of four-dimensional toric Ricci solitons that Firester and Tsiamis studied in 2024.

Rigidity result

Theorem (October 2024)

If $0 < \theta < \pi/2$ is a constant, the non-axisymmetric metric $(\star)$ is a Ricci soliton if and only if A and B are constants and there are functions $S^x(x, y)$ and $S^y(x, y)$ satisfying the system of equations

$$\partial_y S^x + \partial_x S^y = 4q_x q_y, \qquad \partial_x S^x = 2q_x^2, \qquad \partial_y S^y = 2q_y^2,$$

and

$$Aq_{xx} + Bq_{yy} - \frac{Aq_x^2 + Bq_y^2}{q} - \frac{Aq_x S^x + Bq_y S^y}{q^2} = \frac{\lambda}{q}.$$

Description of Einstein metrics

This leads to an explicit description of Einstein metrics of the form $(\star)$.

Corollary (October 2024)

If $0 < \theta < \pi/2$ is a constant, the non-axisymmetric metric $(\star)$ is an Einstein metric if and only if A and B are constants and $q(x, y) = ax + by + c$ for some constants a, b, c. In this case, we have $\lambda = -3(a^2A + b^2B)$ in the Einstein metric equation $\mathrm{Ric}(g) = \lambda g$.

Classification of Ricci solitons

Corollary (October 2024)

If $0 < \theta < \pi/2$ is a constant, a Ricci soliton of the form $(\star)$ where q is a homogeneous function is one of

1. *an Einstein metric as described on the previous page;*
2. *a product of two Ricci soliton surface metrics with the same constant λ defined in the Ricci soliton equation; or*
3. *a product of a flat surface with an Einstein surface, given by*

$$g = \frac{1}{f(x)^2}\left(\frac{dx^2}{A} + \frac{dy^2}{B} + A\,ds^2 + B\,dt^2\right)$$

up to translating, rescaling, and swapping x and y, where $f \in \{\cosh, \sinh, \sin\}$, and A and B are constants.

Acknowledgments

- Mentor: Benjy Firester (MIT)
- MIT PRIMES program
- Raphael Tsiamis

Thanks for your attention!

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