The Classification of Compact Surfaces

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December 8, 2024, MIT PRIMES Mini-Conference

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- We will focus on the classification of compact surfaces.

Classic Definitions and Results

Definition (Regular surface)

$S \subseteq \mathbb{R}^3$ is a **regular surface** if at each $p \in S$, there exists a local parametrization $\mathbf{x} : U \to V \cap S$ that is a C^∞ homeomorphism with an injective differential. Here, $U \subseteq \mathbb{R}^2, V \subseteq \mathbb{R}^3$ are open sets and $p \in V$.

Definition (Regular value)

A point $q \in \mathbb{R}^m$ is a **regular value** of $F : \mathbb{R}^n \supseteq U \to \mathbb{R}^m$ if dF_p is surjective for *all* $p \in U$ such that $F(p) = q$.

Definition (Orientable surface)

A regular surface S is **orientable** if it can be covered with coordinate neighborhoods such that any overlapping regions have coordinate changes with positive Jacobian determinants, defining an **orientation**.

Classic Definitions and Results

Theorem (Inverse Function Theorem)

Let $F : \mathbb{R}^n \supseteq U \to \mathbb{R}^n$ be a differentiable mapping. If dF_p is an isomorphism at $p \in U$, then F has a local differentiable inverse around p.

Classic Definitions and Results

Definition (Smooth map from a surface to $\mathbb{R}$)

If $S \subset \mathbb{R}^3$ is a regular surface, and V is an open subset of S, then a map $f : V \subset S \to \mathbb{R}$ is *smooth* at $p \in V$ if there exists some parametrization $\mathbf{x} : \mathbb{R}^2 \supseteq U \to S$ with $p \in \mathbf{x}(U) \subset V$ such that $f \circ \mathbf{x} : \mathbb{R}^2 \supseteq U \to \mathbb{R}$ is smooth at $\mathbf{x}^{-1}(p)$. We say that f is *smooth* in V if it is smooth at all $p \in V$.

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- If $p_0 \in S$, then the function $f : S \to \mathbb{R}$ given by $f(p) = |p - p_0|^2$ is smooth.

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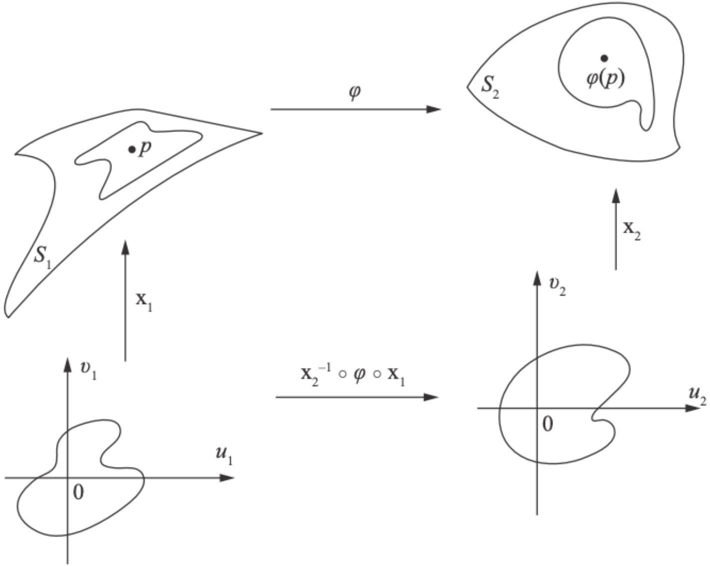
- Let $v \in \mathbb{R}^3$ be some unit vector. Then, the function $h : S \to \mathbb{R}$ given by $h(p) = \langle p, v \rangle_{\mathbb{R}^3}$, is smooth.

Classic Definitions and Results

Definition (Smooth map between surfaces)

If S_1 and S_2 are regular surfaces, then a continuous map $\phi : S_1 \supset V \to S_2$ where V is an open set of S_1, is *smooth* at $p \in V$, if for any parametrizations $\mathbf{x}_1 : \mathbb{R}^2 \supset U_1 \to S_1$ and $\mathbf{x}_2 : \mathbb{R}^2 \supset U_2 \to S_2$ such that $p \in \mathbf{x}_1(U_1)$ and $\phi(\mathbf{x}_1(U_1)) \subset \mathbf{x}_2(U_2)$, the map $\mathbf{x}_2^{-1} \circ \phi \circ \mathbf{x}_1 : U_1 \to U_2$ is smooth at $\mathbf{x}_1^{-1}(p)$.

