THE WARPED TENSOR PRODUCT OF FROBENIUS ALGEBRAS

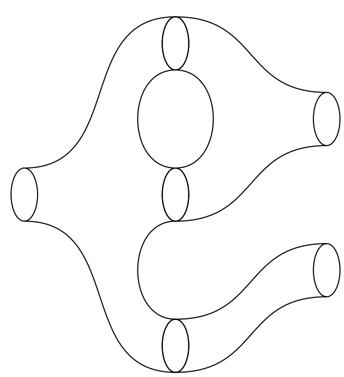
ROHAN DAS

Abstract. Frobenius algebras were first studied in the 1930s due to their importance to the representation theory of finite groups. Recently, they have returned to popularity because commutative Frobenius algebras correspond exactly to two-dimensional Topological Quantum Field Theories, which combine the principles of classical field theory, special relativity, and quantum mechanics. In this paper, we introduce the warped tensor product and use it to build new symmetric monoidal structures on Frobenius algebras.

1. Introduction

Frobenius Algebras. Frobenius algebras were first studied in the 1930s; roughly speaking, they are vector spaces equipped with multiplication, a multiplicative identity, and a pairing, which we more concretely define in Section 2. Their initial applications were in the representation of finite groups, partly because group algebras are Frobenius. Around a decade later, Nakayama discovered a duality theory in [Nak39] and [Nak41] that widely expanded the applications of Frobenius algebras to topics such as homological algebra, algebraic geometry, combinatorics, and number theory. More recently, they have been of particular interest because commutative Frobenius algebras are equivalent to two dimensional Topological Quantum Field Theories (TQFTs).

TQFTs. We first loosely describe TQFTs over vector spaces. A d-manifold is a surface that locally resembles the Euclidean space $\mathbb{R}^d$ everywhere, and cobordisms are $(d+1)$-manifolds that link some d-manifolds, called the inboundary, to some others, called the outboundary. For example, shown below is a cobordism; its inboundary is one circle and its outboundary is the disjoint union of two circles.



(1.1)

Then, a TQFT associates d-manifolds to vector spaces and cobordisms up to diffeomorphism (a topological equivalence) to linear maps. The choices of vector spaces must respect the multiplicative structures: for manifolds M_1 and M_2 sent to V_1 and V_2, the disjoint union is sent to the tensor product $V_1 \otimes V_2$. Note that the cobordism in Diagram 1.1 is the gluing of the following cobordisms,

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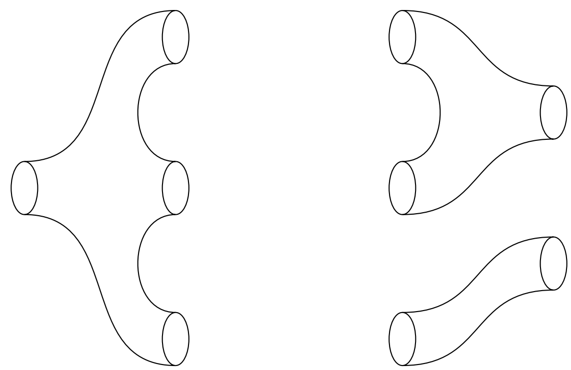


where the right cobordism is itself a disjoint union of smaller cobordisms. Like for the d-manifolds, the disjoint union of cobordisms must be given by the tensor product of the corresponding linear maps. Additionally, the gluing of cobordisms should correspond to function composition.

TQFTs were first axiomatically defined by Atiyah in [Ati88]. Atiyah's definition associated manifolds to modules rather than vector spaces, and cobordisms to module elements rather than linear maps. These choices are subject to additional axioms that better illustrate the connection to physics; we briefly touch on this here and discuss the physics applications in more detail later in this section. For now, two of the axioms require topological properties to be respected by the choices, relating to relativistic invariance, while another two axioms give linear and multiplicative structures, reflecting a quantum nature of the theory.

Higher dimensional TQFTs are extremely complicated. Their classification was conjectured by the cobordism hypothesis in [BD95], suggesting an equivalence between TQFTs and underlying categories formed by discarding noninvertible morphisms. A proof of the cobordism hypothesis has been sketched in [Lur09]. However, lower dimensional TQFTs are both well understood and more applicable.

For the case of two dimensional TQFTs, two key simplifications can be made. First, the only 2-manifolds are the circle, line, half-line, and unit interval, and the circle is the only one relevant to the TQFT consideration. Therefore when assigning vector spaces to surfaces, one choice of V for the circle uniquely determines everything else (e.g. the disjoint union of 2 circles would be $V \otimes V$). All that remains is to pick linear maps for the cobordisms. Second, diffeomorphism becomes a much easier condition: any two cobordisms with the same inboundary, outboundary, and genus (number of holes) are diffeomorphic. In fact, this reformulation of diffeomorphism is equivalent to the relations on commutative Frobenius algebras, hence the correspondence between them.

Applications of TQFTs. Like other quantum field theories, TQFTs unify classical field theory with the principles of special relativity and quantum mechanics. One key difference from other such theories is that TQFTs are not very interesting in the typical consideration, flat Minkowski spacetime, so instead we often consider them over Riemann surfaces. One particular similar concept is the Conformal Field Theory (CFT), axiomatically defined in [Seg88]. The relationship is detailed in [Dij89]. Essentially, they differ in which topological structure they preserve; while CFTs additionally preserve a complex structure, TQFTs preserve the more natural geometric concept of orientation.

While they are primarily used by physicists, TQFTs also have a wide range of applications in pure mathematics. Just to name a few, the properties preserved by the deformations of classical objects can be understood via their (quantum) symmetries, TQFTs produce invariants of closed manifolds, and three-dimensional TQFTs relate closely to knot invariants, especially the Jones polynomial.

Generalizations to categories. Because Atiyah's axioms define TQFTs from a module's elements, they do not easily generalize to TQFTs over a general monoidal category. However, we use a more modern definition which has been described earlier. It can be formalized as follows: let Cob_{d-1}^{d} denote the monoidal category of cobordisms up to diffeomorphism, with multiplication given by the disjoint union, and let Vect_k denote the monoidal category of vector spaces with multiplication given by the tensor product. Then a TQFT is a symmetric monoidal functor $\mathrm{Cob}_{d-1}^{d} \to \mathrm{Vect}_k$. It is easy to generalize this definition to a general monoidal category $\mathcal{C}$: a TQFT is a monoidal functor $\mathrm{Cob}_{d-1}^{d} \to \mathcal{C}$. The exact conditions required of $\mathcal{C}$ used in this way varies; it is usually symmetric or braided, often rigid, and frequently a tensor category. In this work, we require that $\mathcal{C}$ is symmetric monoidal, as defined in Section 2.

This also gives an alternative realization of the connection to quantum relativity: as described in [Koc04], the closed manifolds model space, the cobordisms model space-time, and the target objects of $\mathcal{C}$ model state spaces. Again, the algebraic characterization of these topological objects preserves their physical meaning. For example, the disjoint union is sent to the tensor product, both of which represent the (combined) state space of two independent systems.

Finally, just like TQFTs, Frobenius algebras can be defined over monoidal categories. The definitions are entirely analogous, and by default Frobenius algebras are assumed to be over monoidal categories, not always a field.

Twisted tensor products. The other items of interest are twisted tensor products, which were first studied in [CSV95]. They aim to extend the algebraic realization of the product of two topological spaces to noncommutative differential geometry.

Twisted tensor products were also extended to Frobenius algebras in [OO24], and many important Frobenius algebras were recovered from the twisted tensor products. For example, given a finite group G acting on a finite group H by $\varphi : G \to \mathrm{Aut}(H)$, the twisting map $\tau : g \otimes h \mapsto \varphi(g)(h) \otimes g$ recovers the group algebra $k(H \rtimes_\varphi G)$ and its Frobenius structure as $kG \otimes_\tau kH$.

Motivation. To expand on how TQFTs model spacetime, moving from left to right along a cobordism represents moving forward in time, while moving along a surface in the cobordism corresponds to moving within space. However, TQFTs inherently have commutativity: paths that differ in orientation, or even different orderings of multiple paths, would be considered the same. Therefore, we seek to add noncommutativity through new multiplicative structures on them.

The typical means of adding noncommutativity is the twisted tensor product. However, since that twists the algebra, an operation that lacks a clear topological meaning, we instead consider modifications of the twisted tensor product. As explained in Section 3.1, the dual construction of the twisted tensor product (the cotwisted tensor product) never builds nontrivial structures. Therefore, we define the warped tensor product, a close modification of the twisted tensor product, and classify when it preserves the Frobenius property. Ultimately, we use the warped tensor product to build nontrivial symmetric monoidal structures on Frobenius algebras, and since these structures preserve commutativity (of algebras, separate from the geometric commutativity we seek to remove), they also hold over two-dimensional TQFTs.

We mainly focus on Frobenius algebras from here, but even though TQFTs will rarely be explicitly mentioned, they are central to and motivate this work. While we talk in terms of Frobenius algebras, we define and build my constructions in a way such that most properties that hold for Frobenius algebras automatically also hold for commutative Frobenius algebras, and therefore TQFTs.

Main Results. First, we classify exactly when the warped tensor product of two Frobenius algebras is Frobenius. This result is proved throughout Section 3. For vector spaces, the conditions translate to the warp being multiplication by a central, invertible element of the tensor product $A \otimes B$.

The terminology used in the below theorems, as well as the rest of the paper, is detailed in Section 2, and the notation is explained in Section 1.1.

Theorem 1.1. *Let $(A, \nabla_A, \eta_A, \beta_A)$ and $(B, \nabla_B, \eta_B, \beta_B)$ be Frobenius algebras over a symmetric monoidal category $\mathcal{C}$, with copairings α_A and α_B, respectively. Let $\gamma : B \otimes A \to A \otimes B$ be a morphism in $\mathcal{C}$ with a two-sided inverse. Then the warped tensor product $A \otimes_\gamma B$ is Frobenius if and only if the warp decomposes as*

$$\gamma : BA \xrightarrow{\sigma_{12}} AB \xrightarrow{\cong} AB\mathbb{1} \xrightarrow{1_A 1_B \psi} ABAB \xrightarrow{\nabla_\gamma} AB,$$

for a morphism $\psi : \mathbb{1} \to A \otimes B$ accompanied by another morphism $\psi^* : \mathbb{1} \to A \otimes B$ that makes the following diagrams commute.

$$(1.2) \qquad \begin{array}{ccc} \mathbb{1} & \xrightarrow{\psi\psi^*} & ABAB \\ \downarrow{\scriptstyle \psi^*\psi} & \searrow^{\eta_A\eta_B} & \downarrow{\scriptstyle \nabla_\gamma} \\ ABAB & \xrightarrow{\nabla_\gamma} & AB \end{array} \qquad \begin{array}{ccc} AB & \xrightarrow{11\psi} & ABAB \\ \downarrow{\scriptstyle \psi 11} & & \downarrow{\scriptstyle \nabla_\gamma} \\ ABAB & \xrightarrow{\nabla_\gamma} & AB \end{array} \qquad \begin{array}{ccc} AB & \xrightarrow{11\psi^*} & ABAB \\ \downarrow{\scriptstyle \psi^* 11} & & \downarrow{\scriptstyle \nabla_\gamma} \\ ABAB & \xrightarrow{\nabla_\gamma} & AB \end{array}$$

Necessity follows from Lemmas 3.1, 3.2, and 3.4. Sufficiency follows from Lemmas 3.11 and 3.12.

