ON FINITARY POWER MONOIDS OF LINEARLY ORDERABLE MONOIDS

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Abstract. A commutative monoid M is called a linearly orderable monoid if there exists a total order on M that is compatible with its operation. The finitary power monoid of a commutative monoid M is the monoid consisting of all nonempty finite subsets of M under the so-called sumset. In this paper, we investigated whether certain atomic and divisibility properties ascend from linearly orderable monoids to their corresponding finitary power monoids.

1. Introduction

Let M be a commutative monoid (i.e., a commutative semigroup with an identity element). The power monoid $\mathscr{P}(M)$ of M is the commutative monoid consisting of all nonempty subsets of M under the so-called sumset or Minkowski sum: for any nonempty subsets S and T of M,

$$S + T := \{s + t : s \in S \text{ and } t \in T\}.$$

The finitary power monoid of M, denoted here by $\mathscr{P}_{\text{fin}}(M)$, is the submonoid of $\mathscr{P}(M)$ consisting of all finite nonempty subsets of M. Both power monoids and finitary power monoids have been investigated in the literature of semigroup theory from several decades. For instance, see [25] and its references for results until the eighties, and see [27] and its references for more recent results. In the scope of this paper, the algebraic objects we are interested in are finitary power monoids of linearly orderable monoids (i.e., commutative monoids that can be endowed with a total order compatible with their corresponding operations).

In the setting of power monoids, one problem that has received a great deal of attention is the isomorphism problem: this is the problem of deciding, given a class $\mathscr{C}$ of commutative monoids, whether non-isomorphic monoids in $\mathscr{C}$ induce non-isomorphic (finitary) power monoids. A compendium of progress on the isomorphism problem and further problems in the context of power monoids can be found in [23] as well as in the works cited there. The study of the isomorphism problem is still quite active. For instance, the isomorphism problem for power monoids of rank-1 torsion-free commutative monoids was recently solved in [26]. It follows from [10, Theorem 3.12] that every rank-1 torsion-free cancellative commutative monoid that is not a group can be realized as a Puiseux monoid (i.e., an additive submonoid of $\mathbb{Q}_{\geq 0}$).

Arithmetic and factorization aspects of finitary power monoids were previously studied in [9], while atomic and ideal-theoretical aspects of finitary power monoids were previously studied in [5] in the setting of numerical monoids (i.e., Puiseux monoids consisting of nonnegative integers). Another classical problem in the setting of (finitary) power monoids that has attracted the attention of several semigroup theorists for many year is that of the potential ascent of monoidal properties, which boils down to the following question: does the fact that a commutative monoid M satisfies a given property $\mathfrak{p}$ imply that the (finitary) power monoid of M also satisfies the property $\mathfrak{p}$? As for the isomorphism problem, progress in this direction until the eighties can be found in [25] and in the papers it references.

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Moreover, in the recent paper [13], the authors investigate the ascent of atomic and factorization properties from Puiseux monoids to their corresponding finitary power monoids.

In this paper we investigate the ascent of ideal-theoretical and atomic properties from linearly orderable commutative monoids to their corresponding finitary power monoids. A commutative monoid satisfies the ascending chain on principal ideals (ACCP) if every ascending chain of principal ideals eventually stabilizes. Two condition weaker than the ACCP were introduced in [15]: the quasi-ACCP and the almost ACCP. In Section 3, we prove that these two properties ascend from linearly orderable commutative monoids to their corresponding finitary power monoids. In Section 4, we generalize the following result established in [13]: there exists an atomic Puiseux monoid whose finitary power monoid is not atomic. In order to do so, we produce an atomic Puiseux monoid whose finitary power monoid is not even nearly atomic (near atomicity is a weaker notion of atomicity recently introduced in [20].

2. Preliminary

2.1. **General Notation.** As is customary, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ will denote the set of integers, rational numbers, real numbers, and complex numbers, respectively. We let $\mathbb{N}$ and $\mathbb{N}_0$ denote the set of positive and nonnegative integers, respectively. Also, we let $\mathbb{P}$ denote the set of primes. For $b, c \in \mathbb{Z}$ with $b \le c$, we let $[\![b, c]\!]$ denote the set of integers between b and c:

$$[\![b, c]\!] = \{n \in \mathbb{Z} : b \le n \le c\}.$$

In addition, for $S \subseteq \mathbb{R}$ and $r \in \mathbb{R}$, we set

$$S_{\ge r} := \{s \in S : s \ge r\} \quad \text{and} \quad S_{>r} := \{s \in S : s > r\}.$$

For a nonzero $q \in \mathbb{Q}$, let (n, d) be the unique pair with $n \in \mathbb{Z}$ and $d \in \mathbb{N}$ such that $q = \frac{n}{d}$ and $\gcd(n, d) = 1$. We will denote n and d by $\mathsf{n}(q)$ and $\mathsf{d}(q)$, respectively, setting $\mathsf{d}(S) := \{\mathsf{d}(s) : s \in S\}$ for any subset S of $\mathbb{Q} \setminus \{0\}$. For each $p \in \mathbb{P}$ and $n \in \mathbb{Z} \setminus \{0\}$, we let $v_p(n)$ denote the p-adic valuation of n, that is, the maximum $m \in \mathbb{N}_0$ such that $p^m \mid n$, and for $q \in \mathbb{Q} \setminus \{0\}$, we set $v_p(q) := v_p(\mathsf{n}(q)) - v_p(\mathsf{d}(q))$ (after defining $v_p(0) := \infty$, the map $v_p \colon \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}$ is the p-adic valuation map).

2.2. **Commutative Monoids.** We recall that a monoid is a semigroup with an identity element. Throughout this paper, identity elements are required to be inherited by submonoids and preserved by monoid homomorphisms. Moreover, we will tacitly assume that all monoids we will deal with are commutative and additively written. Let M be a monoid. We set $M^\bullet := M \setminus \{0\}$, and we say that M is *trivial* if $M = \{0\}$. The monoid M is called *cancellative* if for all $a, b, c \in M$, the equality $a + b = a + c$ implies that $b = c$. Also, M is called *torsion-free* if for all $b, c \in M$ and $n \in \mathbb{N}$, the equality $nb = nc$ implies $b = c$. The group of invertible elements of M is denoted by $\mathscr{U}(M)$, and M is called *reduced* if the only invertible element of M is 0. The quotient $M/\mathscr{U}(M)$ is a monoid that is called the *reduced monoid* of M and is denoted by M_{red}.

The group $\text{gp}(M)$ consisting of all the formal differences of elements of M (under the operation naturally extended from that of M) is called the *Grothendieck group* of M. When a monoid M is cancellative it can be minimally embedded into its Grothendieck group and this embedding is minimal in the following sense: $\text{gp}(M)$ is the unique abelian group up to isomorphism such that any abelian group containing an isomorphic copy of M will also contain an isomorphic copy of $\text{gp}(M)$. The *rank* of a cancellative monoid M is defined to be the rank of $\text{gp}(M)$ as a $\mathbb{Z}$-module or, equivalently, the dimension of the $\mathbb{Q}$-vector space $\mathbb{Q} \otimes_{\mathbb{Z}} \text{gp}(M)$. Thus, the rank of a cancellative monoid gives a sense of its size or, more accurately, the size of the smallest vector space that contains one of its isomorphic copies. It follows from [10, Theorem 3.12] that a cancellative torsion-free monoid has rank 1 if and only if it is isomorphic to an additive submonoid of $\mathbb{Q}$. The additive submonoids of $\mathbb{Q}$ that are not nontrivial groups are called *Puiseux monoids* and have been actively investigated during the past decades. They will be helpful in this paper to provide the most significant (counter)examples we need.

Let S be a subset of M. We let $\langle S \rangle$ denote the smallest submonoid of M containing S, and we call $\langle S \rangle$ the submonoid of M *generated* by S. If $M = \langle S \rangle$, then S is called a *generating set* of M, and M is called *finitely generated* provided that M has a finite generating set. Observe that any finitely generated Puiseux monoid is isomorphic to an additive (co-finite) submonoid of $\mathbb{N}_0$ (additive co-finite submonoids of $\mathbb{N}_0$ are called *numerical monoids*).

For $b, c \in M$, we say that c (*additively*) *divides* b if $b = c + d$ for some $d \in M$, in which case we write $c \mid_M b$. A submonoid N of M is said to be *divisor-closed* provided that the only pairs $(b, c) \in N \times M$ with $c \mid_M b$ are those with $c \in N$. A *maximal common divisor* (MCD) of a nonempty subset S of M is a common divisor $d \in M$ of S such that the only common divisors of the set $\{s - d : s \in S\}$ are the invertible elements of M. The monoid M is called an *MCD-monoid* provided that every nonempty finite subset of M has an MCD. Also, for $k \in \mathbb{N}$, we say that M is a *k-MCD-monoid* if every subset of M with cardinality k has a maximal common divisor. Observe that every monoid is a 1-MCD-monoid, while a monoid is an MCD-monoid if and only if it is a k-MCD-monoid for every $k \in \mathbb{N}$. The notion of a k-MCD monoid seems to be introduced by Roitman in [24].

2.3. **Atomicity and Ascending Chains of Principal Ideals.** An element $a \in M \setminus \mathscr{U}(M)$ is called an *atom* (or *irreducible*) if whenever $a = b + c$ for some $b, c \in M$, then either $b \in \mathscr{U}(M)$ or $c \in \mathscr{U}(M)$. The set of atoms of M is denoted by $\mathscr{A}(M)$. An element $b \in M$ is called *atomic* if either $b \in \mathscr{U}(M)$ or b can be written as a sum of finitely many atoms (allowing repetitions). As coined in Cohn [8], the monoid M is *atomic* if every element of M is atomic. Following the more recent paper [20] by Lebowitz-Lockard, we say that M is *nearly atomic* if there exists $c \in M$ such that $b + c$ is atomic for all $b \in M$. It follows directly from the definitions that every atomic monoid is nearly atomic. Following Boynton and Coykendall [6], we say that the monoid M is *almost atomic* (resp., *quasi-atomic*) provided that for each $b \in M$, there exists an atomic element (resp., an element) $c \in M$ such that $b + c$ is atomic. One can verify that every nearly atomic monoid is almost atomic, and it follows directly from the definitions that every almost atomic monoid is quasi-atomic.