Classic Definitions and Results



Classic Definitions and Results

Definition (Diffeomorphism)

A *diffeomorphism* between two surfaces S_1 and S_2 is a smooth map $\phi : S_1 \to S_2$ such that its inverse $\phi^{-1} : S_2 \to S_1$ is also smooth. If there exists a diffeomorphism between two surfaces, then they are said to be *diffeomorphic*.

An example of a diffeomorphism is the map $f : S^2 \to E^2$ given by $f(x, y, z) = (xa, yb, zc)$ where E^2 is the 2-dimensional ellipsoid $\{(x, y, z) \in \mathbb{R}^3 | \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\}$. Its inverse $f^{-1} : E^2 \to S^2$ is given by $f^{-1}(x, y, z) = (\frac{x}{a}, \frac{y}{b}, \frac{z}{c})$.

The Classification of Compact Orientable Surfaces

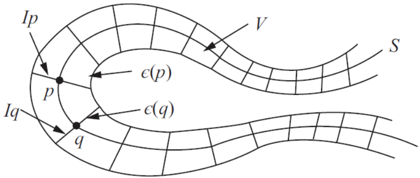
Theorem (Classification of compact surfaces)

Let $S \subset \mathbb{R}^3$ be a regular compact orientable surface. Then there exists an open set $V \subset \mathbb{R}^3$ and a smooth function $g : V \to \mathbb{R}$ which has zero as a regular value and $S = g^{-1}(0)$.

Classic Definitions and Results

Definition (Tubular neighborhood)

$V = \cup_{p\in S} I_p \in \mathbb{R}^3$ is a *tubular neighborhood* of regular surface S, if I_p is an open interval around p of length $2\epsilon_p$ such that $p \neq q \in S$, and $I_p \cap I_q = \emptyset$.



Thus $V = \cup_{p\in S} I_p \subset \mathbb{R}^3$ contains S and through each point of V there passes a unique normal line to S.

Existence of a Local Tubular Neighborhood

Existence of a local tubular neighborhood

Let S be a regular surface and $\mathbf{x} : U \to S$ be a parametrization of a neighborhood of a point $p = \mathbf{x}(u_0, v_0) \in S$. Then there exists a neighborhood $W \subset \mathbf{x}(U)$ of p in S and a number $\epsilon > 0$ such that the segments of the normal lines passing through points $p, q \in W$, with center at p, q and length 2ϵ, are disjoint.

Existence of a Local Tubular Neighborhood

- Let $F : U \times \mathbb{R} \to \mathbb{R}^3$ be given by

$$F(u, v, t) = \mathbf{x}(u, v) + tN(u, v)$$

where $N(u, v)$ is the unit normal vector at the point $\mathbf{x}(u, v)$ on the surface S.

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- By IFT, there exists a parallelepiped $P \times (-\epsilon, \epsilon) \subset U \times \mathbb{R}$ such that $F|_{P\times(-\epsilon,\epsilon)}$ is bijective.
- Thus, for any $p, q \in F(P \times (-\epsilon, \epsilon))$, the normal lines centered at p and q of length 2ϵ are disjoint.

Level Surface Characterization

Level surface characterization for orientable surfaces with global tubular neighborhoods

For an *orientable* regular surface S with a tubular neighborhood V, defining $g(w)$ as the "oriented distance" to S from $w \in V$ along the unique normal line makes g differentiable with 0 as a regular value.

- Consider $F : U \times \mathbb{R} \to \mathbb{R}^3$ as defined previously. By IFT, there exists $\epsilon > 0$ and an open set $V \subset \mathbb{R}^2$ such that

$$F|_{F(V \times (-\epsilon, \epsilon))}$$

has a differentiable inverse $F^{-1} : F(V \times (-\epsilon, \epsilon)) \to V \times (-\epsilon, \epsilon)$.