Using the results of Theorem 1.1, we define new symmetric monoidal structures on the category $\mathrm{Frob}_\mathcal{C}$ of Frobenius algebras over $\mathcal{C}$, which are closed on the full subcategory $\mathrm{cFrob}_\mathcal{C}$ of commutative Frobenius algebras.

Theorem 1.2. *Let ψ be a collection of morphisms $\psi_A : \mathbb{1} \to A$ in $\mathcal{C}$ that is warpable, as defined in Definition 4.4, and let*

$$\Upsilon_{A,B} : AB \xrightarrow{\cong} AB\mathbb{1} \xrightarrow{11\psi_{A,B}} ABAB \xrightarrow{\nabla_\sigma} AB.$$

Consider the class of warps

$$\gamma_{A,B} : BA \xrightarrow{\sigma_{12}} AB \xrightarrow{\Upsilon_{A,B}} AB.$$

Let $\boxtimes$ denote the warped tensor product $A \otimes_\gamma B$, and let α, λ, ρ, and τ denote the associativity, unit, and commutativity constraints of $\mathcal{C}$. Let I be the unit object $\mathbb{1}$ with Frobenius form $\epsilon = \mathrm{id}$. Then $(\mathrm{Frob}_\mathcal{C}, \boxtimes, I, \alpha, \lambda, \rho, \tau)$ and $(\mathrm{cFrob}_\mathcal{C}, \boxtimes, I, \alpha, \lambda, \rho, \tau)$ are symmetric monoidal categories if and only if for all $A, B, C \in \mathrm{Frob}_\mathcal{C}$:

$$(1 \otimes \Upsilon_{B,C})\Upsilon_{A,B\boxtimes C} = (\Upsilon_{A,B} \otimes 1)\Upsilon_{A\boxtimes B,C}; \quad \Upsilon_{I,A} = \Upsilon_{A,I} = 1; \quad \Upsilon_{A,B} = \Upsilon_{B,A}.$$

The proof is given in Section 4.2.

Outline. In Section 2, we establish key definitions for Frobenius algebras that we use throughout this paper. In Section 3, we define the warped tensor product and prove the statement of Theorem 1.1. In Section 4, we discuss implications of Theorem 1.1: first, we explore important properties of Frobenius algebras that are preserved by the warped tensor product; then, we apply Theorem 1.1 to endow the category of Frobenius algebras, as well as its full subcategory of commutative Frobenius algebras, with new symmetric monoidal structures, proving Theorem 1.2; finally, we give a family of solutions to the constraints given in Theorem 1.2.

1.1. **Notation.** This paper will use the following notational conventions. Unless otherwise stated, $\mathcal{C}$ is a symmetric monoidal category with a bifunctor $\otimes$ (called the [standard] tensor product), unit object $\mathbb{1}$, suppressed associativity constraints, unit constraints simply denoted $\cong$, and commutavity constraints σ. In Sections 4.2 and 4.3, all constraints are suppressed. Except for in-line math, tensor products $\otimes$ are indicated by concatenation to save space. In particular, when we say AB in a diagram, we refer to $A \otimes B$. For an object $A \in \mathcal{C}$, the identity morphism is denoted as $1_A : A \to A$, although the subscript is omitted in diagrams to save space.

For some multitensor $X_1 \otimes \cdots \otimes X_n$, we use σ_{ab} to denote the map that sends X_a to position X_b by using the braiding σ without changing the order of the rest of the objects. We explicitly define $\sigma_{a,a+1} = (1)^{(a-1)}\sigma(1)^{(n-a-1)}$, and

$$\sigma_{ab} = \begin{cases} \sigma_{b-1,b} \circ \sigma_{b-2,b-1} \circ \cdots \circ \sigma_{a,a+1} & a < b \\ \sigma_{b,b+1} \circ \sigma_{b+1,b+2} \circ \cdots \circ \sigma_{a-1,a} & a > b \end{cases}.$$

Also, if a map is called invertible, it has a two-sided inverse. Lastly, $\mathrm{Frob}_{\mathcal{C}}$ denotes the category of Frobenius algebras over a symmetric monoidal category $\mathcal{C}$, while $\mathrm{cFrob}_{\mathcal{C}}$ denotes the full subcategory of commutative Frobenius algebras. If something is called a Frobenius algebra without further context, it is a Frobenius algebra over $\mathcal{C}$.

2. Preliminaries

In this section, we introduce definitions and useful information about Frobenius algebras; everything presented here is already known.

2.1. Frobenius Algebras.

Within this subsection, $\mathcal{C}$ denotes a symmetric monoidal category with unit object $\mathbb{1}$. First, we present the definition of an algebra over a category.

Definition 2.1. An **associative unital $\mathcal{C}$-algebra** is a tuple (A, ∇_A, η_A), where $A \in \mathcal{C}$ is an object, and $\nabla_A : A \otimes A \to A$ and $\eta_A : \mathbb{1} \to A$ are morphisms in $\mathcal{C}$ such that the following diagrams commute, indicating left-unitality, right-unitality, and associativity.

(2.1)
$$\begin{array}{ccc} \mathbb{1}A & \xrightarrow{\eta_A 1} & AA \\ & \overset{\cong}{\searrow} & \downarrow{\scriptstyle \nabla_A} \\ & & A \end{array} \qquad \begin{array}{ccc} A\mathbb{1} & \xrightarrow{1\eta_A} & AA \\ & \overset{\cong}{\searrow} & \downarrow{\scriptstyle \nabla_A} \\ & & A \end{array} \qquad \begin{array}{ccc} AAA & \xrightarrow{\nabla_A 1} & AA \\ \downarrow{\scriptstyle 1\nabla_A} & & \downarrow{\scriptstyle \nabla_A} \\ AA & \xrightarrow{\nabla_A} & A \end{array}$$

Throughout this paper, when we refer to an algebra, it is assumed to be associative unital.

To define a Frobenius algebra, also consider a pairing.

Definition 2.2. Let (A, ∇_A, η_A) be a $\mathcal{C}$-algebra. A **pairing** is a morphism $\beta_A : A \otimes A \to \mathbb{1}$. It is **associative** if the following diagram commutes,

(2.2)
$$\begin{array}{ccc} AAA & \xrightarrow{\nabla_A 1} & AA \\ \downarrow{\scriptstyle 1\nabla_A} & & \downarrow{\scriptstyle \beta_A} \\ AA & \xrightarrow{\beta_A} & \mathbb{1} \end{array}$$

and it is **nondegenerate** if for some copairing $\alpha_A : \mathbb{1} \to A \otimes A$, the following diagram commutes.

(2.3)
$$\begin{array}{ccccc} A & \xrightarrow{\cong} & \mathbb{1}A & \xrightarrow{\alpha_A 1} & AAA \\ \downarrow{\scriptstyle \cong} & & & & \\ A\mathbb{1} & & \overset{1}{\searrow} & & \downarrow{\scriptstyle 1\beta_A} \\ \downarrow{\scriptstyle 1\alpha_A} & & & & \\ AAA & & \xrightarrow{\beta_A 1} & & A \end{array}$$

Definition 2.3. A **Frobenius algebra** over $\mathcal{C}$ is a tuple $(A, \nabla_A, \eta_A, \beta_A)$ such that (A, ∇_A, η_A) is a $\mathcal{C}$-algebra and β_A is an associative, nondegenerate pairing.

The default multiplicative structure on Frobenius algebras is given by the standard tensor product.

Definition 2.4. Consider Frobenius algebras $(A, \nabla_A, \eta_A, \beta_A)$ and $(B, \nabla_B, \eta_B, \beta_B)$, with copairings α_A and α_B, respectively. Then the **standard tensor product** is $(A \otimes B, \nabla_\sigma, \eta_\sigma, \beta_\sigma)$ with copairing α_σ as given below.

$$\begin{aligned}
\nabla_\sigma &: ABAB \xrightarrow{\sigma_{23}} AABB \xrightarrow{\nabla_A \nabla_B} AB, \\
\eta_\sigma &: \mathbb{1} \xrightarrow{\cong} \mathbb{1}\mathbb{1} \xrightarrow{\eta_A \eta_B} AB, \\
\beta_\sigma &: ABAB \xrightarrow{\sigma_{23}} AABB \xrightarrow{\beta_A \beta_B} \mathbb{1}\mathbb{1} \xrightarrow{\cong} \mathbb{1}, \\
\alpha_\sigma &: \mathbb{1} \xrightarrow{\cong} \mathbb{1}\mathbb{1} \xrightarrow{\alpha_A \alpha_B} AABB \xrightarrow{\sigma_{23}} ABAB.
\end{aligned}$$

Here, the subscript of σ is used because this is the trivial case of the warped tensor product.

We also use the following standard result, given as Example 3.2.31 in [Koc04].

Proposition 2.1. *The category of Frobenius algebras with the standard tensor product is a symmetric monoidal category.*

A notable corollary, which will be useful in Section 3, is that the standard tensor product of Frobenius algebras is a Frobenius algebra.

Finally, we state a pair of useful standard results. These are given for example in Section 2.3 in [Koc04], as the equivalence to another of the main definitions of Frobenius algebras (in terms of a counit).

Proposition 2.2. *Let $(A, \nabla_A, \eta_A, \beta_A)$ be a Frobenius algebra with copairing α_A. Then the following diagrams commute, and we use the blue arrow to denote these equivalent paths.*

$$\begin{array}{ccc} A & \xrightarrow{\alpha_A 1} & AAA \\ {\scriptstyle 1\alpha_A}\downarrow & \overset{\Delta_A}{\searrow} & \downarrow{\scriptstyle 1\nabla_A} \\ AAA & \xrightarrow{\nabla_A 1} & AA \end{array} \qquad\qquad \begin{array}{ccc} A & \xrightarrow{\eta_A 1} & AA \\ {\scriptstyle 1\eta_A}\downarrow & \overset{\epsilon_A}{\searrow} & \downarrow{\scriptstyle \beta_A} \\ AA & \xrightarrow{\beta_A} & \mathbb{1} \end{array} \tag{2.4}$$

The latter diagram also gives a nice way to express β_A.

Proposition 2.3. *Let $(A, \nabla_A, \eta_A, \beta_A)$ be a Frobenius algebra. Then the following diagram commutes.*

$$\begin{array}{ccc} AA & \xrightarrow{\nabla_A} & A \\ & \underset{\beta_A}{\searrow} & \downarrow{\scriptstyle \epsilon_A} \\ & & \mathbb{1} \end{array} \tag{2.5}$$

3. Warped Tensor Products

In this section, we define the warped tensor product and prove Theorem 1.1.