A subset I of M is said to be an *ideal* of M provided that $I + M := \{b + c : b \in I \text{ and } c \in M\} = I$ (or, equivalently, $I + M \subseteq I$). An ideal I is *principal* if the equality $I = b + M$ holds for some $b \in M$. An element $b \in M$ is said to satisfy the *ascending chain condition on principal ideals* (ACCP) if every ascending chain of principal ideals of M containing the ideal $b + M$ stabilizes. The monoid M is said to satisfy the *ACCP* if every element of M satisfies the ACCP. An ascending chain of principal ideals of M is said *to start* at an element $b \in M$ if the first ideal in the chain is $b + M$. Two ideal-theoretical notions weaker than the ACCP were introduced in [15]: the quasi-ACCP and the almost ACCP. We say that M satisfies the *almost ACCP* (resp., *quasi-ACCP*) if for any nonempty finite subset S of M, there exists an atomic common divisor (resp., a common divisor) $d \in M$ of S such that for some $s \in S$ the element $s - d$ satisfies the ACCP. It follows directly from the definitions that every monoid that satisfies the almost ACCP also satisfies the quasi-ACCP.

It is well known that every cancellative monoid that satisfies the ACCP is atomic (see [11, Proposition 1.1.4]), and so every cancellative monoid that satisfies the ACCP also satisfies the almost ACCP. However, not every atomic monoid satisfies the ACCP, and several examples of cancellative monoids and integral domains witnessing this observation can be found in recent papers, including [15]. Moreover, it follows from [15] that every monoid satisfying the almost ACCP is atomic (in Section 4, we will construct a new rank-one atomic monoid that does not satisfy the almost ACCP). The property of satisfying the quasi-ACCP does not imply that of being atomic as illustrated by the simple Puiseux monoid $\mathbb{Q}_{\geq 0}$.

2.4. **Irreducible Divisors and Factorizations.** We say that M is a *Furstenberg monoid* or satisfies the *Furstenberg property* if every non-invertible element of M is divisible by an atom. The Furstenberg property was introduced by Clark in [7]. It follows from the definitions that every atomic monoid is a Furstenberg monoid. Following Grams and Warner [18], we say that M is an *IDF-monoid* if every element of M is divisible by only finitely many atoms up to associates (two elements of M are *associates* if their differences belong to $\mathscr{U}(M)$). Then we say that M is a *TIDF-monoid* provided that it is a Furstenberg IDF-monoid. The TIDF (tightly irreducible divisor finite) property was introduced and first investigated by Zafrullah and the second author in [16].

Now assume that the monoid M is atomic, which is equivalent to the fact that the reduced monoid M_{red} is atomic. Let $\mathsf{Z}(M)$ be the free (commutative) monoid on the set of atoms $\mathscr{A}(M_{\text{red}})$. The elements of $\mathsf{Z}(M)$ are called *factorizations*. Let $\pi\colon \mathsf{Z}(M) \to M_{\text{red}}$ be the unique monoid homomorphism fixing the set $\mathscr{A}(M_{\text{red}})$. For any $b \in M$, we set $\mathsf{Z}(b) := \pi^{-1}(b + \mathscr{U}(M))$ and call the elements of $\mathsf{Z}(b)$ (*additive*) *factorizations* of b. If $|\mathsf{Z}(b)| < \infty$ for every $b \in M$, then M is called a *finite factorization monoid* (or an FFM for short). It follows from [19, Theorem 2] that a monoid is an FFM if and only if it is an atomic IDF-monoid. In particular, every FFM is a TIDF-monoid. It follows from [11, Corollary 1.4.4] that every FFM satisfies the ACCP.

2.5. **Linearly Ordered Groups and Monoids.** The class consisting of linearly ordered monoids contains all Puiseux monoids and plays a fundamental role in this paper. The monoid M is called *linearly ordered* with respect to a total order relation $\preceq$ on M if $\preceq$ is *compatible* with the operation of M, which means that for all $b, c, d \in M$ the order relation $b \prec c$ ensures that $b + d \prec c + d$. We say that the monoid M is *linearly orderable* provided that M is a linearly ordered monoid with respect to some total order relation on M. More than a century ago, it was proved by Levi [21] that every torsion-free abelian group is a linearly orderable monoid (or, simply, linearly orderable). From this Levi's result one can deduce the following well-known theorem.

Theorem 2.1. *A monoid is linearly orderable if and only if it is cancellative and torsion-free.*

Let G be a linearly ordered abelian group (additively written) with respect to a total order relation $\preceq$. The *nonnegative cone* of G is the submonoid G^+ of G consisting of all nonnegative elements; that is,

$$G^+ := \{g \in G : 0 \preceq g\}.$$

A submonoid of G^+ is called a *positive submonoid* of G. In general, the monoid M is called a *positive monoid* provided that M is isomorphic to a submonoid of the nonnegative cone of a linearly ordered abelian group. Observe that, as a consequence of Theorem 2.1, if the monoid M is cancellative, reduced, and torsion-free, then its Grothendieck group $\text{gp}(M)$ can be turn into a linearly ordered monoid so that M is a positive monoid of $\text{gp}(M)$.

For $g \in G$, we set $|g| := \max\{\pm g\}$. For $g, h \in G$, we write $g = \mathbf{O}(h)$ whenever $|g| \preceq n|h|$ for some $n \in \mathbb{N}$. Now consider the equivalence relation $\sim$ on G obtained as follows: for $g, h \in G$, write $g \sim h$ whenever both equalities $g = \mathbf{O}(h)$ and $h = \mathbf{O}(g)$ hold. Set $\Gamma_G := (G \setminus \{0\})/\sim$ and consider the quotient map $v\colon G \setminus \{0\} \to \Gamma_G$. Then the binary relation $\leq$ on Γ defined by writing $v(g) \leq v(h)$ for any $g, h \in G \setminus \{0\}$ such that $h = \mathbf{O}(g)$ is a total order relation. The elements of Γ_G are called *Archimedean classes* of G, and the quotient map v is called the *Archimedean valuation* on G. The group G is called *Archimedean* provided that Γ_G is a singleton. A monoid is called *Archimedean* if it is a positive monoid of an Archimedean group. According to one of the well-known Hölder's theorems, a linearly orderable abelian group is Archimedean if and only if it is order-isomorphic to a subgroup of the additive group $\mathbb{R}$.

2.6. **Finitary Power Monoid.** Let M be a monoid. We let $\mathscr{P}_{\text{fin}}(M)$ denote the monoid consisting of all nonempty finite subsets of M under the so-called sumset operation: for any nonempty finite subsets S and T of M,

$$S + T := \{s + t : (s, t) \in S \times T\}.$$

The monoid $\mathscr{P}_{\text{fin}}(M)$ is called the *finitary power monoid* of M. To simplify notation, in the scope of this paper we call the monoid $\mathscr{P}_{\text{fin}}(M)$ the *power monoid* of M[1].

We say that the monoid M is *unit-cancellative* provided that for all $b, c \in M$ the equality $b + c = b$ implies that $c \in \mathscr{U}(M)$. It follows from [9, Proposition 3.5] that if M is a linearly orderable monoid, then $\mathscr{P}_{\text{fin}}(M)$ is a unit-cancellative monoid. On the other hand, it is worth emphasizing that power monoids are extremely non-cancellative in the sense that $\mathscr{P}_{\text{fin}}(M)$ is cancellative if and only if the monoid M is trivial. We proceed to prove some preliminary results about power monoids that we will need in the coming sections.

Lemma 2.2. *Let M be a linearly ordered monoid. For any $A, B, C \in \mathscr{P}_{\text{fin}}(M)$ with $A + B = C$, the following statements hold:*

$$\min A + \min B = \min C \quad \text{and} \quad \max A + \max B = \max C.$$

Proof. We only verify that $\min A + \min B = \min C$ as the other identity follows similarly. Let $\preceq$ be the total order relation under which M is a linearly ordered monoid. Since $\min C$ belongs to C and $C = A + B$, we can take some $a \in A$ and $b \in B$ such that $a + b = \min C$. As $\min A \preceq a$ and $\min B \preceq b$, $\min A + \min B \preceq \min C$. On the other hand, the fact that $\min A + \min B \in A + B = C$ ensures that $\min C \preceq \min A + \min B$. □

The following corollary is an immediate consequence of Lemma 2.2.

Corollary 2.3. *Let M be a linearly ordered monoid. If $A, B \in \mathscr{P}_{\text{fin}}(M)$, then $A \mid_{\mathscr{P}_{\text{fin}}(M)} B$ implies that $\min A \mid_M \min B$.*

The following lemma will also be helpful later.

Lemma 2.4. *Let M be a linearly ordered monoid. For $A, B \in \mathscr{P}_{\text{fin}}(M)$ such that $A \mid_{\mathscr{P}_{\text{fin}}(M)} B$, if $\min A = \min B$, then either $A = B$ or $|A| < |B|$.*

Proof. Take $A, B \in \mathscr{P}_{\text{fin}}(M)$ such that $A \mid_{\mathscr{P}_{\text{fin}}(M)} B$, and assume that $\min A = \min B$. If $A = B$, then we are done. Therefore assume that $A \neq B$. Since $A \mid_{\mathscr{P}_{\text{fin}}(M)} B$, we can take $D \in \mathscr{P}_{\text{fin}}(M)$ such that $A + D = B$. By Lemma 2.2, the equality $\min A + \min D = \min B$ holds. Hence the equality $\min A = \min B$ implies that $\min D = 0$. As a result, $A = A + \{0\} \subseteq A + D = B$, and so the inequality $|A| < |B|$ follows from the fact that $A \neq B$. □

[1] In general, the power monoid of M is the larger monoid consisting of all nonempty subsets of M under the same sumset operation.

3. Ascending Chain of Principal Ideals

It is known that if a linearly orderable monoid M satisfies the ACCP, then the power monoid of M also satisfies the ACCP. In this section, we will establish parallel ascent results for the quasi-ACCP and the almost ACCP. Before proving the ascent of these two properties, we need the following preliminary known lemma (we include a proof here for the sake of completeness).

Lemma 3.1. *Let M be a linearly orderable monoid. For any $S, T \in \mathscr{P}_{\text{fin}}(M)$, the following statements hold.*

(1) $|S+T| \ge |S| + |T| - 1 \ge \max\{|S|, |T|\}$.