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- Since $\det(dF_p^{-1}) \neq 0$ for all $p \in F(V \times \{0\})$,

$$\left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right)(p) \neq 0.$$

  which means 0 is a regular value of g.

A Result from Real Analysis

Lebesgue's Number Lemma

If $A \subseteq \mathbb{R}^3$ is compact and $\{U_\alpha\}_{\alpha \in I}$ is a family of open sets U_α such that $A \subset \cup_\alpha U_\alpha$, then there exists a $\delta > 0$, such that for all p, q with $d(p,q) < \delta$, there exists $\alpha \in I$ such that $p, q \in U_\alpha$. This number δ is called the lebesgue number of the family $\{U_\alpha\}$.

- Suppose for the sake of contradiction that such a δ does not exist.

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- Then, for all $n \in \mathbb{N}$, there exist $p_n, q_n \in A$ such that $d(p_n, q_n) < \frac{1}{n}$ but there does not exist α such that $p_n, q_n \in U_\alpha$.

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- By Bolzano-Weierstrass, the sequences p_n and q_n have limit points p and q in A, respectively.

A Result from Real Analysis

- Since $d(p_n, q_n) < \frac{1}{n}$, we have $p = q$. Suppose p is in U_α, so there exists an $\epsilon > 0$ such that $B_\epsilon(p) \subset U_\alpha$.

A Result from Real Analysis

- Since $d(p_n, q_n) < \frac{1}{n}$, we have $p = q$. Suppose p is in U_α, so there exists an $\epsilon > 0$ such that $B_\epsilon(p) \subset U_\alpha$.
- As p is a limit point of p_n, there exists an $N \in \mathbb{N}$ such that for all $n > N$, we have $p_n, q_n \in B_\epsilon(p) \subset U_\alpha$, but this is a contradiction as we assumed p_n and q_n could not belong to the same U_α.

Existence of a Global Tubular Neighborhood

Existence of a global tubular neighborhood

Let $S \subset \mathbb{R}^3$ be a regular, compact, orientable surface. Then there exists a number $\epsilon > 0$ such that whenever $p, q \in S$, the segments of the normal lines of length 2ϵ centered at p and q are disjoint (in other words, S has a tubular neighborhood).

- By the existence of a local tubular neighborhood, for all $p \in S$, there exists a neighborhood W_p and some $\epsilon_p > 0$ such that the segments of the normal lines centered at p, q with length 2ϵ are disjoint for all $q \in W_p$.

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- By the existence of a local tubular neighborhood, for all $p \in S$, there exists a neighborhood W_p and some $\epsilon_p > 0$ such that the segments of the normal lines centered at p, q with length 2ϵ are disjoint for all $q \in W_p$.
- Since S is compact and $\cup_{p \in S} W_p = S$, so we can choose a finite subset $\{W_1, \ldots, W_k\}$ of $\{W_p\}_{p \in S}$ such that $\cup_{i \in [k]} W_i = S$.

Existence of a Global Tubular Neighborhood

- We will show that any

$$\epsilon < \min\left(\epsilon_1, \ldots, \epsilon_k, \frac{\delta}{2}\right)$$

is a working value of ϵ, where δ is the Lebesgue number of $\{W_i\}_{1\leq i\leq k}$ given by Lebesgue's number lemma.

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  is a working value of ϵ, where δ is the Lebesgue number of $\{W_i\}_{1 \leq i \leq k}$ given by Lebesgue's number lemma.
- Let $p, q \in S$. If they are in the same W_i, then the segments formed by the normal lines centered at p and q with length 2ϵ do not intersect as $\epsilon < \epsilon_i$. If p, q are not in the same W_i, then $d(p, q) \geq \delta$, so if the segments intersect at some point P, we have

$$\delta > 2\epsilon \geq d(p, P) + d(P, q) \geq d(p, q) \geq \delta,$$

  which is a contradiction.

Remarks

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2. The theorem also holds if we only assume compactness, as compactness implies orientability for regular surfaces in $\mathbb{R}^n$.

Acknowledgments

We would like to thank the MIT PRIMES program for making this opportunity possible. We would also like to thank our mentor Isaac M. Lopez for his teaching of differential geometry and his guidance in the completion of this presentation.

Citations

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