3.1. **Motivation.** An alternative, prominent definition of Frobenius algebras thinks of them as an algebra and coalgebra with the Frobenius associativity relation. A natural attempt to create new monoidal structures would be to apply the cotwisted tensor product, the dual notion of the well studied twisted tensor product. This would preserve the vector space, multiplication, unit, and counit, while it would twist the coproduct. However, as proved in [Koc04], the four preserved structures uniquely determine the coproduct, so the cotwisted tensor product would never work in the nontrivial case.

Therefore, we define a new concept: the warped tensor product.

Definition 3.1 (Warped Tensor Product)**.** Let $(A, \nabla_A, \eta_A, \beta_A)$ and $(B, \nabla_B, \eta_B, \beta_B)$ be Frobenius algebras with copairings α_A and α_B, respectively. Let $\gamma : B \otimes A \to A \otimes B$ be a morphism in $\mathcal{C}$ with a two-sided inverse; call this the **warp**. Define $\nabla_\gamma, \eta_\gamma, \beta_\gamma, \alpha_\gamma$ as

$$\begin{aligned}
\nabla_\gamma &: ABAB \xrightarrow{\sigma_{23}} AABB \xrightarrow{\nabla_A \nabla_B}, \\
\eta_\gamma &: \mathbb{1} \xrightarrow{\cong} \mathbb{1}\mathbb{1} \xrightarrow{\eta_A \eta_B} AB, \\
\beta_\gamma &: ABAB \xrightarrow{1_A \gamma 1_B} AABB \xrightarrow{\beta_A \beta_B} \mathbb{1}\mathbb{1} \xrightarrow{\cong} \mathbb{1}, \\
\alpha_\gamma &: \mathbb{1} \xrightarrow{\cong} \mathbb{1}\mathbb{1} \xrightarrow{\alpha_A \alpha_B} AABB \xrightarrow{1_A \gamma^{-1} 1_B} ABAB.
\end{aligned}$$

Denote by $A \otimes_\gamma B$ the tuple $(A \otimes B, \nabla_\gamma, \eta_\gamma, \beta_\gamma)$. Call this the **warped tensor product**.

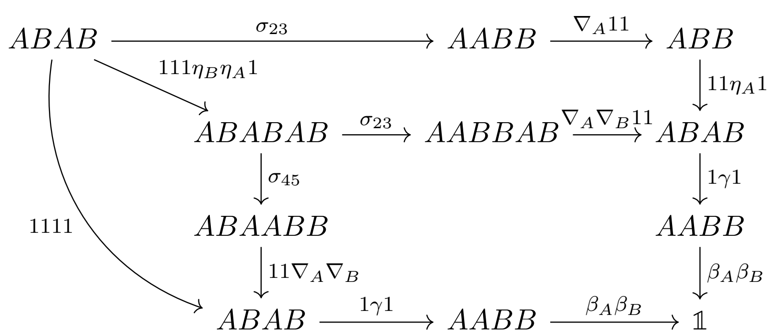
First, we show the necessity of the conditions given in Theorem 1.1.

3.2. **Necessity.** Let $(A, \nabla_A, \eta_A, \beta_A)$ and $(B, \nabla_B, \eta_B, \beta_B)$ be Frobenius algebras with copairings α_A and α_B, respectively. Let $\gamma : B \otimes A \to A \otimes B$ be a morphism in $\mathcal{C}$ with a two-sided inverse. Suppose that $A \otimes_\gamma B$ is a Frobenius algebra with copairing α_γ. Then β_γ is associative, and satisfies the nondegeneracy relation with α_γ.

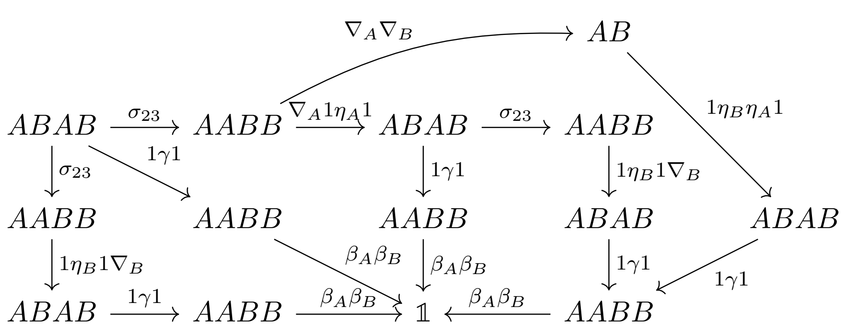
Lemma 3.1. *Suppose β_γ is an associative pairing. Then for some morphism $\psi : \mathbb{1} \to A \otimes B$ in $\mathcal{C}$,*

$$\gamma : BA \xrightarrow{\sigma_{12}} AB \xrightarrow{\cong} A\mathbb{1}B \xrightarrow{1_A \psi 1_B} AABB \xrightarrow{\nabla_A \nabla_B} AB.$$

Proof. Since β_γ is an associative pairing, the following diagram commutes.



Combining this with its flipped diagram (the bottom-left portion in the diagram below, which commutes analogously), the following diagram commutes.



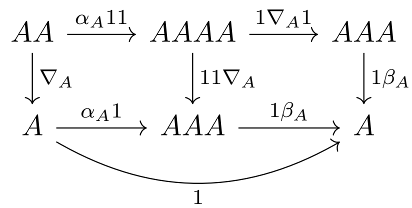
Focus on the clockwise and counterclockwise paths given in the below diagram.

$$\begin{array}{ccccccccc}
ABAB & \xrightarrow{\sigma_{23}} & AABB & \xrightarrow{\nabla_A \nabla_B} & AB & \xrightarrow{1\eta_B\eta_A 1} & ABAB & \xrightarrow{1\gamma 1} & AABB \\
\downarrow{\scriptstyle 1\gamma 1} & & & & & & & & \downarrow{\scriptstyle \beta_A\beta_B} \\
AABB & & & & \xrightarrow{\qquad\beta_A\beta_B\qquad} & & & & \mathbb{1}
\end{array} \tag{3.1}$$

Define $\psi : \mathbb{1} \to A \otimes B$ by

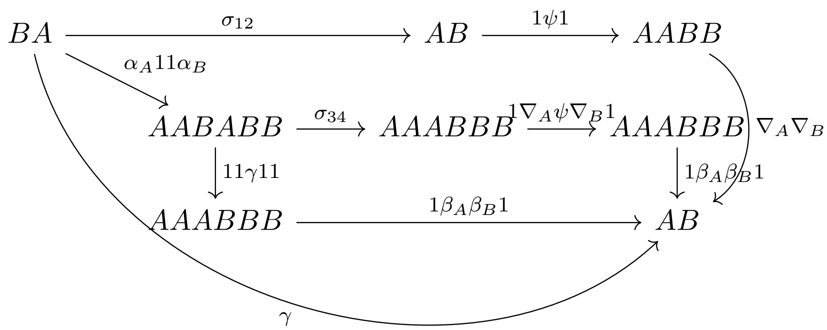
$$\psi : \mathbb{1} \xrightarrow{\cong} \mathbb{1}\mathbb{1} \xrightarrow{\eta_B \eta_A} BA \xrightarrow{\gamma} AB.$$

Intermediately, observe that the following diagram commutes by associativity and left-nondegeneracy.



An analogous but flipped condition holds on B with right-nondegeneracy.

Thus, adding copairings to Diagram 3.1, the following diagram commutes, concluding the proof.



□

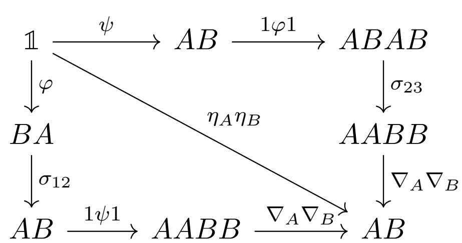
Now, we prove additional requirements using that γ has an inverse γ^{-1}.

Lemma 3.2. *Let (A, ∇_A, η_A) and (B, ∇_B, η_B) be Frobenius algebras, and let $\psi : \mathbb{1} \to A \otimes B$ be a morphism in $\mathcal{C}$. Define*

$$\gamma : BA \xrightarrow{\sigma_{12}} AB \xrightarrow{\cong} A\mathbb{1}B \xrightarrow{1_A \psi 1_B} AABB \xrightarrow{\nabla_A \nabla_B} AB.$$

Then γ has a two-sided inverse γ^{-1} if and only if for some morphism $\varphi : \mathbb{1} \to B \otimes A$ in $\mathcal{C}$, the following diagram commutes (which we call the "inverse condition").

(3.2)



Furthermore, the inverse is given by

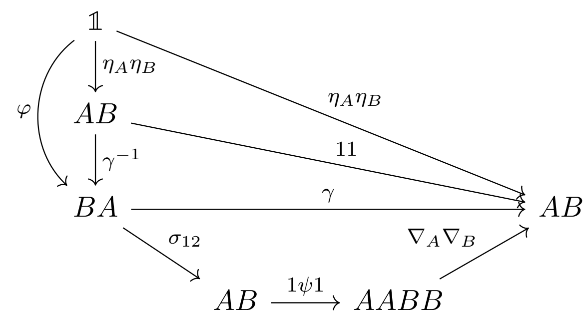
$$\gamma^{-1} : AB \xrightarrow{\cong} A\mathbb{1}B \xrightarrow{1_A \varphi 1_B} ABAB \xrightarrow{\sigma_{23}} AABB \xrightarrow{\nabla_A \nabla_B} AB \xrightarrow{\sigma_{12}} BA. \tag{3.3}$$

Proof. First, suppose γ has a two-sided inverse γ^{-1}.

Define

$$\varphi : \mathbb{1} \xrightarrow{\cong} \mathbb{1}\mathbb{1} \xrightarrow{\eta_B \eta_A} BA \xrightarrow{\gamma^{-1}} AB.$$

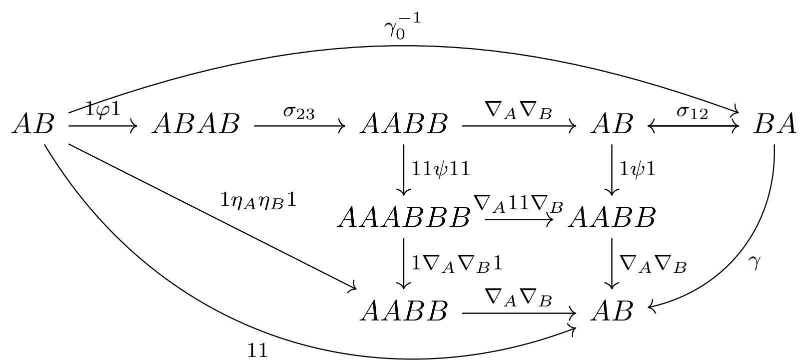
Then the following diagram commutes.