(2) *If $|S| \ge 2$, then $|S+T| > |T|$.*

Proof. (1) Take $S, T \in \mathscr{P}_{\text{fin}}(M)$. The first inequality $|S+T| \ge |S| + |T| - 1$ is [9, Proposition 3.5], and the second inequality follows immediately.

(2) Let $\preceq$ be a total order relation on M turning M into a linearly ordered monoid. Set $s := \min S$ and $t := \min T$. Because $|S| \ge 2$, we can take $r \in S \setminus \{s\}$. Now observe that $s+t \prec r+t = \min(\{r\}+T)$, so $\{s+t\} \notin \{r\} + T$. This means that $|\{s+t\} \cup (\{r\}+T)| = |T| + 1$. Finally, the inclusion $\{s+t\} \cup (\{r\}+T) \subseteq S+T$, along with the fact that T and $\{r\}+T$ have the same cardinality, guarantees that $|S+T| \ge |T| + 1 > |T|$. □

For any monoid M, it is clear that the set consisting of all the singletons of M is a submonoid of $\mathscr{P}_{\text{fin}}(M)$. In light of the second part of the previous lemma, we obtain that such a submonoid is divisor-closed. We record this easy remark here for future reference.

Corollary 3.2. *Let M be a linearly orderable monoid. Then $\{S \in \mathscr{P}_{\text{fin}}(M) : |S| = 1\}$ is a divisor-closed submonoid of $\mathscr{P}_{\text{fin}}(M)$.*

We are in a position to establish the main result of this section, the ascent of the quasi-ACCP and the almost ACCP to power monoids in the class of linearly orderable monoids.

Theorem 3.3. *Let M be a linearly orderable monoid. Then the following statements hold.*

(1) *If M satisfies the quasi-ACCP, then $\mathscr{P}_{\text{fin}}(M)$ also satisfies the quasi-ACCP.*

(2) *If M satisfies the almost ACCP, then $\mathscr{P}_{\text{fin}}(M)$ also satisfies the almost ACCP.*

Proof. To make our notation less cumbersome, we write $\mathscr{P}$ instead of $\mathscr{P}_{\text{fin}}(M)$.

(1) Assume that M satisfies the quasi-ACCP. In order to argue that $\mathscr{P}$ satisfies the quasi-ACCP, fix a nonempty finite subset $\{S_1, \dots, S_n\}$ of $\mathscr{P}$. Now set $S := S_1 \cup \dots \cup S_n$. Because S is a nonempty finite subset of M, the fact that M satisfies the quasi-ACCP allows us to pick a common divisor $d \in M$ of S and also an element $s \in S$ such that $s - d$ satisfies the ACCP in M. Take an index $j \in [\![1, n]\!]$ such that $s \in S_j$. For each $i \in [\![1, n]\!]$, the inclusion $S_i \subseteq S$ ensures that d is a common divisor of S_i in M, and so $\{d\} \mid_{\mathscr{P}} S_i$.

Thus, it suffices to show that $S_j - \{d\}$ satisfies the ACCP in $\mathscr{P}$. To do this, take an ascending chain $(B_n + \mathscr{P})_{n \ge 0}$ of principal ideals of $\mathscr{P}$ starting at $S_j - \{d\}$ and, as B_0 and $S_j - \{d\}$ are associates, we can assume that $B_0 = S_j - \{d\}$. Now set $b_0 := s - d$ and take $b_1 \in B_1$ such that $b_1 \mid_M b_0$, and then note that if $b_0, \dots, b_n$ are elements in M such that $b_i \in B_i$ and $b_i \mid_M b_{i-1}$ for every $i \in [\![1, n]\!]$, then the fact that $B_{n+1} \mid_{\mathscr{P}} B_n$ allows us to take $b_{n+1} \in B_{n+1}$ such that $b_{n+1} \mid_M b_n$. Hence we have inductively constructed a chain $(b_n + M)_{n \ge 0}$ of principal ideals of M with $b_0 = s - d$ such that $b_n \in B_n$ for every $n \in \mathbb{N}_0$. Since the ascending chain $(b_n + M)_{n \ge 0}$ starts at $s - d$, which is an element satisfying the ACCP in M, there exists an index $k_1 \in \mathbb{N}$ such that whenever $n > k_1$ the equality $b_n + M = b_{n-1} + M$ holds, and so $b_{n-1} - b_n \in \mathscr{U}(M)$. On the other hand, it follows from part (1) of Lemma 3.1 that, for each $n \in \mathbb{N}_0$, the divisibility relation $B_{n+1} \mid_{\mathscr{P}} B_n$ implies that $|B_n| \ge |B_{n+1}|$, whence there exists an index $k_2 \in \mathbb{N}$ such that $|B_n| = |B_{k_2}|$ for every $n \ge k_2$. Now, for each $n \in \mathbb{N}$, take a subset C_n of M such that $B_{n-1} = B_n + C_n$. Then by part (2) of Lemma 3.1, for each $n \in \mathbb{N}$ with $n > k_2$ the set C_n must be a singleton, and so $C_n = \{b_{n-1} - b_n\}$. As a consequence, for each $n \in \mathbb{N}$ with $n > \max\{k_1, k_2\}$, we obtain that the element C_n of $\mathscr{P}$ is the singleton containing the invertible element $b_{n-1} - b_n$, and so the chain $(B_n + \mathscr{P})_{n \ge 0}$ must stabilize. Thus, we conclude that $S_j - \{d\}$ satisfies the ACCP.

(2) Suppose now that M satisfies the almost ACCP. As in the previous part, fix a nonempty finite subset $\{S_1, \dots, S_n\}$ of $\mathscr{P}$, and use the fact that M satisfies the almost ACCP to find an atomic common divisor $d \in M$ of $S_1 \cup \dots \cup S_n$ such that $s - d$ satisfies the ACCP in M. If d is an invertible element of M, then $\{d\}$ is an invertible element of $\mathscr{P}$. Otherwise, we can write $d = a_1 + \dots + a_\ell$ for some $a_1, \dots, a_\ell \in \mathscr{A}(M)$, in which case, $\{a_1\}, \dots, \{a_\ell\} \in \mathscr{A}(M)$ and so $\{d\}$ can be written as a sum of atoms in $\mathscr{P}$, namely, $\{d\} = \{a_1\} + \dots + \{a_\ell\}$. Hence $\{d\}$ must be an atomic element in $\mathscr{P}$. Finally, proceeding *mutatis mutandis* as we did in part (1), we can show that $S_j - \{d\}$ satisfies the ACCP in $\mathscr{P}$, where j is an index in $[\![1, n]\!]$ such that $d \in S_j$. Hence we conclude that the power monoid $\mathscr{P}$ also satisfies the almost ACCP. □

4. Atomicity and Maximal Common Divisors

In this first section, we will investigate the ascent of atomicity as well as the ascent of some weaker notions of atomicity from linearly ordered monoids to their corresponding power monoids.

4.1. **Existence of Maximal Common Divisors.** Our next goal is to prove that, for any linearly orderable monoid M, the power monoid $\mathscr{P}_{\text{fin}}(M)$ is atomic if and only if M is an atomic MCD-monoid. This result not only generalizes but also strengths [13, Theorem 3.2], in the sense that it gives a complete characterization of atomic power monoids of linearly ordered monoids (which are more general than Puiseux monoids). Given that the existence of maximal common divisors (MCDs) is essential for our characterization, let us first argue that the existence of MCDs transfers between any linearly orderable monoid and its power monoid.

Proposition 4.1. *For a linearly orderable monoid M, the following conditions are equivalent.*

(a) *M is an MCD-monoid.*

(b) *$\mathscr{P}_{\text{fin}}(M)$ is an MCD-monoid.*

(c) *$\mathscr{P}_{\text{fin}}(M)$ is a k-MCD-monoid for some $k \in \mathbb{N}_{\ge 2}$*

Proof. (a) ⇒ (b): Assuming that M is an MCD-monoid, let us show that each nonempty finite subset $\mathscr{S}$ of $\mathscr{P}_{\text{fin}}(M)$ has an MCD by using induction on $\sum_{S \in \mathscr{S}} |S|$. For the base case, note that if, for a nonempty finite subset $\mathscr{S}$ of $\mathscr{P}_{\text{fin}}(M)$, the equality $\sum_{S \in \mathscr{S}} |S| = 1$ holds, then $\mathscr{S} = \{\{m\}\}$ for some $m \in M$, which implies that $\{m\}$ is an MCD of $\mathscr{S}$ in $\mathscr{P}_{\text{fin}}(M)$. Now fix $n \in \mathbb{N}$ such that every nonempty finite subset $\mathscr{S}$ of $\mathscr{P}_{\text{fin}}(M)$ with $\sum_{S \in \mathscr{S}} |S| \le n$ has an MCD in $\mathscr{P}_{\text{fin}}(M)$. Let $\mathscr{T}$ be a nonempty finite subset of $\mathscr{P}_{\text{fin}}(M)$ with $\sum_{T \in \mathscr{T}} |T| = n + 1$, and let us argue that $\mathscr{T}$ has an MCD in $\mathscr{P}_{\text{fin}}(M)$. Consider the following two cases.

Case 1: $\mathscr{T}$ has a common divisor in $\mathscr{P}_{\text{fin}}(M)$ that is not a singleton. Let D be a common divisor of $\mathscr{T}$ in $\mathscr{P}_{\text{fin}}(M)$ such that $|D| \ge 2$. Then we can write $\mathscr{T} = D + \mathscr{T}'$ for some $\mathscr{T}'$ in $\mathscr{P}_{\text{fin}}(M)$, in which case the inequality $\sum_{T \in \mathscr{T}'} |T| < \sum_{T \in \mathscr{T}} |T|$ holds in light of Lemma 3.1. Therefore our induction hypothesis ensures the existence of an MCD T' of $\mathscr{T}'$ in $\mathscr{P}_{\text{fin}}(M)$. It immediately follows now that $D + T'$ is an MCD of $\mathscr{T}$ in $\mathscr{P}_{\text{fin}}(M)$.

