UNGAR GAMES ON THE YOUNG-FIBONACCI LATTICE AND THE LATTICES OF THE ORDER IDEALS OF SHIFTED STAIRCASES

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ABSTRACT. In 2023, Defant and Li introduced an Ungar move, which sends an element v of a meet-semilattice L to the meet of some subset of the elements covered by v. More recently, Defant, Kravitz, and Williams introduced the Ungar game on L, in which two players take turns making nontrivial Ungar moves starting from an element of L until the player who cannot make a nontrivial Ungar move loses. In this note, we settle two conjectures by Defant, Kravitz, and Williams on the Ungar games on the Young-Fibonacci lattice and the lattices of the order ideals of shifted staircases.

1. INTRODUCTION

Let L be a finite meet-semilattice. In 2023, Defant and Li [2] introduced an *Ungar move*, which sends an element v of L to the meet of $\{v\} \cup T$ for some subset T of the elements that v covers. If $T = \emptyset$, then the Ungar move is *trivial*. For example, in Figure 1, the element a covers the elements $\{b, c, e\}$, and so the set of elements that can be obtained by applying a nontrivial Ungar move is $\{b, c, e, f, g, h\}$. In 2024, Defant, Kravitz, and Williams [1] introduced the Ungar game on a finite meet-semilattice L. In the *Ungar game* on L, Atniss and Eeta alternate turns, starting from an element of L. On each turn, the current player chooses a nonempty subset T of the elements covered by the current element and performs the corresponding Ungar move. The player unable to make a nontrivial Ungar move loses.

An element v of L is an *Atniss win* if Atniss has a winning strategy in the Ungar game on the sublattice $[\hat{0}, v]$ starting from v; otherwise, v is an *Eeta win* (see for example, Figure 1). Let $\mathbf{A}(L)$ and $\mathbf{E}(L)$ be the set of Atniss and Eeta wins in L, respectively. For a graded lattice, let $\mathbf{A}_r(L)$ and $\mathbf{E}_r(L)$ be the set of Atniss and Eeta wins of rank r, respectively.

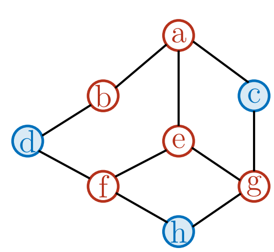

FIGURE 1. A lattice with Atniss wins unshaded and colored red and Eeta wins shaded and colored blue.

In [1], Defant, Kravitz, and Williams studied the Ungar games on the weak order on S_n, the intervals in Young's lattice, and the Tamari lattices. In this note, we settle two conjectures in [1] on the Ungar games on the Young-Fibonacci lattice and the lattices of the order ideals of shifted staircases.

First, we characterize the Eeta wins in the Young-Fibonacci lattice $\mathbb{YF}$. Let $\mathbb{YF}_r$ be the set of elements in $\mathbb{YF}$ of rank r. In what follows, let $|v|$ be the length of a string v, and for $v = v_1 v_2 \dots v_{|v|}$ and $1 \leq i \leq j \leq |v|$, let $v_{i:j} = v_i v_{i+1} \cdots v_j$. If $i > j$, then $v_{i:j} = \emptyset$.

Theorem 1.1. *For $r \geq 0$, an element $v \in \mathbb{YF}_r$ is an Eeta win if and only if*

- $v_{1:|v|-1} = 11 \cdots 1$ *and the number of* 1*s in* v *is even, or*
- $v_{1:|v|-1} \neq 11 \cdots 1$ *and the number of* 1*s to the left of the leftmost* 2 *in* v *is odd.*

Note that $v_{1:|v|-1} = 11 \cdots 1$ is vacuously true when $|v| \leq 1$. Defant, Kravitz, and Williams' conjecture [1, Conjecture 6.1] follows.

Corollary 1.2. *For $r \geq 2$, it holds that $|\boldsymbol{E}_r(\mathbb{YF})| = f_{r-2} + (-1)^r$.*

Next, let $J(\mathrm{SS}_n)$ be the lattice of the order ideals of the n^{th} shifted staircase SS_n, ordered by containment. We characterize the Eeta wins in $J(\mathrm{SS}_n)$, which corrects and settles [1, Conjecture 6.2]. Refer to Section 4 for the natural bijection between an order ideal $v \in J(\mathrm{SS}_n)$ and a binary string $s \in \{0,1\}^n$.