Define

$$\gamma_0^{-1} : AB \xrightarrow{\cong} A\mathbb{1}B \xrightarrow{1_A\varphi 1_B} ABAB \xrightarrow{\sigma_{23}} AABB \xrightarrow{\nabla_A\nabla_B} AB \xrightarrow{\sigma_{12}} BA.$$

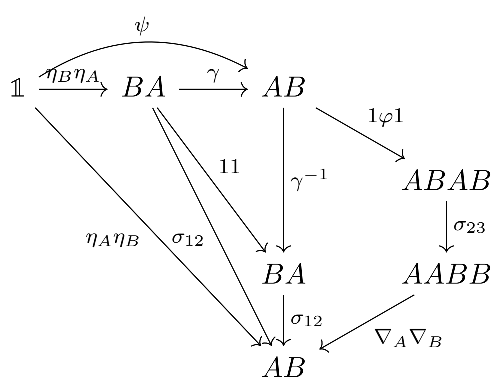
Then the following diagram commutes, so $\gamma \circ \gamma_0^{-1}$ is the identity on $A \otimes B$.



Using the standard argument, by associativity of composition,

$$\gamma_0^{-1} = (\gamma^{-1} \circ \gamma) \circ \gamma_0^{-1} = \gamma^{-1} \circ (\gamma \circ \gamma_0^{-1}) = \gamma^{-1}.$$

Next, the following diagram commutes, giving the final condition.

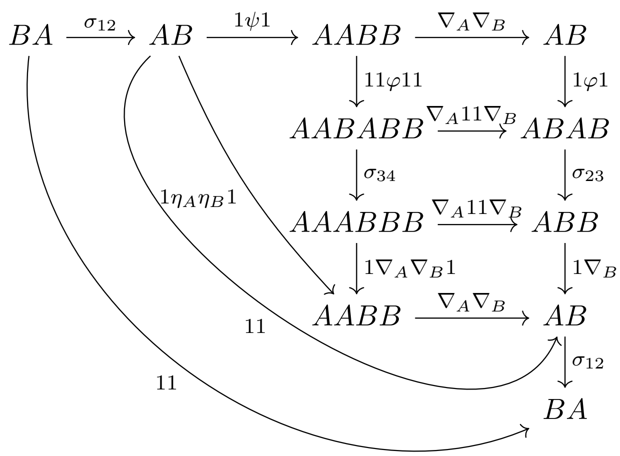


Now, suppose that for some morphism $\varphi : \mathbb{1} \to B \otimes A$ in $\mathcal{C}$, Diagram 3.2 commutes. Define

$$\gamma_0^{-1} : AB \xrightarrow{\cong} A\mathbb{1}B \xrightarrow{1_A\varphi 1_B} ABAB \xrightarrow{\sigma_{23}} AABB \xrightarrow{\nabla_A\nabla_B} AB \xrightarrow{\sigma_{12}} BA.$$

The commutativity of the same diagram we used for $\gamma \circ \gamma_0^{-1}$ in the proof of the other direction again implies that $\gamma \circ \gamma_0^{-1}$ is the identity, so γ_0^{-1} is an inverse on one side.

Also, $\gamma_0^{-1} \circ \gamma$ is the identity due to the commutativity of the following diagram, so γ_0^{-1} is indeed a two-sided inverse.



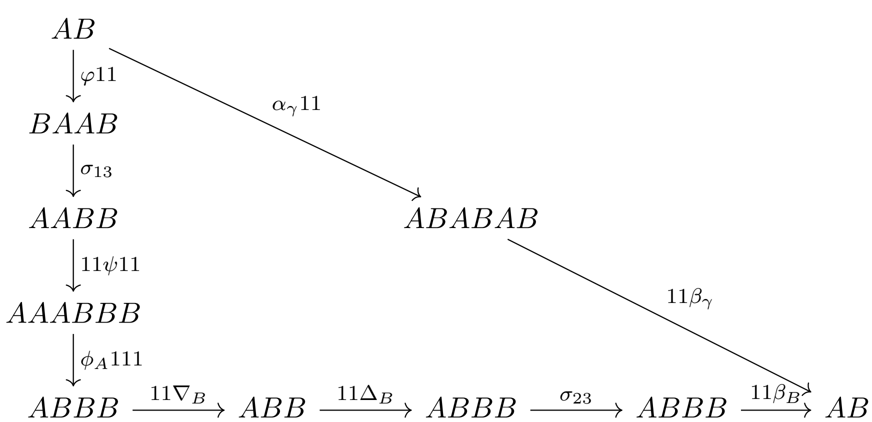
Therefore, if for some $\varphi : \mathbb{1} \to A \otimes B$, Diagram 3.2 commutes, then γ has a two sided inverse

$$\gamma^{-1} : AB \xrightarrow{\cong} A\mathbb{1}B \xrightarrow{1_A \varphi 1_B} ABAB \xrightarrow{\sigma_{23}} AABB \xrightarrow{\nabla_A \nabla_B} AB \xrightarrow{\sigma_{12}} BA.$$

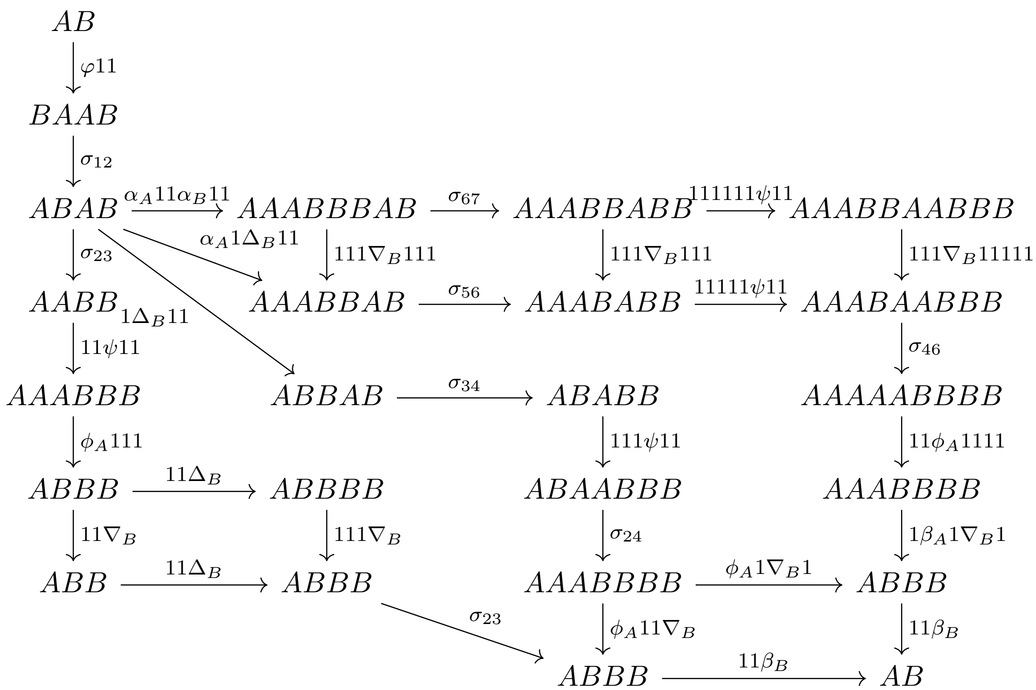
□

We pivot to the nondegeneracy relations. We must first make a simplification.

Lemma 3.3. *Consider Frobenius algebras $(A, \nabla_A, \eta_A, \beta_A)$ and $(B, \nabla_B, \eta_B, \beta_B)$, and let $\gamma : B \otimes A \to A \otimes B$ be a warp given by ψ as before. Let γ^{-1} be the inverse, given by φ as before. Then the following diagram commutes.*



Proof. In the following commutative diagram, the clockwise outer path is $(1_A \otimes 1_B \otimes \beta_\gamma) \circ (\alpha_\gamma \otimes 1_A \otimes 1_B)$, so we conclude.

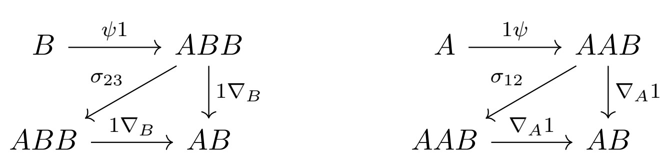


∎

Now, we show that the nondegeneracy relations imply conditions indicating a form of centrality; these will be revisited in Definition 3.2.

Lemma 3.4. *Let $(A, \nabla_A, \eta_A, \beta_A)$ and $(B, \nabla_B, \eta_B, \beta_B)$ be Frobenius algebras, and let $\gamma, \gamma^{-1}, \varphi, \psi$ be given as before. Suppose that $A \otimes_\gamma B$ is a Frobenius algebra with copairing α_γ (on both sides). Then the following diagrams commute.*

(3.4)

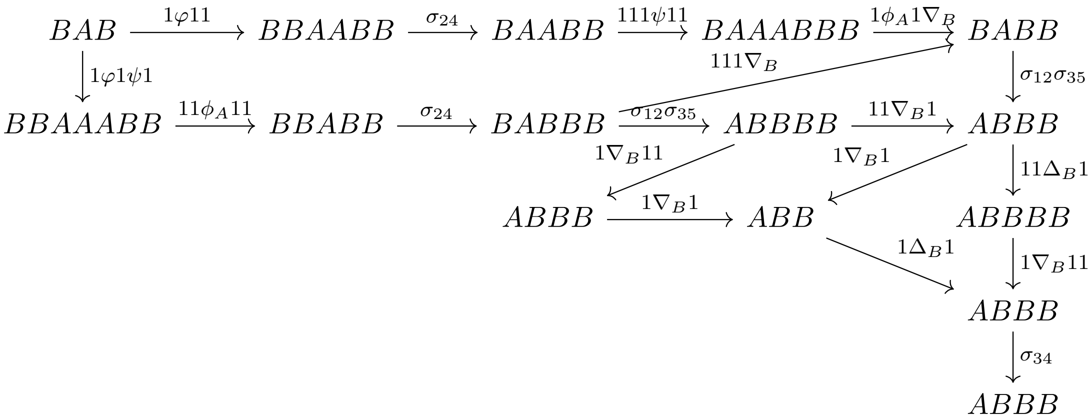


Proof. Assuming the nondegeneracy relation on $A \otimes_\gamma B$, the counterclockwise path given in Lemma 3.3 is the identity. For clarity in describing approach, let $\chi = 1_A \otimes 1_B$ denote this counterclockwise path. We first consider the composition

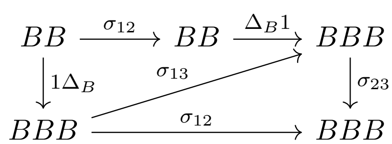
$$\chi_1 : B \xrightarrow{\cong} \mathbb{1}B \xrightarrow{\psi 1_B} ABB \xrightarrow{\sigma_{12}} BAB \xrightarrow{1_B \chi} BAB \xrightarrow{\sigma_{12}} ABB \xrightarrow{1_A \nabla_B} AB.$$

We aim to derive an intermediate result, with which we can simplify χ.