CASE 2: Each common divisor of $\mathscr{T}$ in $\mathscr{P}_{\text{fin}}(M)$ is a singleton. Because M is an MCD-monoid, the finite nonempty subset $U := \bigcup_{T \in \mathscr{T}} T$ of M must have an MCD, namely, $m_0 \in M$. As m_0 is a common divisor of T in M for all $T \in \mathscr{T}$, it follows that $\{m_0\}$ is a common divisor of $\mathscr{T}$. To argue that $\{m_0\}$ is indeed an MCD of $\mathscr{T}$, take a nonempty finite subset D' of M such that $\{m_0\} + D'$ is a common divisor of $\mathscr{T}$. By assumption, $\{m_0\} + D'$ is a singleton, and so $D' = \{d\}$ for some $d \in M$. Now the fact that m_0 is an MCD of U ensures that $d \in \mathscr{U}(M)$. Hence $\{m_0\}$ is an MCD of $\mathscr{T}$ in $\mathscr{P}_{\text{fin}}(M)$.

(b) $\Rightarrow$ (c): This is straightforward.

(c) $\Rightarrow$ (a): Assume that $\mathscr{P}_{\text{fin}}(M)$ is a k-MCD monoid for some $k \in \mathbb{N}_{\ge 2}$, and so a 2-MCD-monoid. To show that M is an MCD-monoid, it suffices to fix a finite subset S of M with $|S| \ge 2$ and prove that S has an MCD in M. Take $s_0 \in S$, and then set $\mathscr{T} := \{\{s_0\}, S \setminus \{s_0\}\}$, which is a subset of $\mathscr{P}_{\text{fin}}(M)$. Because $\mathscr{T}$ contains a singleton, it follows from Lemma 3.1 that all common divisors of $\mathscr{T}$ in $\mathscr{P}_{\text{fin}}(M)$ are singletons. In addition, as $\mathscr{P}_{\text{fin}}(M)$ is a 2-MCD-monoid and $|\mathscr{T}| = 2$, we can take $m_0 \in M$ such that $\{m_0\}$ is an MCD of $\mathscr{T}$ in $\mathscr{P}_{\text{fin}}(M)$. This immediately implies that m_0 is an MCD of S in M. □

For a monoid M, we say that an element S of $\mathscr{P}_{\text{fin}}(M)$ is *indecomposable* if whenever we can write $S = U + V$ for some $U, V \in \mathscr{P}_{\text{fin}}(M)$ either $|U| = 1$ or $|V| = 1$. In order to establish the primary result of this section, we need the following lemma.

Lemma 4.2. *Let M be a linearly orderable monoid, and let S be a finite subset of M with $|S| \ge 2$. Then the following statements hold.*

1. *If $\min S + \max S > 0$, then $S \cup \{4 \max S\}$ is indecomposable.*
2. *If $\min S + \max S < 0$, then $S \cup \{4 \min S\}$ is indecomposable.*

Proof. Take $s_1, \dots, s_n \in M$ with $s_1 < \dots < s_n$ such that $S = \{s_1, \dots, s_n\}$.

(1) Assume that $\min S + \max S \ge 0$, and set $T := S \cup \{4s_n\}$. From $s_1 + s_n > 0$, we obtain that $s_n > 0$. Assume, towards a contradiction, that we can pick U and V to be non-singletons nonempty finite subsets of M such that $T = U + V$ in $\mathscr{P}_{\text{fin}}(M)$. Now set $u_1 := \max U$ and then set $u_2 := \max(U \setminus \{u_1\})$. Similarly, set $v_1 := \max V$ and then set $v_2 := (V \setminus \{v_1\})$. Therefore, we see that $4s_n = \max T = \max(U + V) = u_1 + v_1$ and

$$s_n = \max(T \setminus \{4s_n\}) = \max\big((U + V) \setminus \{u_1 + v_1\}\big) \in \{u_1 + v_2, u_2 + v_1\}.$$

Assume, without loss of generality, that $s_n = u_2 + v_1$. Then $(u_1 + v_1) - (u_2 + v_1) = 3s_n$ and, therefore, $u_1 = u_2 + 3s_n$. Similarly, it follows from $s_n \ge u_1 + v_2$ that $(u_1 + v_1) - (u_1 + v_2) \ge 3s_n$, and so $v_2 + 3s_n \le v_1$. Thus, $u_2 + v_2 + 6s_n \le u_1 + v_1 = 4s_n$, which implies that $u_2 + v_2 + 2s_n \le 0$. On the other hand, we find that

$$u_2 + v_2 + 2s_n \ge \min U + \min V + 2s_n \ge (s_1 + s_n) + s_n \ge s_n > 0,$$

which contradicts the inequality $u_2 + v_2 + 2s_n \le 0$. As a consequence, T is an indecomposable element of $\mathscr{P}_{\text{fin}}(M)$.

(2) Now assume that $\min S + \max S < 0$. This part is symmetric to part (1): indeed, after setting $T := S \cup \{4s_1\}$, we can take the elements u_1 and u_2 (resp., v_1 and v_2) to be the smallest element and second smallest element of U (resp., V), and then we can repeat the argument already given in the proof of the previous part. □

We are in a position to characterize the atomic power monoids of linearly orderable monoids.

Theorem 4.3. *For any linearly orderable monoid M, the following conditions are equivalent.*

(a) *M is an atomic MCD-monoid.*
(b) *$\mathscr{P}_{\text{fin}}(M)$ is an atomic monoid.*
(c) *$\mathscr{P}_{\text{fin}}(M)$ is an atomic MCD-monoid.*
(d) *$\mathscr{P}_{\text{fin}}(M)$ is an atomic k-MCD-monoid for some $k \in \mathbb{N}_{\ge 2}$.*

Proof. Let M be a linearly orderable monoid, and set $\mathscr{M} := \{\{m\} : m \in M\}$, which is a divisor-closed submonoid of $\mathscr{P}_{\text{fin}}(M)$.

(a) $\Rightarrow$ (c): The proof that $\mathscr{P}_{\text{fin}}(M)$ is an atomic monoid follows the line of the proof of [13, Theorem 3.2] *mutatis mutandis*. Then, it follows from Proposition 4.1 that $\mathscr{P}_{\text{fin}}(M)$ is an atomic MCD-monoid.

(c) $\Rightarrow$ (d): This is straightforward.

(d) $\Rightarrow$ (b): This is also straightforward.

(b) $\Rightarrow$ (a): Because $\mathscr{M}$ is a divisor-closed submonoid of $\mathscr{P}_{\text{fin}}(M)$, from the fact that $\mathscr{P}_{\text{fin}}(M)$ is atomic one obtains that $\mathscr{M}$ is atomic. As a consequence, M is also atomic as it is naturally isomorphic to $\mathscr{M}$.

To prove that M is an MCD-monoid, we fix a nonempty finite subset S of M and argue that S has an MCD in M. As every singleton has an MCD in M, we can assume that $|S| \ge 2$. On the other hand, after replacing S by a suitable subset, we can further assume that $s + t \neq 0$ for all $s, t \in S$. Take $s_1, \dots, s_n \in M$ with $s_1 < \dots < s_n$ such that $S = \{s_1, \dots, s_n\}$. Now set $T := S \cup \{t\}$, where $t := 4s_n$ if $s_1 + s_n > 0$ and $t := 4s_1$ if $s_1 + s_n < 0$.

Since $\mathscr{P}_{\text{fin}}(M)$ is an atomic monoid, we can write $T = A_1 + \dots + A_\ell$ for some atoms $A_1, \dots, A_\ell$ of $\mathscr{P}_{\text{fin}}(M)$. In light of Lemma 4.2, we can assume that A_i is a singleton if and only if $i \in [\![1, \ell - 1]\!]$ (at least one of $A_1, \dots, A_\ell$ is not a singleton because T is not a singleton). Write $A_i = \{a_i\}$ for every $i \in [\![1, \ell - 1]\!]$. Thus, the fact that $\{a_1\}, \dots, \{a_{\ell-1}\}$ are atoms of $\mathscr{P}_{\text{fin}}(M)$ implies that they are also atoms of the divisor-closed submonoid $\mathscr{M}$ of $\mathscr{P}_{\text{fin}}(M)$, whence the natural isomorphism between $\mathscr{M}$ and M ensures that $a_1, \dots, a_{\ell-1} \in \mathscr{A}(M)$. Set $A := \{a_1 + \dots + a_{\ell-1}\}$ and $\mathscr{T} := \{S, \{t\}\}$, and let us argue the following claim.

Claim. A is an MCD of $\mathscr{T}$ in $\mathscr{P}_{\text{fin}}(M)$.

Proof of Claim. Since $A + A_\ell = T = \{s_1, \dots, s_n\} \cup \{t\}$ and A is a singleton, we see that A divides both $S = \{s_1, \dots, s_n\}$ and $\{t\}$ in $\mathscr{P}_{\text{fin}}(M)$, so A is a common divisor of $\mathscr{T}$. Now suppose that $A + D$ divides both S and $\{t\}$ in $\mathscr{P}_{\text{fin}}(M)$ for some nonempty finite subset D of M. Then D must be a singleton because it divides the singleton $\{t\}$ in $\mathscr{P}_{\text{fin}}(M)$. Because $A + D$ is a singleton and also a common divisor of S and $\{t\}$ in $\mathscr{P}_{\text{fin}}(M)$, it follows that $A + D$ divides T in $\mathscr{P}_{\text{fin}}(M)$. Now take a nonempty finite subset D' of M such that $(A + D) + D' = T = \{a_1 + \dots + a_{\ell-1}\} + A_\ell$. This implies that $D + D' = A_\ell$ (every singleton is a cancellative element in $\mathscr{P}_{\text{fin}}(M)$). Now the fact that A_ℓ is an atom of $\mathscr{P}_{\text{fin}}(M)$ guarantees that either D or D' is invertible in $\mathscr{P}_{\text{fin}}(M)$. Since D' is not a singleton (because A_ℓ is not a singleton), it follows that D is invertible in $\mathscr{P}_{\text{fin}}(M)$. Hence we conclude that A is an MCD of $\mathscr{T}$ in $\mathscr{P}_{\text{fin}}(M)$, and the claim is established.

It follows from the established claim that $s := a_1 + \dots + a_{\ell-1}$ is a common divisor of S in M. To argue that s is an MCD of S in M, take $d \in M$ such that $s + d$ is a common divisor of S in M. Then $A + \{d\}$ divides both S and $\{t\}$ in $\mathscr{P}_{\text{fin}}(M)$, and so the established claim implies that $\{d\}$ is invertible in $\mathscr{P}_{\text{fin}}(M)$. Therefore $d \in \mathscr{U}(M)$, and so s is an MCD of S in M. We can now conclude that M is an atomic MCD-monoid, which concludes our proof. □

4.2. **Non-Ascent of Atomicity.** Next we construct an atomic rank-one torsion-free monoid M whose power monoid $\mathscr{P}_{\text{fin}}(M)$ is not atomic, confirming the result first given in [13, Section 3] that the property of being atomic does not ascend to power monoids. First, we argue the following lemma.

