Incidence Geometry in Euclidean and p-Adic Spaces

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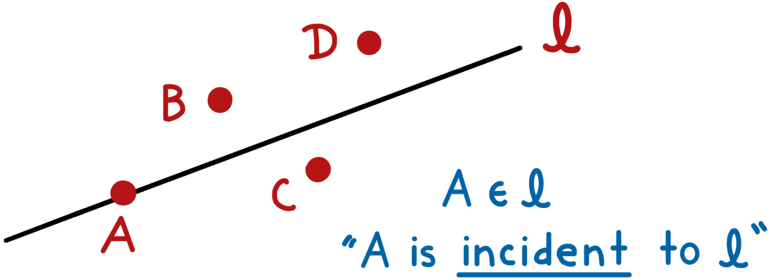
Winter PRIMES Mini-Conference
MIT PRIMES Local Reading

12/8/2024

Introduction to Incidence Geometry

- Incidence Geometry focuses on the study of geometric objects (points, lines, and other shapes) and their relationships.
- When a point lies on a line, we call it **incident** to that line.



The Szemerédi–Trotter Theorem

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Theorem (Szeméredi-Trotter)

Let P be a set of m points in $\mathbb{R}^2$ and L a set of n lines in $\mathbb{R}^2$. Let the set of incidences $I(P, L)$ between P and L be defined as

$$I(P, L) = \{(p, \ell) \in P \times L \mid p \in \ell\}.$$

Then, the following inequality holds for some absolute constant c:

$$|I(P, L)| \leq c(m^{2/3}n^{2/3} + m + n).$$

ST Theorem: Tightness of Bound

Example

For any $M \in \mathbb{N}$, consider the set of points on the integer lattice

$$P = \{(x, y) \in \mathbb{Z}^2 \mid 1 \leq x \leq M; 1 \leq y \leq 2M^2\},$$

and the set of lines

$$L = \{(x, ax + b) \in \mathbb{R}^2 \mid a, b \in \mathbb{Z}; 1 \leq a \leq M; 1 \leq b \leq M^2\}.$$

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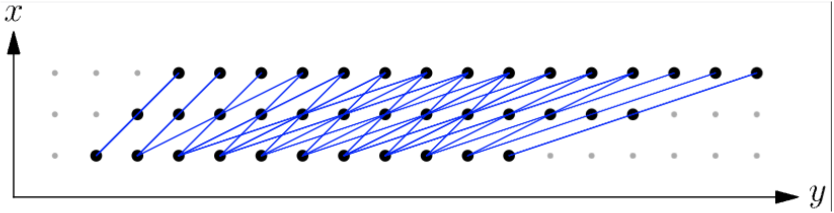

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We have $m = |P| = 2M^3$, $n = |L| = M^3$. Notice that every line in L intersects M points in P, at $x = 1, 2, \dots M$. This gives M^4 incidences total. Indeed, the ST theorem gives

$$|I(P, L)| \leq c(m^{2/3}n^{2/3} + m + n) \sim O(M^4).$$

Talk Overview

1. Proof sketch of the Szeméredi-Trotter theorem over the reals
2. The unit and distinct distances problems over the reals
3. Introduction to the p-adic integers
4. Szeméredi-Trotter theorem, unit distances problem, and distinct distances problem over the p-adics

ST Theorem: Cell Partitioning Lemma

- Cell Partitioning Lemma:
  - We are able to divide the plane into regions (cells) bounded by a relatively small subset of lines from L such that each cell has not too many lines passing through it.
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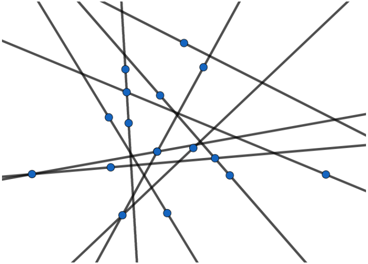