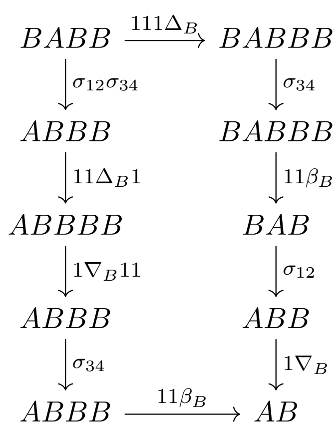
Due to the associativity of multiplication and the Frobenius associativity relation, the following diagram commutes.



Furthermore, because the following diagram commutes,



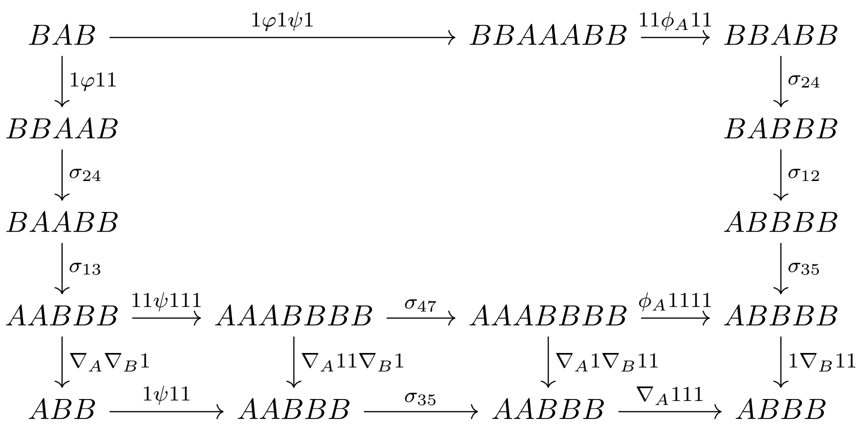
the following rectangle also commutes.



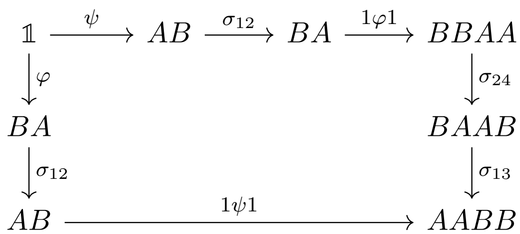
Overlaying this at the right of the prior diagram, the clockwise path is

$$BAB \xrightarrow{1_B\chi} BAB \xrightarrow{\sigma_{12}} ABB \xrightarrow{1_A\nabla_B} AB.$$

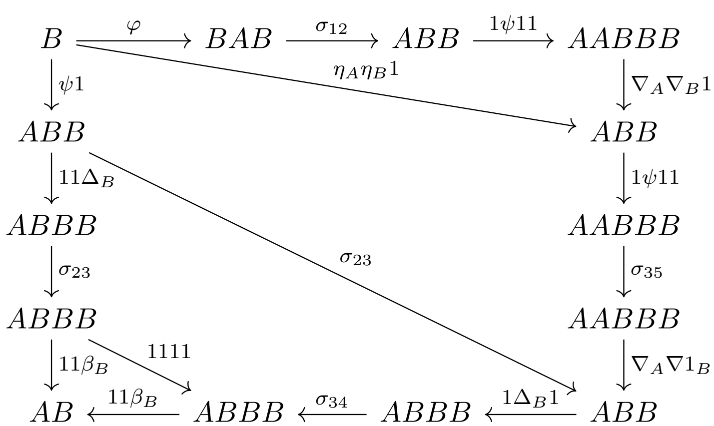
The following diagram commutes, essentially reordering the counterclockwise path in the overlaid diagram.



The counterclockwise path in the above diagram is much easier to work with, especially because part of it is related to the inverse conditions given in Diagram 3.2 by commutativity of the following diagram.



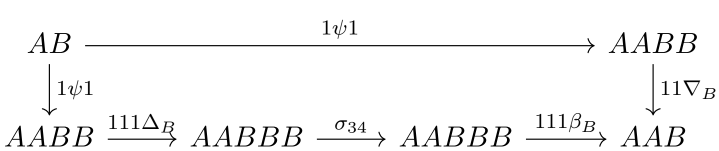
Collecting our progress so far, in the following diagram, which commutes by unitality and the inverse conditions given by Diagram 3.2, the clockwise path is χ_1.



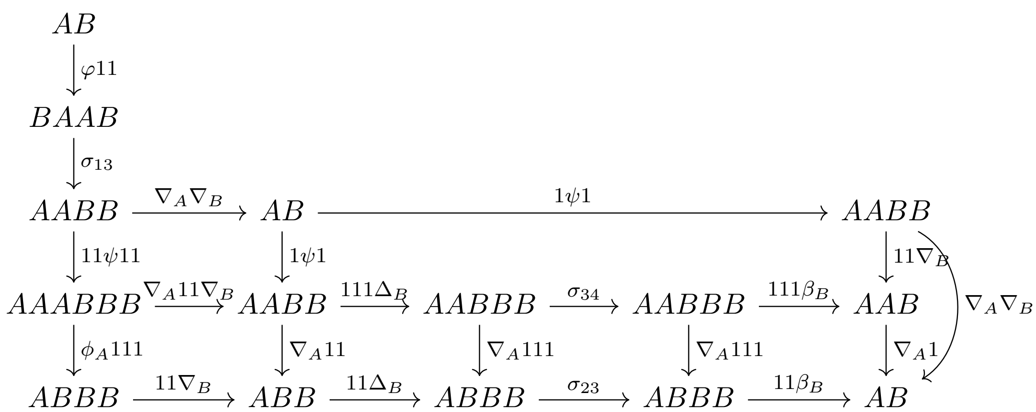
Since χ is just the identity by nondegeneracy, χ_1 is given by

$$\chi_1 : B \xrightarrow{\cong} \mathbb{1}B \xrightarrow{\psi 1_B} ABB \xrightarrow{1_A \nabla_B} AB.$$

Therefore, the commutativity of the above diagram implies the commutativity of the following diagram.



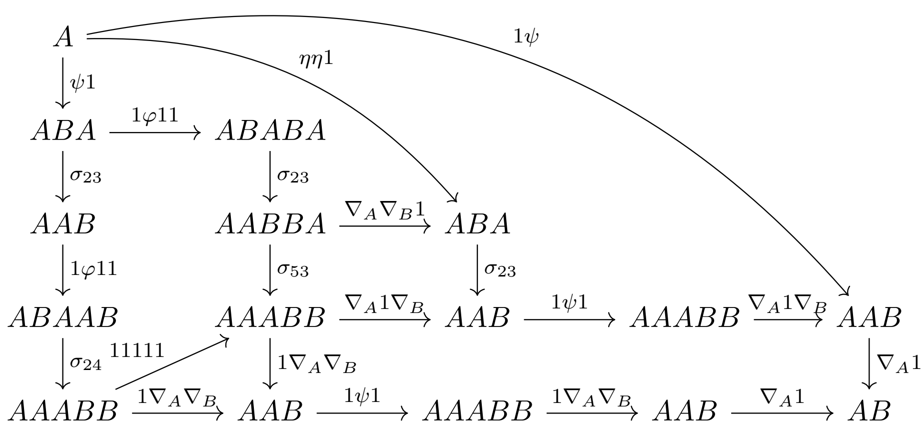
Now that we have this intermediate result, we can simplify χ due to the commutativity of the following diagram (where χ is the counterclockwise path).



Let χ' denote the clockwise path here; it is also the identity, but this distinction improves clarity. We now consider the composition

$$\chi_2 : A \xrightarrow{\cong} \mathbb{1}A \xrightarrow{\psi 1_A} ABA \xrightarrow{\sigma_{23}} AAB \xrightarrow{\chi'} AAB \xrightarrow{\nabla_A 1_B} AB.$$

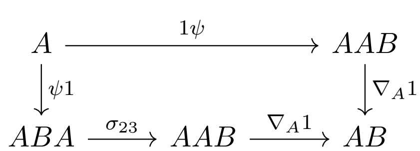
The following diagram commutes by associativity of multiplication and the inverse conditions given in Diagram 3.2.



The counterclockwise path here is χ_2. As before, since χ' is the identity, χ_2 is given by

$$\chi_2 : A \xrightarrow{\cong} \mathbb{1}A \xrightarrow{\psi 1_A} ABA \xrightarrow{\sigma_{23}} AAB \xrightarrow{\nabla_A 1_B} AB.$$

Therefore, the following diagram commutes.



This is almost the desired condition; to finish, note that the following diagram commutes by naturality.

$$\begin{array}{ccc} A & \xrightarrow{1\psi} & AAB \\ \big\downarrow{\scriptstyle \psi 1} & \overset{\sigma_{31}}{\nearrow} & \big\downarrow{\scriptstyle \sigma_{12}} \\ ABA & \xrightarrow{\sigma_{23}} & AAB \end{array} \tag{3.5}$$

Combining these, we finally reach the first of the two conditions claimed in Diagram 3.4. The other follows analogously; in particular, it is recovered by reversing the order of tensor products in each object and morphism in the above proof, and swapping A with B. ☐

So far, we have shown that a form of inverse and centrality conditions are necessary. These are not the same as the ones given in Theorem 1.1, but as demonstrated in the next subsection, these are enough to imply the desired conditions.

3.3. Centrality Conditions.

Let Frobenius algebras $(A, \nabla_A, \eta_A, \beta_A)$ and $(B, \nabla_B, \eta_B, \beta_B)$ be Frobenius algebras with copairings α_A and α_B, respectively. Let $\psi : \mathbb{1} \to A \otimes B$ and $\varphi : \mathbb{1} \to B \otimes A$ be morphisms in $\mathcal{C}$, and consider γ and γ^{-1} given by

$$\gamma : BA \xrightarrow{\sigma_{12}} AB \xrightarrow{\cong} A\mathbb{1}B \xrightarrow{1_A \psi 1_B} AABB \xrightarrow{\nabla_A \nabla_B} AB,$$

and

$$\gamma^{-1} : AB \xrightarrow{\cong} A\mathbb{1}B \xrightarrow{1_A \varphi 1_B} ABAB \xrightarrow{\sigma_{23}} AABB \xrightarrow{\nabla_A \nabla_B} AB \xrightarrow{\sigma_{12}} BA.$$

The conditions forced by the nondegeneracy relations as in the prior subsection essentially say that ψ and φ are central, and this allows for the simplification of many previous diagrams and operations. In this regard, it inconvenient that φ maps to $B \otimes A$ instead of $A \otimes B$; accordingly, define

$$\varphi' : \mathbb{1} \xrightarrow{\varphi} BA \xrightarrow{\sigma_{12}} AB.$$

First, the centrality conditions should be restated and related to what we have shown is necessary.