Lemma 4.4. *Let M be a Puiseux monoid generated by a set S, and let (p, a) be the only pair in $\mathbb{P} \times S$ such that $p \mid \mathsf{d}(s)$. Then the following statements hold.*

(1) $a \in \mathscr{A}(M)$.

(2) *For each $q \in M$, the following set is a singleton:*

$$\big\{c + p\mathbb{Z} : q = ca + r \text{ for some } c \in \mathbb{N}_0 \text{ and } r \in \big\langle S \setminus \{a\}\big\rangle\big\}. \tag{4.1}$$

Proof. (1) This follows immediately as if $a \in \langle S \setminus \{a\}\rangle$, then we could take $c_1, \dots, c_\ell \in \mathbb{N}$ and $s_1, \dots, s_\ell \in S \setminus \{a\}$ such that $a = c_1 s_1 + \dots + c_\ell s_\ell$, which is not possible because $v_p(a) \le -1$ while $v_p(c_1 s_1 + \dots + c_\ell s_\ell) \ge 0$.

(2) Set $N := \langle S \setminus \{a\}\rangle$, and observe that $v_p(r) \ge 0$ for every $r \in \text{gp}(N)$. Now fix $q \in M$, and let $\mathscr{C}_q$ be the set described in (4.1). Let $c_1 + p\mathbb{Z}$ and $c_2 + p\mathbb{Z}$ be two elements of $\mathscr{C}_q$, and then write $q = c_1 a + r_1 = c_2 a + r_2$ for some $r_1, r_2 \in N$. Since $(c_1 - c_2)a = r_2 - r_1 \in \text{gp}(N)$, we see that

$$v_p(c_1 - c_2) \ge 1 + v_p((c_1 - c_2)a) = 1 + v_p(r_2 - r_1) \ge 1,$$

which means that $p \mid c_1 - c_2$ or, equivalently, $c_1 + p\mathbb{Z} = c_2 + p\mathbb{Z}$. As a consequence, we can conclude that $\mathscr{C}_q$ is a singleton for each $q \in M$. □

Based on part (2) of Lemma 4.4, we introduce the following notation.

Notation. Let the notation be as in Lemma 4.4. For each $q \in M$, we let $c_{a,p}(q)$ denote the unique element of $\mathbb{Z}/p\mathbb{Z}$ contained in the singleton (4.1), and we set $c_a(q)$ instead of $c_{a,p}(q)$ when p is unique given a. This satisfies some useful properties: for instance, one can check that

$$c_{a,p}(q + r) = c_{a,p}(q) + c_{a,p}(r) \quad \text{for all} \quad q, r \in M.$$

Also note that $c_{a,p}(q)$ is the same for any S we choose where a satisfies the desired condition. Thus, as long as S exists, the notation is well defined.

We proceed to produce an example of an atomic Puiseux monoid that is not 2-MCD. The following example, which is motivated by [13, Example 3.3], not only improves the choice of the generating set of the later, but will also help us resolve the question of whether near atomicity ascends to power monoids in the class of linearly orderable monoids.

Example 4.5. (cf. [13, Example 3.3]) Let p_n be the n^{th} prime in $\mathbb{P}_{\ge 5}$, and consider the Puiseux monoid M generated by the set $A_1 \cup A_2$, where

$$A_1 := \left\{ \frac{1}{2^n p_{2n+2}} : n \in \mathbb{N}_0 \right\} \quad \text{and} \quad A_2 := \left\{ \frac{1}{p_{2n+1}} \left(\frac{1}{3} + \frac{1}{2^n} \right) : n \in \mathbb{N}_0 \right\}.$$

By part (1) of Lemma 4.4, every element of $A_1 \cup A_2$ is an atom of M and, therefore, $\mathscr{A}(M) = A_1 \cup A_2$. Hence M is an atomic monoid. We will argue that M is not a 2-MCD monoid by showing that the subset $\big\{1, \frac{4}{3}\big\}$ of M does not have an MCD (both 1 and $\frac{4}{3}$ belong to M).

For $z \in \mathsf{Z}(M)$ and $a \in \mathscr{A}(M)$, we let z_a be the number of copies of a that appear in z, and let p_a be the unique prime in $\mathbb{P}_{\ge 5}$ dividing $\mathsf{d}(a)$. Observe that, for each $r \in M$, if the equality $v_{p_a}(r) = 0$ holds, then for any $z \in \mathsf{Z}(r)$ there exists $d_a \in \mathbb{N}_0$ such that $z_a = d_a p_a$.

We claim that $a \nmid_M 1$ for any $a \in A_2$. Suppose, by way of contradiction, that $a \mid_M 1$ for some $a \in A_2$, and fix $z \in \mathsf{Z}(1)$ with $z_a \ge 1$. The equality $v_{p_a}(1) = 0$ implies that $d_a := \frac{z_a}{p_a} \in \mathbb{N}_0$, and so that $d_a \ge 1$. Now the fact that $1 \ge z_a a = d_a\big(\frac{1}{3} + \frac{1}{2^n}\big)$ for some $n \in \mathbb{N}_0$ ensures that $d_a \in \{1, 2\}$. Thus, one can deduce from $v_3(1) = 0$ that (as an atom of A_2 appears in z) there must exist pairwise distinct atoms $a_1, a_2, a_3 \in A_2$ at least two of them appearing in z such that $d_{a_1} + d_{a_2} + d_{a_3} = 3k$ for some $k \in \mathbb{N}$. As a consequence,

$$\sum_{i=1}^{3} z_{a_i} a_i \ge \sum_{i=1}^{3} d_{a_i} \left(\frac{1}{3} + \frac{1}{2^{n_i}} \right) > 1$$

for some $n_1, n_2, n_3 \in \mathbb{N}_0$, which contradicts that $1 \ge \sum_{i=1}^{3} z_{a_i} a_i$ (as $\sum_{i=1}^{3} z_{a_i} a_i$ is a factorization of a divisor of 1 in M). Therefore $a \nmid_M 1$ for any $a \in A_2$, as claimed.

Let us prove now that $\left\{1, \frac{4}{3}\right\}$ has no MCD in M. Fix a common divisor $q \in M$ of $\left\{1, \frac{4}{3}\right\}$, and let us find a positive common divisor of $\left\{1 - q, \frac{4}{3} - q\right\}$ in M. Since $q \mid_M 1$, it follows from the claim proved in the previous paragraph that $a \nmid_M q$ for any $a \in A_2$. Let z be a factorization of $\frac{4}{3} - q$. Since $v_3\left(\frac{4}{3} - q\right) = -1$, the inclusion $v_3(A_1) \subseteq \mathbb{N}_0$ implies that at least one atom from A_2 appears in z. As the sum of any four atoms of A_2 is larger than $\frac{4}{3}$, at most three pairwise distinct atoms of A_2 can appear in z. Let $\{a_1, a_2, a_3\}$ be a 3-subset of A_2 containing the atoms of A_2 that appear in z. Then $v_3\left(\frac{4}{3} - (z_{a_1} a_1 + z_{a_2} a_2 + z_{a_3} a_3)\right) \ge 0$, and so

$$v_3\left(\frac{4}{3} - \frac{d_{a_1} + d_{a_2} + d_{a_3}}{3}\right) = v_3\left(\frac{4}{3} - \sum_{i=1}^{3} d_{a_i}\left(\frac{1}{3} + \frac{1}{2^{n_i}}\right)\right) = v_3\left(\frac{4}{3} - \sum_{i=1}^{3} z_{a_i} a_i\right) \ge 0.$$

for some $n_1, n_2, n_3 \in \mathbb{N}_0$. Thus, $d_{a_1} + d_{a_2} + d_{a_3} \in 1 + 3\mathbb{N}_0$, and so $d_{a_1} + d_{a_2} + d_{a_3} = 1$ because the inequality $d_{a_1} + d_{a_2} + d_{a_3} \ge 4$ implies that $z_{a_1} a_1 + z_{a_2} a_2 + z_{a_3} a_3 > \frac{4}{3}$. Therefore there is a unique index $n \in \mathbb{N}_0$ such that the atom $a_n := \frac{1}{p_{2n+1}}\left(\frac{1}{3} + \frac{1}{2^n}\right) \in A_2$ appears in z, and it appears exactly p_{2n+1} times. Then we can take $r \in \langle A_1 \rangle$ such that $\frac{4}{3} - q = p_{2n+1} a_n + r = \left(\frac{1}{3} + \frac{1}{2^n}\right) + r$, which implies that

$$1 - q = \frac{1}{2^{n+1}} + \left(\frac{1}{2^{n+1}} + r\right) \quad \text{and} \quad \frac{4}{3} - q = \frac{1}{2^{n+1}} + \left(\frac{1}{3} + \frac{1}{2^{n+1}}\right) + r.$$

Hence $\frac{1}{2^{n+1}}$ is a positive common divisor of the set $\left\{1 - q, \frac{4}{3} - q\right\}$ in M, and so we conclude that $\left\{1 - q, \frac{4}{3} - q\right\}$ does not have an MCD in M. Thus, M is not an MCD-monoid.

The following remark is an immediate consequence of Theorem 4.3.

Remark 4.6. Although the monoid constructed in Example 4.5 is atomic, its power monoid is not atomic.

5. Notions Weaker than Atomicity

In this section, we study three natural generalizations of atomicity that have been recently considered in the literature: near atomicity, almost atomicity, and quasi-atomicity.

5.1. Near Atomicity.

Motivated by the construction provided in Example 4.5, in this section we will produce an atomic Puiseux monoid whose power monoid is not even nearly atomic, giving a negative answer to the question of whether near atomicity ascends from linearly orderable monoids to their corresponding power monoids.