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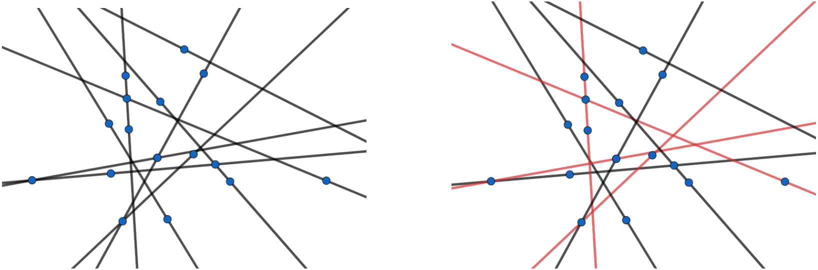

ST Theorem: Cell Partitioning Proof Sketch

- Bounding incidences within cells:
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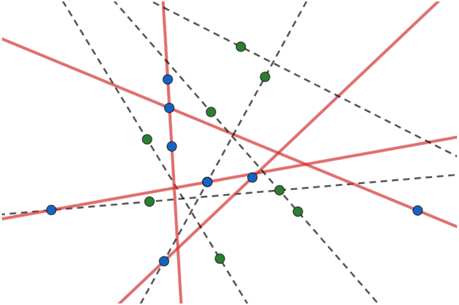
- Summing over partition:
  - Careful choice of the size of the set of partitioning lines.
  - Add results from all cells and handle incidences along boundaries via recursion to finish the proof of the Szemerédi–Trotter bound.



Applications of ST Theorem

- Many proofs linking different areas:
  - Combinatorial geometry (Cell Partitioning).
  - Graph theory (Crossing Numbers)
  - Topological and Algebraic Methods (Polynomial Ham Sandwich theorem)

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- Many proofs linking different areas:
  - Combinatorial geometry (Cell Partitioning).
  - Graph theory (Crossing Numbers)
  - Topological and Algebraic Methods (Polynomial Ham Sandwich theorem)
- Connection to other problems in combinatorial geometry (Erdős problems)
  - Erdős distinct distances problem
  - Erdős unit distances problem

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The Distinct Distances Problem Over the Reals

Problem (the distinct distances problem in $\mathbb{R}^2$)

- *Use the Euclidean distance:* $d(p_i, p_j) = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$.
- *Let* $P = \{p_1, p_2, \ldots, p_n\}$ *be a set of* n *points in* $\mathbb{R}^2$.
- *What is the minimum number of distinct distances* $d(p_i, p_j)$ *determined by* n *points?*

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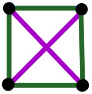

Distances: $1, 1, 1, 1, \sqrt{2}, \sqrt{2}$

Distinct Distances: $1, \sqrt{2}$

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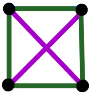

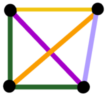

Distinct Distances Problem Over the Reals

Theorem (Erdős; 1946. Guth and Katz; 2015.)

Asymptotically, for some positive constants c_1 and c_2, the minimal number of distinct distances, $f(n)$, is

$$c_1\left(\frac{n}{log(n)}\right) \leq f(n) \leq c_2\left(\frac{n}{\sqrt{log(n)}}\right).$$

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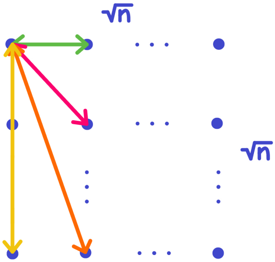
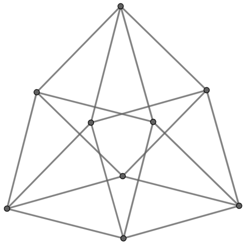
The upper bound is given by a $\sqrt{n} \times \sqrt{n}$ square grid.



Problem (the unit distance problem in $\mathbb{R}^2$)

1. *We are once again given n points $P = \{p_1, p_2, \ldots, p_n\}$ in the real plane for some positive integer n.*
2. *Define $g(n)$ to be the maximum number of pairs $1 \leq i < j \leq n$ such that $d(p_i, p_j) = 1$.*



This optimal graph of squares and equilateral triangles shows $g(9) = 18$.