Definition 3.2. Let $(A, \nabla_A, \eta_A, \beta_A)$ and $(B, \nabla_B, \eta_B, \beta_B)$ be Frobenius algebras, and let $\psi : \mathbb{1} \to A \otimes B$ and $\varphi : \mathbb{1} \to B \otimes A$ be morphisms in $\mathcal{C}$. Call the commutativity of the following two diagrams the **condensed centrality conditions**,

$$\begin{array}{ccc} AB & \xrightarrow{11\psi} & ABAB \\ \big\downarrow{\scriptstyle \psi 11} & & \big\downarrow{\scriptstyle \nabla_\gamma} \\ ABAB & \xrightarrow{\nabla_\gamma} & AB \end{array} \qquad \begin{array}{ccc} AB & \xrightarrow{11\varphi'} & ABAB \\ \big\downarrow{\scriptstyle \varphi' 11} & & \big\downarrow{\scriptstyle \nabla_\gamma} \\ ABAB & \xrightarrow{\nabla_\gamma} & AB \end{array}$$

and call the commutativity of the following four diagrams the **decomposed centrality conditions**.

$$\begin{array}{ccc} B & \xrightarrow{\psi 1} & ABB \\ & \overset{\sigma_{23}}{\swarrow} & \big\downarrow{\scriptstyle 1\nabla_B} \\ ABB & \xrightarrow{1\nabla_B} & AB \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{1\psi} & AAB \\ & \overset{\sigma_{12}}{\swarrow} & \big\downarrow{\scriptstyle \nabla_A 1} \\ AAB & \xrightarrow{\nabla_A 1} & AB \end{array}$$

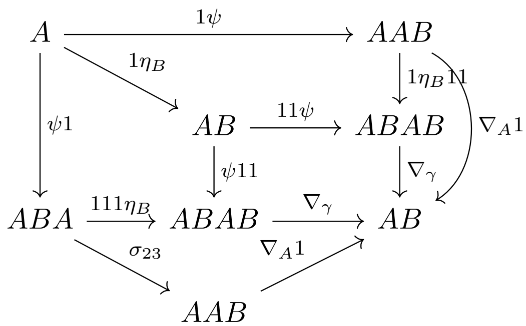
$$\begin{array}{ccc} B & \xrightarrow{\varphi' 1} & ABB \\ & \overset{\sigma_{23}}{\swarrow} & \big\downarrow{\scriptstyle 1\nabla_B} \\ ABB & \xrightarrow{1\nabla_B} & AB \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{1\varphi'} & AAB \\ & \overset{\sigma_{12}}{\swarrow} & \big\downarrow{\scriptstyle \nabla_A 1} \\ AAB & \xrightarrow{\nabla_A 1} & BA \end{array}$$

Lemma 3.5. *The two decomposed centrality conditions relating to ψ are equivalent to the condensed centrality condition for ψ, and the same holds for φ'.*

Proof. We prove this for ψ; the result follows analogously for φ'.

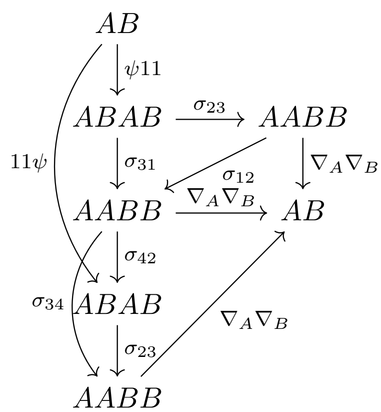
First, assume the condensed centrality condition. We show that the decomposed centrality condition for A holds, and it follows analogously for B.

Due to the condensed centrality condition and unitality, the following diagram commutes.



Recalling that Diagram 3.5 commutes, this implies the decomposed centrality condition on A.

Now, assume the decomposed centrality conditions for ψ. The condensed centrality condition holds by the commutativity of the following diagram.

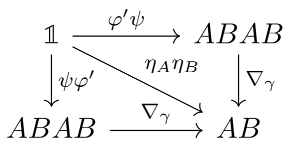


∎

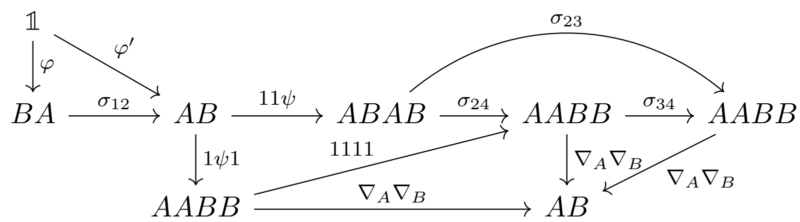
Throughout the rest of the section, assume the condensed centrality condition for ψ, because the two decomposed conditions follow due to Lemma 3.4.

Now, we rewrite the inverse relations on ψ and φ.

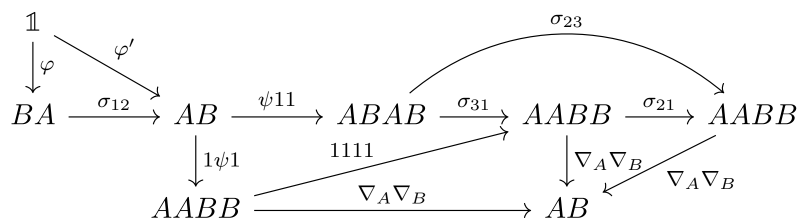
Lemma 3.6. *Let $(A, \nabla_A, \eta_A, \beta_A)$ and $(B, \nabla_B, \eta_B, \beta_B)$ be Frobenius algebras, and let ψ, φ, and φ' be defined as before. Suppose that the condensed centrality condition on ψ holds. Then the inverse relations on ψ and φ in Diagram 3.2 imply to the commutativity of the following diagram.*



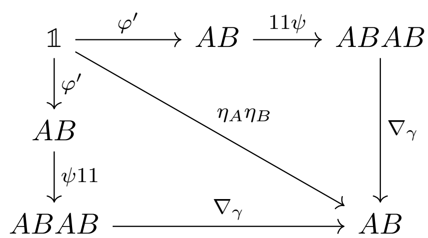
Proof. By the decomposed centrality conditions on ψ, the following diagram commutes.



Similarly, the following diagram also commutes.



The inverse relations on ψ and φ in Diagram 3.2 state that the counterclockwise paths are just $\mathbb{1} \xrightarrow{\cong} \mathbb{1}\mathbb{1} \xrightarrow{\eta_A\eta_B} AB$, so the following diagram commutes, concluding the proof.

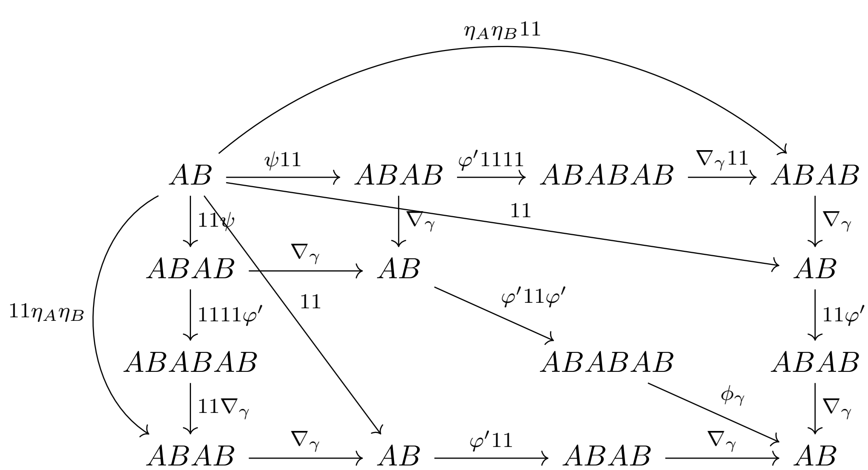


$\square$

Using the new inverse relations from the above result, the centrality condition on φ' can be recovered.

Lemma 3.7. *Consider Frobenius algebras $(A, \nabla_A, \eta_A, \beta_A)$ and $(B, \nabla_B, \eta_B, \beta_B)$, and let ψ, φ, and φ' be defined as before. Suppose that the condensed centrality condition on ψ holds. Then the condensed centrality condition on φ' holds.*

Proof. The condensed centrality condition on φ' follows by essentially multiplying the centrality condition on ψ by φ' on both sides and using associativity, explicitly given by the commutativity of the following diagram.



□

By now, we have fully shown that the inverse and centrality conditions as formulated in Theorem 1.1 are necessary. In the remainder of this subsection, we reformulate key maps assuming the inverse and centrality conditions but not necessarily that $A \otimes_\gamma B$ is a Frobenius algebra. This will vastly simplify the proof of sufficiency, and it will also help to better understand what the possibilities for $A \otimes_\gamma B$ look like.

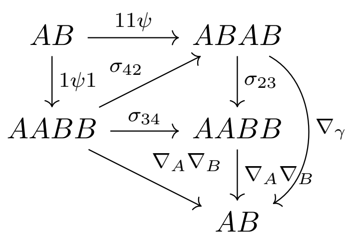
Lemma 3.8. *Consider Frobenius algebras $(A, \nabla_A, \eta_A, \beta_A)$ and $(B, \nabla_B, \eta_B, \beta_B)$, and let γ, γ', ψ, φ, and φ' be defined as before. Suppose that the condensed centrality condition on ψ holds. Then the warp is given by*

$$\gamma : BA \xrightarrow{\sigma_{12}} AB \xrightarrow{\cong} AB\mathbb{1} \xrightarrow{1_A 1_B \psi} ABAB \xrightarrow{\nabla_\gamma} AB,$$

and its inverse is given by

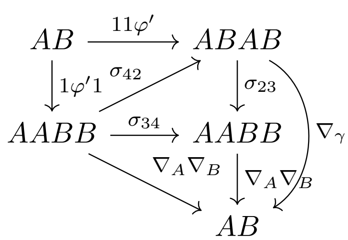
$$\gamma^{-1} : AB \xrightarrow{\cong} AB\mathbb{1} \xrightarrow{1_A 1_B \varphi'} ABAB \xrightarrow{\nabla_\gamma} AB \xrightarrow{\sigma_{12}} BA.$$

Proof. To reformulate γ, first note that the following diagram commutes by the decomposed centrality conditions on ψ.



Because γ is the counterclockwise path precomposed with σ_{12}, we conclude.

The proof for γ^{-1} is similar: the following diagram commutes by the decomposed centrality conditions on φ', and γ^{-1} is the counterclockwise path composed with σ_{12}.



□

Note that due to the condensed centrality conditions, it is equivalent to say

$$\gamma : BA \xrightarrow{\sigma_{12}} AB \xrightarrow{\cong} \mathbb{1}AB \xrightarrow{\psi 1_A 1_B} ABAB \xrightarrow{\nabla_\gamma} AB,$$

and similar for γ^{-1}.

With this new formulation of γ and γ^{-1}, the pairing and the copairing can also be rewritten. To do this, we use that the standard tensor product $(A \otimes B, \nabla_\sigma, \eta_\sigma, \beta_\sigma)$ is a Frobenius algebra with copairing α_σ, as stated in Proposition 2.1. By definition, ∇_σ is the same as ∇_γ, and η_σ is the same as η_γ. This will help in reformulating β_γ and α_γ, which is done below.