Lemma 5.1. *Let M be a Puiseux monoid, and let T be an element of $\mathscr{P}_{\text{fin}}(M)$. Suppose that any divisor S of T in $\mathscr{P}_{\text{fin}}(M)$ is such that for any singleton $\{x\}$ dividing S in $\mathscr{P}_{\text{fin}}(M)$ there exists a singleton $\{y\} \notin \{\{0\}, S\}$ dividing $S - \{x\}$ in $\mathscr{P}_{\text{fin}}(M)$. Then T is not divisible by any atoms.*

Proof. Suppose, by way of contradiction, that S is an atom of $\mathscr{P}_{\text{fin}}(M)$ that divides T. Then $\{0\} \mid_{\mathscr{P}_{\text{fin}}(M)} S$, so there is some $\{x\} \mid_{\mathscr{P}_{\text{fin}}(M)} S - \{0\} = S$ where $\{x\} \notin \{\{0\}, S\}$. Since M is reduced, $\{x\}$ and $S - \{x\}$ are not invertible elements, which contradicts that S is an atom. □

Now we present a fact which will become useful to prove the main theorem of this section.

Lemma 5.2. *Let M be a Puiseux monoid, and let p be a prime such that there exists a unique atom a in M whose denominator is divisible by p. If the map $c_{a,p}$ is constant on S for some $S \in \mathscr{P}_{\text{fin}}(M)$, then $c_{a,p}$ is constant on any divisor D of S in $\mathscr{P}_{\text{fin}}(M)$.*

Proof. Let S be an element of $\mathscr{P}_{\text{fin}}(M)$ such that the map $c_{p,a}$ is constant on S, and let D be a divisor of S in $\mathscr{P}_{\text{fin}}(M)$. Write $S = C + D$ for some $C \in \mathscr{P}_{\text{fin}}(M)$. Then for any $r, s \in D$ and $t \in C$, we see that

$$c_{a,p}(r) + c_{a,p}(t) = c_{a,p}(r+t) = c_{a,p}(s+t) = c_{a,p}(s) + c_{a,p}(t).$$

Hence $c_{a,p}(r) = c_{a,p}(s)$, and so we conclude that $c_{a,p}$ is also constant on D. □

We proceed to prove that near atomicity does not ascend from linearly orderable monoids to their corresponding power monoids. This is the primary result of this section.

Theorem 5.3. *There exists an atomic linearly orderable monoid whose power monoid is not even nearly atomic.*

Proof. Let N, D, Q, and the terms of the sequence $(P_k)_{k \ge 1}$ be pairwise disjoint infinite subsets of $\mathbb{P}_{\ge 3}$. Also, let the sequences $(n_i)_{i \ge 1}$, $(d_i)_{i \ge 1}$, and $(q_i)_{i \ge 1}$ be strictly increasing enumerations of the sets N, D, and Q, respectively, and for each $k \in \mathbb{N}$, let $(p_{k,i})_{i \ge 1}$ be a strictly increasing enumeration of P_k. Now let M be the Puiseux monoid generated by the set $\bigcup_{n \in \mathbb{N}_0} A_k$, where

$$A_0 := \left\{ \frac{1}{2^i q_i} : i \in \mathbb{N} \right\} \quad \text{and} \quad A_k := \left\{ \frac{1}{p_{k,i}} \left(\frac{n_k}{d_k} + \frac{1}{2^i} \right) : i \in \mathbb{N} \right\}$$

for every $k \in \mathbb{N}$. Now set $a_{0,i} := \frac{1}{2^i q_i}$ and $a_{k,i} := \frac{1}{p_{k,i}} \big(\frac{n_k}{d_k} + \frac{1}{2^i} \big) \in A_k$ for all $(k,i) \in \mathbb{N} \times \mathbb{N}$. After replacing, for each $k \in \mathbb{N}$, the sequence $(p_{k,i})_{i \ge 1}$ by a suitable subsequence, one can assume that $p_{k,i} > 2^i n_k + d_k$ for every $i \in \mathbb{N}$. Hence the $p_{k,i}$-valuation of $a_{k,i}$ is negative for all $(k,i) \in \mathbb{N} \times \mathbb{N}$. Thus, it follows from part (1) of Lemma 4.4 that $\mathscr{A}(M) = \bigcup_{k \in \mathbb{N}_0} A_k$ and, therefore, M is atomic. For all $r \in M$ and $i \in \mathbb{N}$, we set

$$c_{a_{0,i}}(r) := c_{a_0, q_i}(r) \quad \text{and} \quad c_{a_{k,i}}(r) := c_{a_{k,i}, p_{k,i}}(r).$$

Now set $b_{k,i} := \frac{n_k}{d_k} + \frac{1}{2^i}$ for all $(k,i) \in \mathbb{N} \times \mathbb{N}$, and observe that $N := \big\langle b_{k,i} : (k,i) \in \mathbb{N} \times \mathbb{N} \big\rangle$ is a submonoid of M. In addition, it follows from part (1) of Lemma 4.4 that N is atomic with $\mathscr{A}(N) = \{ b_{k,i} : (k,i) \in \mathbb{N} \times \mathbb{N} \}$.

We will prove that $\mathscr{P} := \mathscr{P}_{\text{fin}}(M)$ is not nearly atomic. Fix $S \in \mathscr{P}$ and let us find $T \in \mathscr{P}$ such that $S + T$ is not atomic. Because S is a finite set, we can take $j \in \mathbb{N}$ large enough so that $p_{k,1} > \max \mathsf{d}(S)$ for all $k \ge j$, while $d_j > \max \mathsf{d}(S)$ and $n_j > \max S$. Now set $T := \{t_j, t_{j+1}\}$, where $t_j := b_{j,1}$ and $t_{j+1} := b_{j+1,1}$. Fix $(s,t) \in S \times T$ and $c \in \{0,1\}$, and then write

$$s + ct = r_j + \sum_{k \ge j} \sum_{i \in \mathbb{N}} c_{k,i} a_{k,i}, \tag{5.1}$$

where $r_j \in M$ and $\{c_{k,i} : (k,i) \in \mathbb{N}_{\ge j} \times \mathbb{N}\} \subseteq \mathbb{N}_0$ (with $c_{k,i} = 0$ for almost all $(k,i) \in \mathbb{N}_{\ge j} \times \mathbb{N}$) such that $d_k \nmid \mathsf{d}(r_j) \ge 0$ and $p_{k,i} \nmid \mathsf{d}(r_j) \ge 0$ for all $(k,i) \in \mathbb{N}_{\ge j} \times \mathbb{N}$. For any $(k,i) \in \mathbb{N}_{\ge j} \times \mathbb{N}$, we see that $p_{k,i} \ge p_{k,1} > \max \mathsf{d}(S)$ and so $p_{k,i} \nmid \mathsf{d}(s)$. Thus, for each $(k,i) \in \mathbb{N}_{\ge j} \times \mathbb{N}$, we obtain that $p_{k,i} \nmid \mathsf{d}(s + ct)$, and so $p_{k,i} \mid c_{k,i}$. Therefore

$$s + ct = r_j + \sum_{k \ge j} \sum_{i \in \mathbb{N}} \frac{c_{k,i}}{p_{k,i}} \Big(\frac{n_k}{d_k} + \frac{1}{2^i} \Big) = r_j + r'_j + \sum_{k \ge j} c_k \frac{n_k}{d_k}, \tag{5.2}$$

where r'_j is a nonnegative dyadic rational and $c_k := \sum_{i \in \mathbb{N}} \frac{c_{k,i}}{p_{k,i}} \in \mathbb{N}_0$. For each index $k \ge j$, the fact that $d_k \ge d_j > \max \mathsf{d}(S)$ implies that $d_k \nmid \mathsf{d}(s)$. Thus, if $c = 0$, then for each index $k \ge j$ we obtain that $\frac{c_k}{d_k} \in \mathbb{N}_0$ and so the fact that $n_k \ge n_j > \max S$ (along with (5.2)) implies that $c_k = 0$, whence

$c_{k,\ell} = 0$ for every $\ell \in \mathbb{N}$. Thus, any factorization of s in M contains no copy of the atom $a_{k,\ell}$ for any pair $(k,\ell) \in \mathbb{N}_{\ge j} \times \mathbb{N}$. Proceeding similarly, we can argue that if $c = 1$, then $a_{k,\ell} \nmid s + t$ for each pair $(k,\ell) \in \mathbb{N}_{\ge j+2} \times \mathbb{N}$, from which we can deduce that any factorization z of $s + t_j$ (resp., $s + t_{j+1}$) contains no copy of the atom $a_{k,\ell}$ for any pair $(k,\ell) \in \mathbb{N}_{\ge j+2} \times \mathbb{N}$ and also that we can pick an index $i \in \mathbb{N}$ such that z has either 0 or $p_{j,i}$ (resp., $p_{j+1,i}$) copies of the atom $a_{j,i}$ (resp., $a_{j+1,i}$). Now, for each $s \in S$, define

$$c_{a_{j,i}p_{j,i}}(s) = c_{d_j, a_{j,i}p_{j,i}}\left(s - \sum_{m=1}^{n} c_{a_{j,m}}(s) a_{j,m}\right),$$

where $v_{a_{j,m}}(s) = 0$ whenever $m > n$ (a finite n exists as the element s is atomic). Note that this is equal for large enough $i \in \mathbb{N}$. Before proceeding, we need to establish the following claim.

