General Representation Theory and Representations of Finite Groups

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Outline

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Algebra

Definition

An associative algebra A is a vector space over a field k with an associative bilinear multiplication

$$\cdot : A \times A \to A.$$

Furthermore, we say that A is unital if A has an element 1 such that

$$a \cdot 1 = 1 \cdot a = a, \quad \forall a \in A$$

We will only work with unital algebras.

Examples

- The **matrix algebra** $Mat_n(k)$ over k with basis $E_{ij}, 1 \leq i, j \leq n$, such that $E_{ij} \cdot E_{kl} = \delta_{jk} E_{il}$ ($\delta_{jk} = 1$ if $j = k$ and zero otherwise).
- The **free algebra** $A = k\langle x_1, \ldots, x_n \rangle$ has a basis of words in letters $x_1, \ldots, x_n$. The product of two words is given by concatenation.

Representation

Definition

A finite dimensional representation of an associative algebra A is a finite dimensional vector space V with homomorphism of algebras

$$\rho : A \to \operatorname{End} V.$$

In other words, $\rho(*)$ is a k-linear map that preserves multiplication and unit.

Examples

- $V = A$, for $\rho : A \to EndA$, $\rho(a)$ is the operator of left multiplication by a. This is known as the **regular** representation.

Definition

A subrepresentation U of V of an algebra A is a vector subspace $U \subset V$ invariant under operators $\rho(a) : V \to V$, $\forall a \in A$.

Representation (continued)

Definition

A representation V is called irreducible if its only subrepresentations are V and 0.

Definition

A direct sum of representations (V_1, ρ_1) and (V_2, ρ_2) is the vector space $V_1 \oplus V_2 = \{(v_1, v_2) | v_1 \in V_1, v_2 \in V_2\}$ with ρ defined by $\rho(a)(v_1, v_2) = (\rho_1(a)v_1, \rho_2(a)v_2)$ where $v_1 \in V_2$, $v_2 \in V_2$, and $a \in A$.

Definition

A representation V is called indecomposable if it cannot be written as the direct sum of two nonzero subrepresentations.

Remark

The main goals in representation theory are to:

- *Classify all irreducible representations of an algebra A.*
- *Classify all indecomposable representations of A.*

We will only work with finite dimensional algebras and representations.

Schur's Lemma

Definition

A homomorphism between two representations V_1 and V_2 denoted by $\phi : V_1 \to V_2$ is a linear map that commutes with the action of A, so $\phi(av) = a\phi(v)$ for any $v \in V_1$ and $a \in A$.

Lemma (Schur's lemma)

Suppose V_1, V_2 are representations of an algebra A. Let $\phi : V_1 \to V_2$ be a nonzero homomorphism of representations. Then:

a) If V_1 is irreducible, then ϕ is injective.

b) If V_2 is irreducible, then ϕ is surjective.

c) If V_1 and V_2 are both irreducible, ϕ is an isomorphism.

Semisimple Representations

Definition

A semisimple representation, also known as completely reducible, is a direct sum of irreducible representations

Radicals

Definition

The radical of a finite dimensional algebra A, denoted $Rad(A)$, is the set of elements in A that act by 0 in all irreducible representations of A.

Definition

- A left ideal I of an algebra A is a vector subspace of A that satisfies the condition that for every $a \in A$ and $x \in I$, $a \cdot x \in I$.
- A right ideal I is the subspace of A with the condition that $x \cdot a \in I$ for all $a \in A, x \in I$.
- A two-sided ideal is a subspace of A which is both a left and a right ideal.

Remark

A radical is necessarily a two-sided ideal.

Semisimple Algebras

Definition

A finite dimensional algebra A is semisimple if $Rad(A) = 0$.

Theorem

The following are equivalent for finite dimensional algebra A:
1. *A is semisimple;*
2. *$\sum_i (dimV_i)^2 = dimA$, with V_i's being the distinct irreducible representations of A.*
3. *$A \cong \bigoplus_i Mat_{d_i}(k)$ for certain d_i.*
4. *Any finite dimensional representation of A is semisimple, i.e. completely reducible (hence why these algebras are also known as semisimple).*
5. *A is completely reducible representation of A.*

Outline

1. Associative unital algebras
   - Algebra
   - Representations
   - Radicals

2. Group Representations
   - Groups
   - Representations
   - Characters

Group Theory

- A group represents the **symmetries** of an object.