Lemma 3.9. *Consider Frobenius algebras $(A, \nabla_A, \eta_A, \beta_A)$ and $(B, \nabla_B, \eta_B, \beta_B)$ with copairings α_A and α_B, respectively. Let $\psi : \mathbb{1} \to A \otimes B$ and $\varphi' : \mathbb{1} \to A \otimes B$ be morphisms in $\mathcal{C}$ as before, satisfying* *the inverse and centrality conditions. Define γ and γ^{-1} as before, or equivalently as in Lemma 3.8. Then the pairing β_γ is given by*

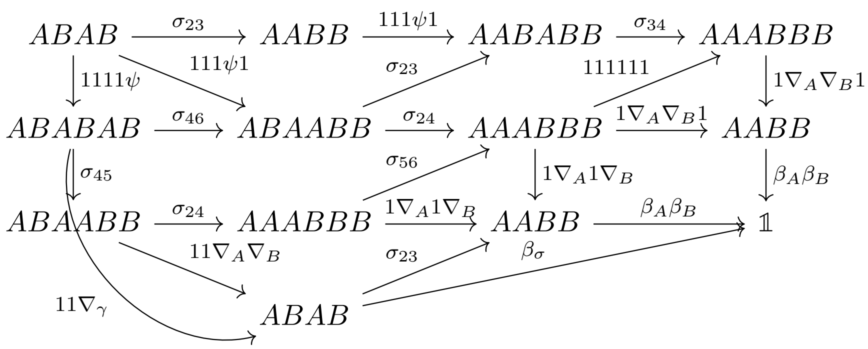
$$\beta_\gamma : ABAB \xrightarrow{\cong} ABAB\mathbb{1} \xrightarrow{1_A 1_B 1_A 1_B \psi} ABABAB \xrightarrow{1_A 1_B \nabla_\gamma} ABAB \xrightarrow{\beta_\sigma} \mathbb{1},$$

and the copairing α_γ is given by

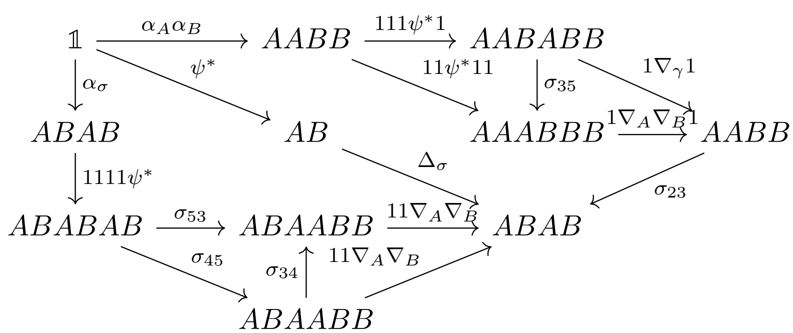
$$\alpha_\gamma : \mathbb{1} \xrightarrow{\alpha_\sigma} ABAB \xrightarrow{\cong} ABAB\mathbb{1} \xrightarrow{1_A 1_B 1_A 1_B \varphi'} ABABAB \xrightarrow{1_A 1_B \nabla_\gamma} ABAB.$$

Proof. For β_γ, we will use the equivalent formulations of γ given in Lemma 3.8

The following diagram commutes by the associativity of β_σ and the centrality conditions on ψ. Because the clockwise path is β_γ, it gives the desired reformulation.



For α_γ, we will use the original formulation of γ^{-1} given in Equation 3.3. Because Diagram 2.4 commutes for Frobenius algebras, the following diagram commutes.



□

Again, the $\mathbb{1}$ factor can be placed on the left here. This does not immediately follow from the centrality conditions, so a proof is included in Lemma 3.10 in Section 3.4.

So far, we have proved that if the warped tensor product of two Frobenius algebras is Frobenius, then all of the conditions listed in Theorem 1.1 are necessary. The map φ' used so far corresponds to ψ^* in the theorem.

3.4. **Sufficiency.** We now show that the conditions given in Theorem 1.1 sufficient for the warped tensor product of two Frobenius algebras to be Frobenius.

Consider Frobenius algebras $(A, \nabla_A, \eta_A, \beta_A)$ and $(B, \nabla_B, \eta_B, \beta_B)$ with copairings α_A and α_B, respectively. Let C denote $A \otimes B$. Let $\psi : \mathbb{1} \to C$ and $\psi^* : \mathbb{1} \to C$ be morphisms in $\mathcal{C}$ satisfying the centrality and inverse conditions as in Theorem 1.1. Define γ and γ^{-1} as in Lemma 3.8.

The tuple $(C, \nabla_\gamma, \eta_\gamma)$ is the standard tensor product of (A, ∇_A, η_A) and (B, ∇_B, η_B). Since the tensor product of algebras is an algebra, $A \otimes_\gamma B$ indeed has an algebra structure. Therefore, to be a Frobenius algebra, only the associativity and nondegeneracy of the pairing β_γ must be satisfied.

Recall from Lemma 3.9 that the pairing and copairing are given by

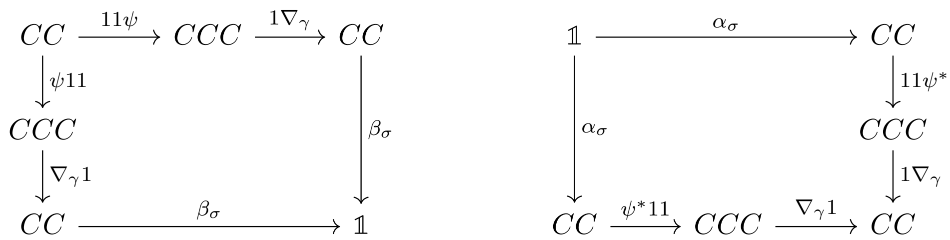
$$\beta_\gamma : CC \xrightarrow{\cong} CC\mathbb{1} \xrightarrow{1_C 1_C \psi} CCC \xrightarrow{1_C \nabla_\gamma} CC \xrightarrow{\beta_\sigma} \mathbb{1},$$

and

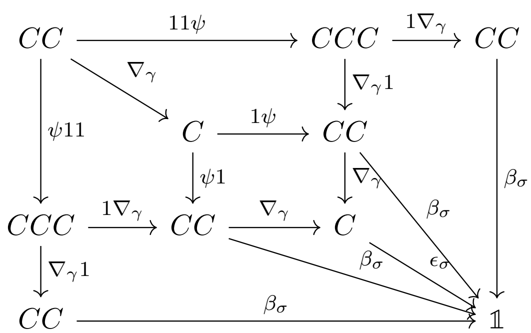
$$\alpha_\gamma : \mathbb{1} \xrightarrow{\alpha_\sigma} CC \xrightarrow{\cong} CC\mathbb{1} \xrightarrow{1_C 1_C \psi^*} CCC \xrightarrow{1_C \nabla_\gamma} CC,$$

and recall that the $\mathbb{1}$ factor can be placed on the other side without changing anything (as formally stated below).

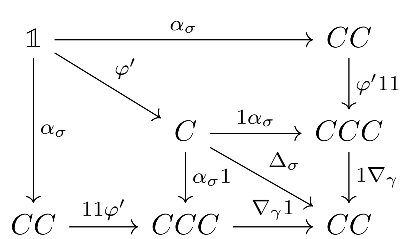
Lemma 3.10. *The following two diagrams commute.*



Proof. Recall from Proposition 2.1 that $(C, \nabla_\gamma, \eta_\gamma, \beta_\sigma)$ is a Frobenius algebra with copairing α_σ. Let Δ_σ and ϵ_σ be as given by Proposition 2.4. Then using the associativity of the pairing and the definition of the pairing given in Proposition 2.3, the following diagram commutes, giving the first result.



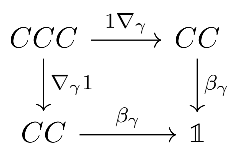
Using Diagram 2.4, the following diagram commutes, giving the other result.



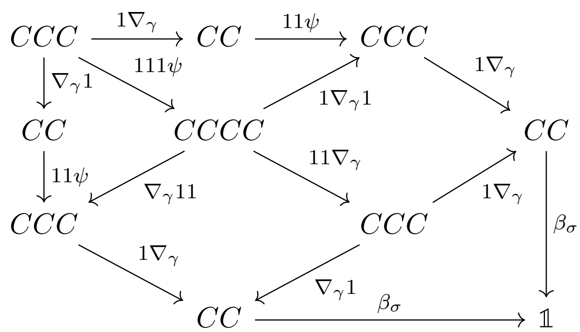
□

Finally, we show these conditions are sufficient to guarantee that $A \otimes_\gamma B$ is a Frobenius algebra.

Lemma 3.11. *The pairing β_γ is associative, so the following diagram commutes.*



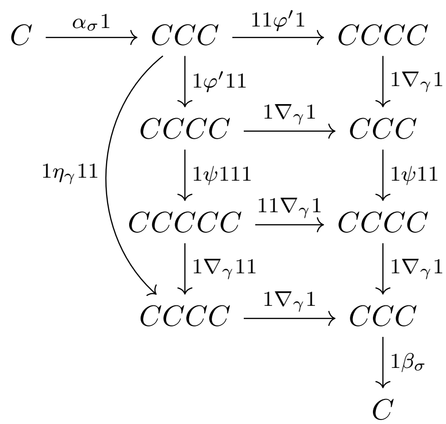
Proof. Because β_σ is an associative pairing and ∇_γ is an associative multiplication, the following diagram commutes, concluding.



□

Lemma 3.12. *The pairing β_γ is nondegenerate with copairing α_γ, so the following two diagrams commute.*

Proof. By the associativity of ∇_γ and the inverse and centrality conditions, the following diagram commutes.



The clockwise path is $(1_C \otimes \beta_\gamma) \circ (\alpha_\gamma \otimes 1_C)$, while by unitality, the counterclockwise path is $(1_C \otimes \beta_\sigma) \circ (\alpha_\sigma \otimes 1_C)$. Since β_σ is nondegenerate with copairing α_σ, this shows the first nondegeneracy relation.

The second diagram follows analogously – in particular, by flipping the order of the tensor products in each object and morphism in the diagram, since due to Lemma 3.10, the clockwise path is still $(\beta_\gamma \otimes 1_C) \circ (1_C \otimes \alpha_\gamma)$. □

Therefore, the conditions listed in Theorem 1.1 are sufficient to ensure that the warped tensor product of Frobenius algebras is Frobenius, concluding the proof.

4. Implications of Theorem 1.1

In this section, we discuss important consequences of Theorem 1.1.

4.1. **Properties preserved by warps.** We state the definitions of important properties of Frobenius algebras and classify when the warped tensor product preserves them.