Claim. Whenever we write $S + T$ as a finite sum of elements of $\mathscr{P}$, there is a summand S' satisfying the following property: for any $s \in S'$, there is another element $t \in S'$ such that one of the following inequalities holds for any sufficiently large index $i \in \mathbb{N}$:

$$c_{a_{j,i}p_{j,i}}(s) > c_{a_{j,i}p_{j,i}}(t) \quad \text{or} \quad c_{a_{j+1,i}p_{j+1,i}}(s) > c_{a_{j+1,i}p_{j+1,i}}(t). \tag{5.3}$$

Proof of Claim. First, observe that T satisfies the property in our claim because $c_{a_{j,i}p_{j,i}}(t_k) = c_{a_{j+1,i}p_{j+1,i}}(t_{j+1}) = 1$ and $c_{a_{j+1,i}p_{j+1,i}}(t_j) = c_{a_{j,i}p_{j,i}}(t_{j+1}) = 0$. Now observe that S does not have any factors from A_j or A_{j+1}, and so $S + T$ also satisfies the desired property. Now notice that from bounding, these values are actually the exact multiplicities for every factorization. This means that for any $B, C \in \mathscr{P}$ such that $B + C \mid_{\mathscr{P}} S + T$, if $a, b \in B$ and $c \in C$, then Lemma 5.2 (along with the previous observation) ensures that the term $\sum_{m=1}^{n} c_{a_{j,m}}(s) a_{j,m}$ does not depend on a and b and, as a consequence,

$$c_{a_{j,i}p_{j,i}}(a + c) - c_{a_{j,i}p_{j,i}}(b + c) = c_{a_{j,i}p_{j,i}}(a) - c_{a_{j,i}p_{j,i}}(b).$$

Now write $S + T = S_1 + \cdots + S_\ell$ for some $S_1, \dots, S_\ell \in \mathscr{P}$, and then take an index m such that $c_{a_{j,m}p_{j,m}}(s) = 0$ for any $s \in S_1 \cup \cdots \cup S_\ell$. If for each summand S_i there exists an element $s_i \in S_i$ for which no corresponding t exists (as in the property of our claim), then $s_1 + \cdots + s_\ell$ is an element in $S + T$ with no corresponding t, which is not possible because $S + T$ satisfies the desired property. This is because, for each $i \in [\![1, \ell]\!]$, the coefficients $v_{a_{j,m}p_{j,m}}(s_i)$ and $c_{a_{j+1,m}p_{j+1,m}}(s_i)$ are minimum in S_i, and so the corresponding coefficients for $s_1 + \cdots + s_\ell$ are minimum in $S_1 + \cdots + S_\ell$. Hence the claim is established.

Now let the summand be S' and take any $s \in S'$. Then the established claim guarantees the existence of $t \in S'$ such that for any sufficiently large $i \in \mathbb{N}$ one of the inequalities in (5.3) holds. Assume, without loss of generality, that for any factorization, $c_{a_{j,i}p_{j,i}}(s) > c_{a_{j,i}p_{j,i}}(t)$ holds for any large enough $i \in \mathbb{N}$. This, together with Lemma 4.4, implies that the set of atoms from A_j in any factorization of s and t is the empty set or a set of $p_{j,i}$ copies of the atom $a_{j,i}$ for some index i.

Thus, if d is a common divisor of $\{s, t\}$ in M, then the set of atoms from A_j in any factorization of d is the empty set; this means that, for each index $i \in \mathbb{N}$, there exist $p_{j,i}$ copies of the atom $a_{j,i}$ in any factorization of $s - d$, whence $\frac{n_j}{d_j} + \frac{1}{2^i} \mid_M s$ or $\frac{1}{2^{i+1}} \mid_M s$. Therefore, for any $\{d\} \mid_{\mathscr{P}} S'$, there exists $k \in \mathbb{N}$ large enough such that $\{\frac{1}{2^k}\} \mid_{\mathscr{P}} S' - \{d\}$. As a consequence, it follows from Lemma 5.2 that $S + T$ is not an atomic element of $\mathscr{P}$. Hence we can conclude that the power monoid $\mathscr{P}$ is not nearly atomic. $\square$

5.2. **Almost Atomicity and Quasi-Atomicity.** Finally, we explore the property of almost atomicity and quasi-atomicity, and we construct a rank-2 linearly orderable monoid that is almost atomic but its power monoid is not even quasi-atomic. With this construction, we provide a negative answer to the ascent of both almost and quasi-atomicity to power monoids.

For any $q \in \text{gp}(Q)$, define $v_{1/p}(q) = \frac{c}{p}$, where c is the unique element in $[\![0, p-1]\!]$ such that $v_p\big(q - \frac{c}{p}\big) \geq 0$. Now, define $k(x) = 2 - x + \sum_{p \in P} v_{1/p}(x) \in \mathbb{Z}$. Then consider the Puiseux monoid $Q := \big\langle \frac{1}{p} : p \in \mathbb{P} \setminus \{2\} \big\rangle$, and set $N := \text{gp}(Q) \cap (2,3)$. The most important object for the rest of this section is the following additive submonoid of $\mathbb{Q}^2$:

$$M := \big\langle A \cup B \cup T \big\rangle, \tag{5.4}$$

where

$$A = \Big\{ \Big(\frac{1}{5}, x + \frac{1}{2^{k(x)}} \Big) : x \in N \Big\}, \quad B = \Big\{ \Big(\frac{1}{7}, x + \frac{1}{2^{k(x)}} \Big) : x \in N \Big\}, \text{ and } T = \Big\{ \Big(0, \frac{1}{2^i} \Big) : i \in \mathbb{N} \Big\}.$$

Note that $\mathscr{A}(M) = A \cup B$ and $k(x) \geq 0$ for $x \in N$.

Lemma 5.4. *The monoid M is almost atomic and any non-atom is divisible by some element of T.*

Proof. We first show M is almost atomic. It suffices to show for any element $(0, \frac{1}{2^n}) \in T$, there is an atomic element a where $a + (0, \frac{1}{2^n})$ is also atomic. Take $x, y \in N$ such that $k(x) = n-1$, and $p \in \mathbb{P} \setminus \{2\}$ such that $v_{1/p}(x) = v_{1/p}(y) = 0$ and $x - \frac{1}{p}, y + \frac{1}{p} \in \text{gp}(Q) \cap [2,3)$. Then $k\big(x - \frac{1}{p}\big) = k(x) + 1 = n$ and $k\big(y + \frac{1}{p}\big) = k(y)$. Thus, $b + (0, \frac{1}{2^n}) = a$, where

$$a = \Big(\frac{1}{5}, x + \frac{1}{2^{k(x)}} \Big) + \Big(\frac{1}{5}, y + \frac{1}{2^{k(y)}} \Big) \quad \text{and} \quad b = \Big(\frac{1}{5}, x - \frac{1}{p} + \frac{1}{2^{k(x - \frac{1}{p})}} \Big) + \Big(\frac{1}{5}, y + \frac{1}{p} + \frac{1}{2^{k(y + \frac{1}{p})}} \Big)$$

However, both a and b are atomic, so M is almost atomic. We now show any non-atom of M is divisible by an element T. It suffices to show that any element which is the sum of two atoms satisfies this, as every non-atom must be divisible by an element of T or the sum of two elements of $A \cup B$. Let the element be the sum of $a = \big(k_1, x + \frac{1}{2^{k(x)}}\big)$ and $b = (k_2, y + \frac{1}{2^{k(y)}})$ for some $x, y \in N$. Then from the argument before, $\frac{1}{2^{k(x)+1}} \mid_M a + b$, as desired. □

Lemma 5.4 shows that every atom of $\mathscr{P}_{\text{fin}}(M)$ contains an atom or 0, as otherwise it is divisible by $\{(0, \frac{1}{2^k})\}$ for some k. Call a set that satisfies this property a *semi-atom*. An element of $\mathscr{P}_{\text{fin}}(M)$ is atomic only if it can be expressed as the sum of atoms, and therefore it must be expressed as a sum of semi-atoms. We are now at a position to prove that $\mathscr{P}_{\text{fin}}(M)$ is not quasi-atomic.

Theorem 5.5. *There exists an almost atomic submonoid of $\mathbb{Q}^2$ such that its power monoid is not quasi-atomic.*

Proof. We define M as above. We first note that $(\frac{1}{5}, \frac{10}{3}) = (\frac{1}{5}, 2 + 1/3 + \frac{1}{2^{k(2+1/3)}}) \in A$, and similarly $(\frac{1}{7}, \frac{10}{3}) \in B$. It suffices to show that no S exists so that $S + \{(\frac{2}{5}, \frac{20}{3}), (\frac{3}{7}, 10)\}$ is atomic. For each $x \in M$, let $\pi(x)$ be the first coordinate of x.

By way of contradiction, suppose $\{(\frac{2}{5}, \frac{20}{3}), (\frac{3}{7}, 10)\} \mid_M \sum_{i=1}^n F_i$ for some atoms F_i. For each F_i, let $f_i \in F_i$ so that $\pi(f_i)$ is minimum. Then because each F_i is an atom and therefore a semi-atom, we see $\pi(f_i) \in \{0, \frac{1}{5}, \frac{1}{7}\}$. This means that for any $x \in F_i$, either $\pi(x) = \pi(f_i)$ or $\pi(x) - \pi(f_i) \geq \frac{2}{35}$. Thus, for any $x \in \sum_{i=1}^n F_i$, either $\pi(x) = \pi\big(\sum_{i=1}^n f_i\big)$ or $\pi(x) - \pi\big(\sum_{i=1}^n f_i\big) \geq \frac{2}{35}$. In other words, $\big|\pi(x) - \pi\big(\sum_{i=1}^n f_i\big)\big| \neq \frac{1}{35} = \frac{3}{7} - \frac{2}{5}$. However, $\sum_{i=1}^n f_i \in \sum_{i=1}^n F_i$, contradicts the fact that $\{(\frac{2}{5}, \frac{20}{3}), (\frac{3}{7}, 10)\} \mid_M \sum_{i=1}^n F_i$. Therefore, we conclude that M is not quasi-atomic. □

Unlike the cases of atomicity and near atomicity, we could not find a rank-one torsion-free almost atomic (resp., quasi-atomic) monoid whose power monoid is not almost atomic (resp., quasi-atomic). Aiming to motivate the search for such rank-one torsion-free monoid, we conclude this section with the following open question.

Question 5.6. *Can we construct a rank-one torsion-free almost atomic (resp., quasi-atomic) monoid whose power monoid is not almost atomic (resp., quasi-atomic)?*

6. The Furstenberg Property

In this final section we turn our attention to the Furstenberg property, which is a property defined in terms of divisibility by atoms and it is also a notion weaker than atomicity.

6.1. **Furstenberg and Weaker Notions.** Similar to atomicity, there are notions of almost and quasi-Furstenberg, which were introduced and studied in [20] in the setting of integral domains and were then investigated in [22] in the setting of Puiseux monoids. Let M be a monoid. We say that M is *quasi-Furstenberg* if for each non-invertible $b \in M$, there exists $c \in M$ and $a \in \mathscr{A}(M)$ such that $a \mid_M b + c$ but $a \nmid_M c$. On the other hand, we say that M is *almost Furstenberg* if for each non-invertible $b \in M$ there exists an atomic element $c \in M$ and $a \in \mathscr{A}(M)$ such that $a \mid_M b + c$ but $a \nmid_M c$. Finally, we say that M is *nearly Furstenberg* if there exists $c \in M$ such that for each non-invertible $b \in M$, there exists $a \in \mathscr{A}(M)$ such that $a \mid_M b + c$ but $a \nmid_M c$. It turns out that each of the nearly, almost, and the quasi-Furstenberg properties ascends from linearly orderable monoids to their corresponding power monoids. Before proving this, we need the following lemma.