Definition

A group G is a set with an operation $\cdot : G \times G \to G$ (multiplication), satisfying the following requirements:

- **Associative:** $(a \cdot b) \cdot c = a \cdot (b \cdot c)$,
- **Identity:** There is an $e \in G$ such that $a \cdot e = e \cdot a = a$,
- **Inverse:** There is an $a^{-1} \in G$ such that $a \cdot a^{-1} = a^{-1} \cdot a = e$,

for all $a, b, c \in G$.

- Implicitly, a group is closed under multiplication.

Definition

A group homomorphism $f : G \to H$ for two groups G, H is a map f such that $f(g_1 \cdot g_2) = f(g_1) \cdot f(g_2)$ for all $g_1, g_2 \in G$. It's an isomorphism if f is bijective.

Example: S_3

- S_3 is the group of permutations of three elements. So each $\sigma \in S_3$ maps $\sigma : \{1, 2, 3\} \to \{1, 2, 3\}$
- Multiplication: compose permutations.
- Identity permutation e maps $x \mapsto x$.
- We can describe permutations in terms of cycles.

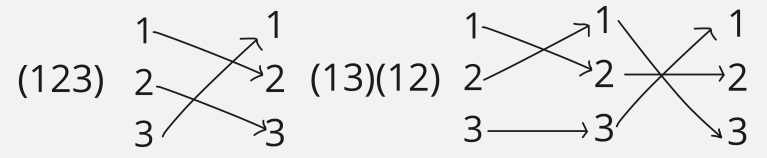

Fig.1. Some permutations in S_3.

- Sign of permutation: even/odd number of transpositions.

Example: D_3

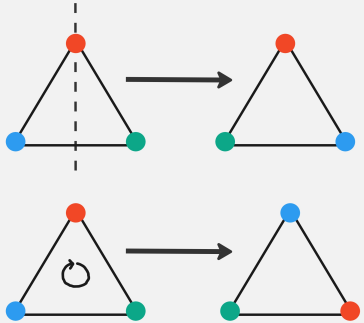

Fig.2. Flips and Rotations in D_3.

- Combining these flips and rotations, we can get 6 possible configurations. So $|D_3| = 6$.
- It turns out that all possible permutations of vertices are obtainable in D_3. So $D_3 \cong S_3$.

Group Representations

Definition

The group $GL(V)$ of all invertible linear maps from a finite-dimensional k-vector space V to itself is called the general linear group of V.

Definition

Let G be a finite group. A representation of G is a finite-dimensional k-vector space V, with a homomorphism $\rho_V : G \to GL(V)$.

- Informally, a representation describes how a group acts on a vector space.
- Such a representation is equivalent to a finite-dimensional representation of the **group algebra** $k[G]$, with basis $\{a_g | g \in G\}$ and the multiplication rule $a_g \cdot a_h = a_{gh}$.
- Let us fix a field $k = \mathbb{C}$.

Example: Cyclic Groups

- Consider the group C_n of order n generated by the element g. It contains the elements

$$g, g^2, g^3, \ldots, g^n = e$$

  We call such a group **cyclic**.

- There are precisely n non-isomorphic irreducible representations of G, all of which are one-dimensional.
- Let ζ be a primitive n-th root of unity. Then we have the irreducible representations $\mathbb{C}_1, \ldots, \mathbb{C}_n$, with homomorphism

$$\rho_k(g) := \zeta^k,$$

  and hence in general,

$$\rho_k(g^a) = \zeta^{ak}.$$

Example: Representations of S_3

- Consider S_3, the symmetric group, describing permutations of 3 elements.
- There are two one-dimensional representations of S_3:
  - The **trivial** representation, $\mathbb{C}_+$, with $\rho(g) := 1$.
  - The **sign** representation $\mathbb{C}_-$, with $\rho(g) := \text{sign}(g)$.
- Recall that $S_3 \cong D_3$. So we have the **dihedral** representation $\mathbb{C}^2$, with $\rho : S_3 \to GL_2(\mathbb{C})$:

$$\rho((12)) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$\rho((123)) = \begin{pmatrix} \cos\frac{2\pi}{3} & -\sin\frac{2\pi}{3} \\ \sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} \end{pmatrix}$$

- The **tautological** representation T: $\rho : S_3 \to GL_3(\mathbb{C})$, which sends each element to the corresponding permutation 3×3 permutation matrix.