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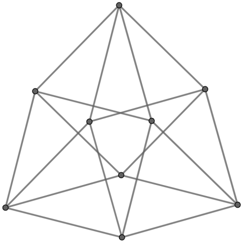
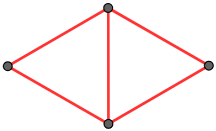


This optimal graph of squares and equilateral triangles shows $g(9) = 18$.
(Erdős, 1946) finds a construction proving $g(n) \geq n^{1+\frac{c}{\log(\log n)}}$.

The Unit Distance Problem Over the Reals

How do we find an upper bound? Just turn it into an incidence problem!

1. If we draw the unit circle centered at each point, we create $2g(n)$ incidences between n points and n unit circles.
2. A variant of the ST theorem then proves that $g(n) \leq cn^{\frac{4}{3}}$.



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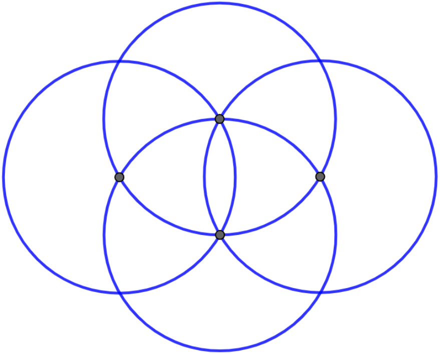

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Introducing p-Adics

Definition ($v_p(x)$–p-adic valuation)

We call the ***p*-adic valuation** of x, $v_p(x)$, the largest nonnegative integer such that $p^{v_p(x)}$ divides x.

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$$1 + 1 + 1\cdot 2^1 + 1\cdot 2^2 + \ldots = |2^\infty|_2 = \frac{1}{2^\infty} = 0$$

$$v_p(2^\infty) = \infty$$

Introducing p-Adics

Definition (p-adic integer)

A ***p*-adic integer** is an element of the ring $\mathbb{Z}_p$, which consists of all formal power series of the form:

$$x = \sum_{n=0}^{\infty} a_n p^n,$$

where $a_n \in \{0, 1, 2, \ldots, p-1\}$ for each n.

- The series converges in p-adic space

Introducing p-Adics

Definition (p-adic distance)

The ***p*-adic distance** between two p-adic integers a and b is written $d_p(a,b)$ and defined as

$$d_p(a,b) = |a-b|_p = p^{-v_p(a-b)}.$$

Example

What is $d_5(38, 23)$?

- $38 - 23 = 15$ so we need to evaluate $|15|_5$. Observe that 5 divides 15, but 25 does not. Therefore, $v_5(15) = 1$, so $|15|_5 = 5^{-1} = \frac{1}{5}$.

Notice that the distance between two integers never exceeds 1! In fact, $d_p(a,b) = 1$ if and only if $a \not\equiv b \pmod{p}$.

- This is because if $a \not\equiv b$, then we know $p \nmid a - b$, so $v_p(a-b) = 0$, meaning $p^{-v_p(a-b)} = p^0 = 1$, while if $a \equiv b$, then we know $p | a - b$, so $v_p(a-b) \geq 1$, meaning $p^{-v_p(a-b)} \leq p^{-1} < 1$.

Below is a representation of the 3-adic integers, numbers of the form $x = a_0 + a_1 \cdot 3 + a_2 \cdot 3^2 + \cdots$.

Overview of the p-Adic Integers

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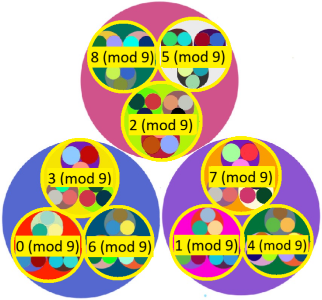

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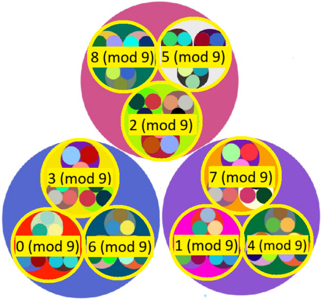

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- The integers within a subcircle of a given circle have a particular 3-adic distance from integers in another subcircle of that circle.