Throughout this subsection, let $(A, \nabla_A, \eta_A, \beta_A)$ and $(B, \nabla_B, \eta_B, \beta_B)$ be Frobenius algebras. Call the standard tensor product $(A \otimes B, \nabla_\sigma, \eta_\sigma, \beta_\sigma)$. Let $\gamma = \Upsilon \circ \sigma$ be a warp satisfying the conditions in Theorem 1.2, where

$$\Upsilon : AB \xrightarrow{\cong} \mathbb{1}AB \xrightarrow{\psi 11} ABAB \xrightarrow{\nabla_\sigma} AB.$$

Call the warped tensor product's pairing β_γ and let σ_{AB} denote the braiding on $(AB)(AB)$.

First consider commutativity. The proof itself is less interesting; since this is a property of algebras, it is essentially the same as for the standard tensor product. However, preservation of commutativity is the most important, because it shows that the new symmetric monoidal structures are closed over commutative Frobenius algebras, which are equivalent to TQFTs.

Definition 4.1. A Frobenius algebra $(A, \nabla_A, \eta_A, \beta_A)$ is **commutative** if the following diagram commutes.

$$\begin{tikzcd} A \otimes A \arrow[r, "\sigma_{12}"] \arrow[dr, "\nabla_A"'] & A \otimes A \arrow[d, "\nabla_A"] \\ & A \end{tikzcd}$$

Proposition 4.1. *If Frobenius algebras A and B (as written earlier) are commutative, then $A \otimes_\gamma B$ is commutative.*

Proof. Since multiplication is the same as for the standard tensor product, the proof works the same way. Because the following diagram commutes (where the counterclockwise path is ∇_σ), $A \otimes_\gamma B$ is commutative.

$$\begin{tikzcd} ABAB \arrow[r, "\sigma_{14}"] \arrow[rr, bend left, "\sigma_{AB}"] \arrow[d, "\sigma_{23}"] & BABA \arrow[r, "\sigma_{14}"] & ABAB \arrow[d, "\sigma_{23}"] \arrow[dd, bend left=60, "\nabla_\sigma"] \\ AABB \arrow[r, "\sigma_{12}"] \arrow[drr, "\nabla_A \nabla_B"'] & AABB \arrow[r, "\sigma_{34}"] \arrow[dr, "\nabla_A \nabla_B"] & AABB \arrow[d, "\nabla_A \nabla_B"] \\ & & AB \end{tikzcd}$$

□

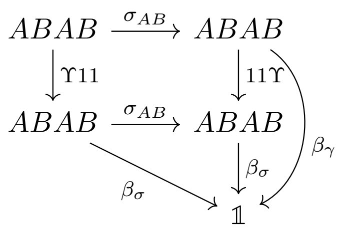
The next property is symmetry, a generaliztion of commutativity.

Definition 4.2. A Frobenius algebra $(A, \nabla_A, \eta_A, \beta_A)$ is **symmetric** if the following diagram commutes.

$$\begin{tikzcd} A \otimes A \arrow[r, "\sigma_{12}"] \arrow[dr, "\beta_A"'] & A \otimes A \arrow[d, "\beta_A"] \\ & \mathbb{1} \end{tikzcd}$$

Proposition 4.2. *If Frobenius algebras A and B are symmetric, then $A \otimes_\gamma B$ is symmetric.*

Proof. As with commutativity, the standard tensor product of two symmetric Frobenius algebras is symmetric. The proof is standard and similar as before, so we omit it. Then, the the result follows because the following diagram commutes.

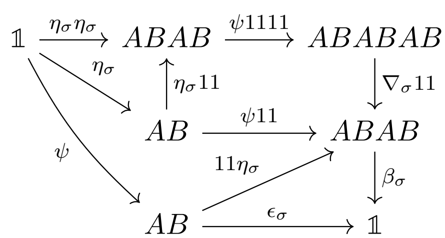


∎

Definition 4.3. A Frobenius algebra $(A, \nabla_A, \eta_A, \beta_A)$ is **special** if $\beta_A \circ (\eta_A \otimes \eta_A) = 1$.

Proposition 4.3. *Let $\epsilon_\sigma = \beta_\sigma \circ (1 \otimes \eta_\sigma)$. If Frobenius algebras A and B are special, then $A \otimes_\gamma B$ is special if and only if $\epsilon_\sigma \circ \psi = 1$.*

Proof. Again, we omit the proof that the standard tensor product $A \otimes_\sigma B$ is special. The following diagram commutes (where the clockwise path is $\beta_\gamma \circ (\eta_\sigma \otimes \eta_\sigma)$), so $A \otimes_\gamma B$ is special.



∎

The condition $\epsilon_\sigma \circ \psi = 1$ can not be simplified much further; in particular, if ψ_0 has $\epsilon_\sigma \circ \psi_0 \neq 0$, then $\psi = \frac{\psi_0}{\epsilon_\sigma \circ \psi_0}$ would have $\epsilon_0 \circ \psi = 1$.

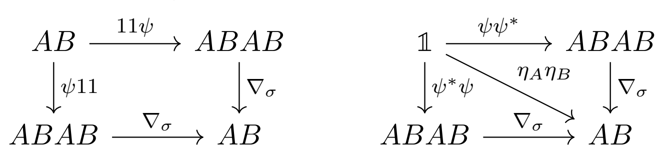
4.2. **New symmetric monoidal structures.** We ask when the warped tensor product forms a symmetric monoidal structure. If we instead were to consider a general monoidal structure, the condition $\beta_{A \boxtimes B} = \beta_{B \boxtimes A}$ would be omitted, so $\Upsilon_{A,B} = \Upsilon_{B,A}$ would no longer be necessary. In the family of examples provided in Section 4.3, $\cong$ would not involve the trivial twisting map σ anymore, so it would denote equivalence up to associativity and unitality.

We introduce terminology to describe classes $\psi_{A,B}$ that create valid warps.

Definition 4.4 (Warpable Classes)**.** A class of morphisms $\psi_{A,B} : \mathbb{1} \to A \otimes B$ is **warpable** if defining

$$\Upsilon_{A,B} = AB \xrightarrow{\cong} AB\mathbb{1} \xrightarrow{11\psi_{A,B}} ABAB \xrightarrow{\nabla_\sigma} AB,$$

there exists a class of morphisms $\psi^*_{A,B} : \mathbb{1} \to A \otimes B$ such that the below diagrams commute.



In the above diagrams, ψ is short for $\psi_{A,B}$, and ψ^* similarly.

Now, consider some warpable class $\psi_{A,B}$ and the corresponding $\Upsilon_{A,B}$.

As before, suppress the isomorphism constraints of $\mathcal{C}$ as a monoidal category with $\otimes$.

Question 4.4. *When does $\boxtimes$ respect the suppressed constraints? In other words, when are $\beta_{A \boxtimes (B \boxtimes C)} = \beta_{(A \boxtimes B) \boxtimes C}$, $\beta_{I \boxtimes A} = \beta_A$, $\beta_{A \boxtimes I} = \beta_A$, and $\beta_{A \boxtimes B} = \beta_{B \boxtimes A}$?*

Recall that $\beta_{A\boxtimes B} = \beta_{A\otimes B} \circ (1 \otimes \Upsilon_{A,B})$.

Suppose some morphisms $f, g : C \to C$ satisfy $\beta_C \circ (1 \otimes f) = \beta_C \circ (1 \otimes g)$. Then

$$f = (1 \otimes \beta_C) \circ (\alpha_C \otimes f) = (1 \otimes \beta_C) \circ (\alpha_C \otimes g) = g.$$

Clearly $f = g$ implies the former condition. Therefore, $\beta_C \circ (1 \otimes f) = \beta_C \circ (1 \otimes g)$ if and only if $f = g$.

For the first condition,

$$\beta_{A\boxtimes(B\boxtimes C)} = \beta_{A\otimes(B\boxtimes C)} \circ (1 \otimes \Upsilon_{A,B\boxtimes C}) = \beta_{A\otimes(B\otimes C)} \circ (1 \otimes (\Upsilon_{B,C}\Upsilon_{A,B\boxtimes C})),$$

and similarly

$$\beta_{(A\boxtimes B)\boxtimes C} = \beta_{(A\otimes B)\otimes C} \circ (1 \otimes (\Upsilon_{A,B}\Upsilon_{A\boxtimes B,C})).$$

Then $\beta_{A\boxtimes(B\boxtimes C)} = \beta_{(A\boxtimes B)\boxtimes C}$ when $\Upsilon_{B,C}\Upsilon_{A,B\boxtimes C} = \Upsilon_{A,B}\Upsilon_{A\boxtimes B,C}$. Proceeding similarly for the other conditions, the resulting conditions are:

$$\Upsilon_{B,C}\Upsilon_{A,B\boxtimes C} = \Upsilon_{A,B}\Upsilon_{A\boxtimes B,C}; \quad \Upsilon_{I,A} = 1; \quad \Upsilon_{A,I} = 1; \quad \Upsilon_{A,B} = \Upsilon_{B,A}.$$

This answers Question 4.4 and concludes the proof of Theorem 1.2.

4.3. **A family of solutions.** For two Frobenius algebras A and B, say that $A \cong B$ if they can be related solely by the suppressed isomorphism constraints. Define this for algebra in the same way.

Also, let F denote the forgetful functor $\mathrm{F} : \mathrm{Frob}_{\mathcal{C}} \to \mathrm{Alg}_{\mathcal{C}}$. This functor essentially outputs the same vector space, multiplication, and unit, forgetting the additional structure of the pairing.

Consider a collection of maps $\theta_A : \mathbb{1} \to A$ satisfying the same conditions as $\psi_{A,B}$, and let θ_A^* denote the "multiplicative inverses." Then define

$$\varphi_A : A \xrightarrow{\cong} A\mathbb{1} \xrightarrow{1\theta_A} AA \xrightarrow{\nabla_A} A.$$

Suppose that $\varphi_A = \varphi_B$ whenever $\mathrm{F}(A) \cong \mathrm{F}(B)$. Then a family of working ψ is given by the multiplication $\psi_{A,B} = \theta_{A\otimes B}\theta_A^*\theta_B^*$.

This family is a concrete example of a new multiplicative structure on Frobenius algebras. While there are not any clear special properties about this family in particular, it gives assurance that although the conditions listed in Theorem 1.2 are still complicated, they have nontrivial solutions.

The condition that $\mathrm{F}(A) \cong \mathrm{F}(B)$ is much more restrictive than that $A \cong B$, but is used here to ensure that $\theta_{A\otimes B} = \theta_{A\boxtimes B}$. As a future direction, we pose the question of whether such a condition is necessary in general.

Question 4.5. *Are there any classes of morphisms ψ inducing a symmetric monoidal structure which detect the Frobenius structure? In other words, does there exist a warpable class ψ which satisfies the conditions in Theorem 1.2 such that for some Frobenius algebras A, B, and C, $\mathrm{F}(A) \cong \mathrm{F}(B)$ but $\psi_{A,C} \neq \psi_{B,C}$?*

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(Das) Basis Independent Silicon Valley, San Jose, CA 95126, USA