Maschke's Theorem

Theorem (Maschke)

Let G be a finite group. Then $\mathbb{C}[G]$ is semisimple. Equivalently, if V is a representation of G then we can write

$$V = V_1 \oplus \cdots \oplus V_n,$$

where the V_i are (not necessarily distinct) irreducible representations of G.

Corollary (Sum-of-squares)

Let $V_1, \ldots, V_s$ be the non-isomorphic irreducible representations of G. Then

$$|G| = \sum_{i=1}^{s} \dim(V_i)^2.$$

Characters

Definition

Let V be a representation of G. Then define the character $\chi_V : G \to \mathbb{C}$ as

$$\chi_V(g) := \mathrm{Tr}|_V(\rho_V(g)).$$

- The irreducible characters form an basis of $F_c(G, \mathbb{C})$, the space of class functions $G \to \mathbb{C}$.

Orthogonality Relations

- The irreducible characters form an **orthonormal** basis of $F_c(G, \mathbb{C})$, under the form $\langle\ ,\ \rangle$, defined as

$$\langle f_1, f_2 \rangle := \frac{1}{|G|} \sum_{g \in G} f_1(g)\overline{f_2(g)}.$$

Theorem (Orthogonality Relations)

Let V and W be representations of G. Then

$$\langle \chi_V, \chi_W \rangle = \dim \operatorname{Hom}_G(W, V)$$

Moreover, if V and W are irreducible, then

$$\langle \chi_V, \chi_W \rangle = \begin{cases} 1 & \text{if } V \cong W \\ 0 & \text{if } V \ncong W \end{cases}$$

Orthogonality Relations (continued)

Corollary

Let V be a representation of G. If $\langle \chi_V, \chi_V \rangle = 1$, then V is irreducible.

Proof.

$\mathrm{Hom}_G(V, V) = \langle \chi_V, \chi_V \rangle = 1$ is trivial, so there are no proper subrepresentations $W \subset V$. □

- This gives us an easy way to check if a representation is irreducible.

Corollary

The number of irreducible representations of G is equal to the number of conjugacy classes of G.

Character tables

- Character tables let us visualize these orthogonality relations.

S_3	e	(12)	(123)
$\#$	1	3	2
$\mathbb{C}_+$	1	1	1
$\mathbb{C}_-$	1	-1	1
$\mathbb{C}^2$	2	0	-1

Fig.3. Irreducible characters of S_3.

- Characters evaluated on representative elements.
- This shows that $\mathbb{C}_+$, $\mathbb{C}_-$, and $\mathbb{C}^2$ are the **only** irreducible representations (up to an isomorphism) of S_3, since:
  - $\langle \chi_i, \chi_i \rangle = 1$
  - Sum-of-squares satisfied.
- We also have the Second Orthogonality Relations, for the columns of the character table.

Burnside's Theorem

Definition

- We say a group is abelian if its elements commute. For example, the group $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ under multiplication.
- A subgroup $H \subset G$ is normal if for all $g \in G$, $h \in H$, we have $ghg^{-1} = H$. Then we write $H \triangleleft G$.
- A group G is solvable if there exists a sequence of proper normal subgroups
$$\{1\} = G_1 \triangleleft \cdots \triangleleft G_n = G$$
such that the successive quotients G_{i+1}/G_i are all abelian.

Theorem (Burnside)

Suppose G is a group of order $|G| = p^a q^b$, where p and q are primes, and $a, b \geq 0$. Then G is solvable.

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Bibliography

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