Lemma 6.1. *Let M be a linearly orderable monoid. Then for each non-invertible $S \in \mathscr{P}_{\text{fin}}(M)$ there exists either an atom A of $\mathscr{P}_{\text{fin}}(M)$ with $|A| \ge 2$ such that A divides S in $\mathscr{P}_{\text{fin}}(M)$ or a non-invertible $d \in M$ such that $\{d\}$ divides S in $\mathscr{P}_{\text{fin}}(M)$.*

Proof. Suppose that M is a linearly ordered monoid under the total order relation $\le$, and let S be a nonempty finite subset of M. Assume that $\{d\}$ does not divide S in $\mathscr{P}_{\text{fin}}(M)$ for any non-invertible $d \in M$. If S is an atom of $\mathscr{P}_{\text{fin}}(M)$, then S cannot be a singleton and so we can take $A := S$. Assume, otherwise, that S is not an atom, and write $S = A + B$ for some non-invertible A and B in $\mathscr{P}_{\text{fin}}(M)$. Among all such sum decompositions, assume that we have chosen one minimizing $|A|$. Since $|A| \ge 2$ and $|B| \ge 2$, it follows from Lemma 3.1 that $|A| < |S|$ and $|B| < |S|$. In this case, A must be an atom of $\mathscr{P}_{\text{fin}}(M)$ as otherwise we could write $A = A' + B'$ for some non-invertible elements A' and B' in $\mathscr{P}_{\text{fin}}(M)$ and the fact that $|B'| \ge 2$ would imply that $|A'| < |A|$, contradicting the minimality of $|A|$. □

We are in a position to establish the ascent of all the Furstenberg-like properties introduced earlier from linearly orderable monoids to their corresponding power monoids.

Theorem 6.2. *Let M be a linearly ordereable monoid. Then the following statements hold.*

1. *If M is a Furstenberg monoid, then $\mathscr{P}_{\text{fin}}(P)$ is a Furstenberg monoid.*
2. *If M is quasi-Furstenberg, then $\mathscr{P}_{\text{fin}}(P)$ is also quasi-Furstenberg.*
3. *If M is almost Furstenberg, then $\mathscr{P}_{\text{fin}}(P)$ is also almost Furstenberg.*
4. *If M is nearly Furstenberg, then $\mathscr{P}_{\text{fin}}(P)$ is also nearly Furstenberg.*

Proof. Set $\mathscr{P} := \mathscr{P}_{\text{fin}}(M)$.

(1) For each $S \in \mathscr{P}$, either there exists an atom $A \in \mathscr{P}$ with $|A| > 1$ such that $A \mid_{\mathscr{P}} S$ or $\{d\} \mid_{\mathscr{P}} S$ for some non-invertible $d \in M$. In the first case, we are done because A is an atom of $\mathscr{P}$ such that $A \mid_{\mathscr{P}} S$. In the second case, since M is a Furstenberg monoid, we can take $a \in \mathscr{A}(M)$ such that $a \mid_M d$ and, therefore, we obtain that $\{a\}$ is an atom of $\mathscr{P}$ such that $\{a\} \mid_{\mathscr{P}} S$.

(2) This follows similar to part (1).

(3) This follows similar to part (1).

(4) This follows similar to part (1). □

6.2. **The TIDF Property.** In this last section, we consider TIDF-monoids, which are a special type of Furstenberg monoids. Recall that a monoid is a TIDF-monoid provided that the set of divisors of each non-invertible element is nonempty and finite (up to associate). We first prove that the TIDF property ascends to power monoids in the class of positive Archimedean monoids. This is a consequence of the fact that for positive Archimedean monoids the TIDF property implies atomicity. We proceed to prove this last statement.

Proposition 6.3. *Let M be an positive Archimedean monoid. If M is a TIDF-monoid, then M is atomic.*

Proof. Since M is positive Archimedean monoid, we can assume that its Grothendieck group $\text{gp}(M)$ is a linearly ordered group under the total order $\preceq$, and also that M is a submonoid of the nonnegative cone $\text{gp}(M)^+$ (we can also use Hölder's theorem, and assume that $M \subseteq \mathbb{R}_{\geq 0}$).

Suppose, by way of contradiction, that M is not atomic. Then there exists $q_0 \in M$ that is not atomic. Now set $A_1 := \{a \in \mathscr{A}(M) : a \mid_M q_0\}$. Note that A_1 is nonempty and finite because M is a reduced TIDF-monoid and $q_0 \neq 0$. Thus, A_1 has a minimum element. Now set $q_1 := q_0 - \min A_1$. Now suppose we have produced, for some $n \in \mathbb{N}$, a finite descending chain $(A_i)_{i=1}^n$ of nonempty finite subsets of $\mathscr{A}(M)$ and a finite sequence $(q_i)_{i=1}^n$ with terms in $M^\bullet$ such that $q_i = q_{i-1} - \min A_i$ for every $i \in [\![1, n]\!]$. The equality

$$q_0 = q_n + \sum_{i=1}^{n} \min A_i,$$

along with the fact that q_0 is not atomic, guarantees that q_n is not atomic. Now set $A_{n+1} := \{a \in \mathscr{A}(M) : a \mid_M q_n\}$. Because $q_n \mid_M q_{n-1}$, the inclusion $A_{n+1} \subseteq A_n$ must hold. In addition, observe that A_{n+1} is a nonempty and finite set because M is a reduced TIDF-monoid and $q_n \neq 0$. Then we can set $q_{n+1} := q_n - \min A_{n+1}$. After repeating this process indefinitely, we obtain a descending chain $(A_n)_{n\geq 1}$ of nonempty finite subsets of $\mathscr{A}(M)$ and a sequence $(q_n)_{n\geq 0}$ with terms in M such that $q_n - q_{n+1} = \min A_{n+1}$ for every $n \in \mathbb{N}_0$. For each $n \in \mathbb{N}$, we can now write

$$q_0 = q_n + \sum_{j=0}^{n-1}(q_j - q_{j+1}) = q_n + \sum_{j=0}^{n-1} \min A_{j+1} \succeq n \min A_1.$$

Since $\min A_1 \succ 0$, the fact that $q_0 \succeq n \min A_1$ for every $n \in \mathbb{N}$ contradicts that M is a positive Archimedean monoid. □

We obtain the following corollary on the ascent of both the FFM and the TIDF property.

Corollary 6.4. *Let M be a positive Archimedean monoid. Then the following statements hold.*

(1) *If M is an FFM, then its power monoid is also an FFM.*

(2) *If M is a TIDF-monoid, then its power monoid is also a TIDF-monoid.*

Proof. Set $\mathscr{P} := \mathscr{P}_{\text{fin}}(M)$.

(1) Assume first that M is an FFM. Since M is a positive Archimedean monoid, we can take a linearly ordered abelian group G such that M is a submonoid of the nonnegative cone of G. According to Hölder's theorem, G is order-isomorphic to a subgroup of the additive group $\mathbb{R}$. Therefore M is order-isomorphic to some additive submonoid of $\mathbb{R}_{\geq 0}$. Since the power monoids of isomorphic monoids are isomorphic, we can assume that M is a submonoid of $\mathbb{R}_{\geq 0}$. In order to prove now that the power monoid $\mathscr{P}$ is also an FFM, it suffices to follow, *mutatis mutandis*, the argument given in the proof of [13, Theorem 4.2].

(2) Assume now that M is a TIDF-monoid. Since M is a positive Archimedean monoid, it follows from Proposition 6.3 that M is atomic. As M is atomic and every element of M is only divisible by finitely many atoms (up to associate), [19, Theorem 2] ensures that M is an FFM. Thus, it follows from the previous part that $\mathscr{P}$ is also an FFM and, as a consequence, a TIDF-monoid. □

We now show that, in general, the TIDF property does not ascend to power monoids in the more general class of linearly orderable monoids.

Theorem 6.5. *There exists a linearly orderable monoid satisfying the TIDF property whose finite power monoid does not satisfy the IDF property.*

Proof. Consider the monoid

$$\mathscr{M} = (\mathbb{Z} \cdot a) \oplus (\mathbb{Z} \cdot b) \oplus (\mathbb{Z} \cdot y) \oplus (\mathbb{Z} \cdot z) \oplus \bigoplus_{i=1}^{\infty} (\mathbb{Z} \cdot x_i).$$

Let $A = \mathbb{N}_0 a$, $B = \mathbb{N}_0 b$, $X_i = \mathbb{N}x + \text{gp}(A)$, $Y_i = \mathbb{N}(x_i - y) + \text{gp}(B)$, and $Z = \mathbb{N}z + \text{gp}(\langle \bigcup_{i \in \mathbb{N}} X_i \cup Y_i \rangle)$. Then let M be the submonoid of $\mathscr{M}$ generated by $A \cup B \cup (\cup_{i \in \mathbb{N}} X_i \cup Y_i) \cup Z$. We have M is linearly orderable because $\mathscr{M}$ is cancellative and torsion-free, hence M is cancellative and torsion-free, which implies M is linearly orderable. The only atoms in M are a and b, and each element of M is divisible by at least one of these. Therefore, M is a TIDF monoid and is Furstenberg.

Now we claim that $\{z, z - y\} \in \mathscr{P}_{\text{fin}}(M)$ has infinitely many atom divisors, hence $\mathscr{P}_{\text{fin}}(M)$ is not a TIDF monoid. Note that $\{x_i, x_i - y\} \mid_{\mathscr{P}_{\text{fin}}(M)} \{z, z - y\}$ for all i. Thus, it suffices to show $\{x_i, x_i - y\}$ is an atom for each i. However, if $\{x_i, x_i - y\} = S + T$, then by Lemma 3.1, at least one of S and T is a singleton. Assume $S = \{s\}$ is the singleton. Then s is a common divisor of x_i and $x_i - y$. But x_i is not divisible by b, and $x_i - y$ is not divisible by a. Therefore, s does not have any atom divisors, so because M is Furstenberg, we see s is a unit. As a result, $\{x_i, x_i - y\}$ is an atom for all i, which means $\mathscr{P}_{\text{fin}}(M)$ is not IDF. □

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