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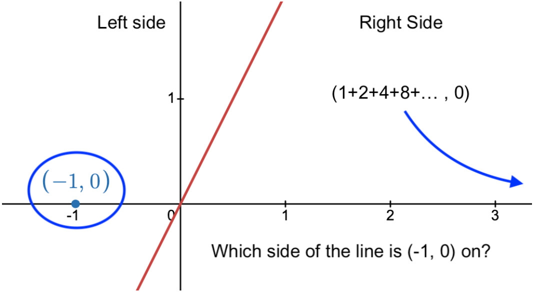
The ST Theorem Over the p-Adic Plane $\mathbb{Z}_p^2$

- All points in $\mathbb{Z}_p^2$ lie within the complex plane $\mathbb{C}^2$, and Szemeredi-Trotter in $\mathbb{C}^2$ was already proven (Tóth, 2003).
- Can we find a direct proof of the ST theorem in $\mathbb{Z}_p^2$?
- While the aforementioned $\mathbb{R}^2$ proof utilizes cell partitioning, in $\mathbb{Z}_p^2$ a line doesn't actually divide the plane into two sides like it does in $\mathbb{R}^2$, so partitioning doesn't make sense.

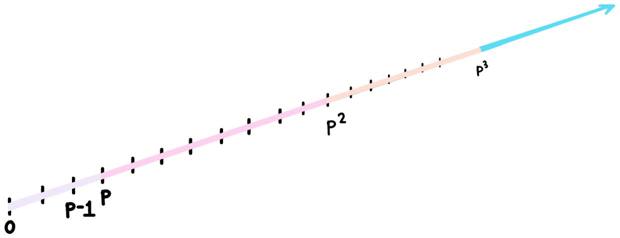


The Distinct Distances Problem Over the p-Adics

Theorem (Carratu, Tatar, Xu; 2024)

The minimum number of distinct distances for n points in $\mathbb{Z}_p$ is $\lceil \log_p n \rceil$.

- Key idea: the number of distinct distances is determined by the unique k for which $p^{k-1} < n \leq p^k$.



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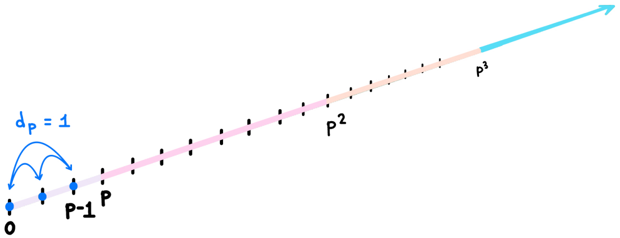

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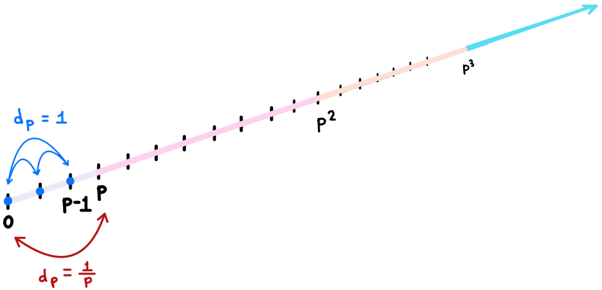

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The result generalizes for d-dimensional p-adic space:

The minimum number of distinct distances for n points in $\mathbb{Z}_p^d$ is $\lceil \log_p n^{\frac{1}{d}} \rceil$.

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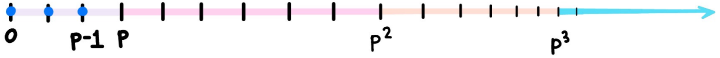

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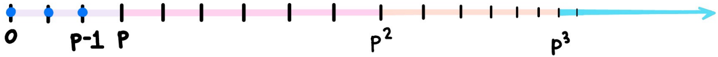

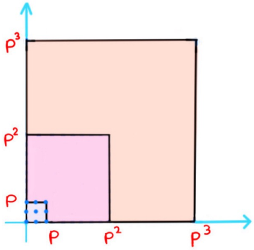

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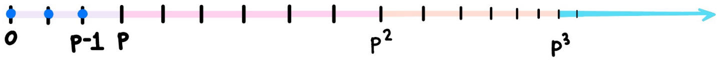

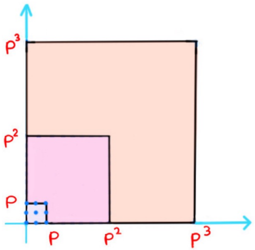

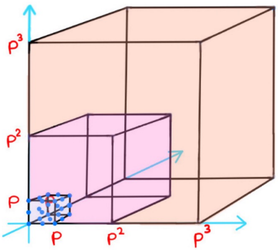

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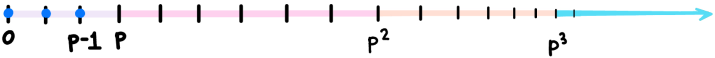

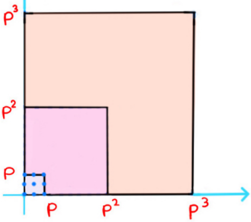

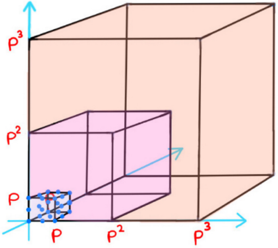

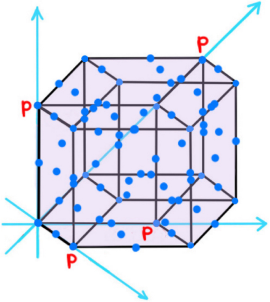

The Unit Distances Problem Over the p-Adics

Theorem (Carratu, Tatar, Xu; 2024)

The maximum number of unit distances for n points in $\mathbb{Z}_p$ is

$$\frac{1}{2}(n^2 - n\alpha - \alpha\beta - \beta)$$

where $\alpha, \beta \in \mathbb{Z}_{\geq 0}$ are such that $\beta < p$ and $n = \alpha p + \beta$.

To prove this we observe the following:

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To prove this we observe the following:

When a and b have the same remainder mod p, we know $d_p(a, b) < 1$.

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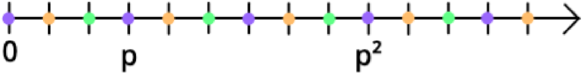
This inspires us to color the p-adic number line with p colors:



Above is the coloring for $p = 3$.

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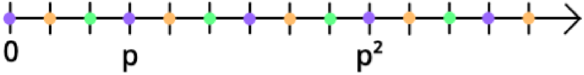

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Observe that the only pairs of points that AREN'T unit distance apart are those of the same color! Thus, $f(n)$=(total pairs)-(same-color pairs)

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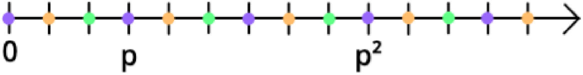

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Letting b_i be the number of points of color i, we find

$$f(n) = \frac{1}{2}(n^2 - \sum b_i^2)$$

and because $\sum b_i = n$, a single application of the Cauchy-Schwarz inequality finishes.

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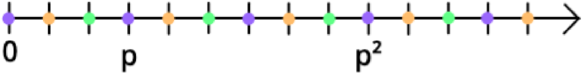

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(Note: this argument easily generalizes to d dimensions by replacing p colors with p^d colors.)

Thank you to our mentor Dr. Manik Dhar, Prof. Etingof, Dr. Gerovitch, Dr. Khovanova, Dr. Gotti, and our parents!

Questions?

PRIMES