Theorem 1.3. *An order ideal $v \in J(\mathrm{SS}_n)$ with binary representation $s \in \{0,1\}^n$ is an Eeta win if and only if*

- $s_{|s|} = 0$, *and*
- *there are no odd-length sequences of* 1*s followed by an odd-length sequence of* 0*s in* $s_{1:|s|-1}$.

The sequence $|\mathbf{E}(J(\mathrm{SS}_n))|$ follows directly.

Corollary 1.4. *The sequence $|\mathbf{E}(J(\mathrm{SS}_n))|$ is OEIS A061279 [4].*

The rest of this note is as follows. Section 2 is preliminaries. In Sections 3 and 4, we prove Theorems 1.1 and 1.3, respectively.

2. Preliminaries

A *meet-semilattice* L is a poset in which every pair of elements has a greatest lower bound (called their *meet* and notated by $\wedge$). Due to the commutative and associative properties of the meet operation, each subset X of finite meet-semilattice L has a greatest lower bound, or meet. We call L a *lattice* if it is a meet-semilattice in which any two elements also have a least upper bound (called their *join* and notated by $\vee$). For $\{u, v\} \subseteq L$ such that $u \leq v$, the interval $[u, v]$ is the set of all $w \in L$ such that $u \leq w \leq v$. If $|[u, v]| = 2$, then say that v *covers* u and write that $u \lessdot v$. An *order ideal* of L is a subposet $I \subseteq L$ such that for any element $u \in L$ and any element $v \in I$, if $u \leq v$, then $u \in I$. For the rest of this note, all order ideals of all posets are finite. Let $\hat{0}$ be the minimal element of a lattice L. A *graded lattice* L has a rank function ρ such that $\rho(\hat{0}) = 0$ and $\rho(u) + 1 = \rho(v)$ if $u \lessdot v$.

Given an element v in meet-semilattice L, an *Ungar move* sends v to the meet of $\{v\} \cup T$, where T is a subset of the elements that v covers, chosen by the player making the move. In the *Ungar game* on L, Atniss starts on a given element of L and alternates with Eeta in making nontrivial Ungar moves. The game continues until a player reaches the minimal element $\hat{0}$ and cannot make a nontrivial Ungar move, and that player loses. Let $\mathrm{Ung}(v)$ be the set of elements in L that can be obtained by applying an Ungar move to v. The sets of Atniss and Eeta wins in L can be determined recursively, as stated in Defant, Kravitz, and Williams' paper [1].

Lemma 2.1 (Defant, Kravitz, and Williams [1, Section 1]). *An element v in meet-semilattice L is an Eeta win if and only if every element of* $\mathrm{Ung}(v) \setminus \{v\}$ *is an Atniss win. Otherwise,* $v \in \mathbf{A}(L)$.

3. The Young-Fibonacci Lattice

3.1. **Definitions.** The Young-Fibonacci lattice $\mathbb{YF}$ was introduced by Fomin [3] and Stanley [5]. The elements of $\mathbb{YF}$ are the words of the alphabet $\{1, 2\}$. Note that any substring of an element in $\mathbb{YF}$ is also an element of $\mathbb{YF}$. The rank of $v \in \mathbb{YF}$ is defined by $\rho(v) := \sum_{i=1}^{|v|} v_i$. Let $\mathbb{YF}_r$ be the set of all elements in $\mathbb{YF}$ of rank r. Let $u \in \mathbb{YF}$. The lattice $\mathbb{YF}$ is defined such that $u \lessdot v$ if and only if

- $v = u_{1:i} 1 u_{i+1:|u|}$ such that $u_{1:i}$ contains no 1s, or
- $v = u_{1:i-1} 2 u_{i+1:|u|}$ such that u_i is the leftmost 1 in u.

(see for example, Figure 2 for $\mathbb{YF}$ up to rank 5).

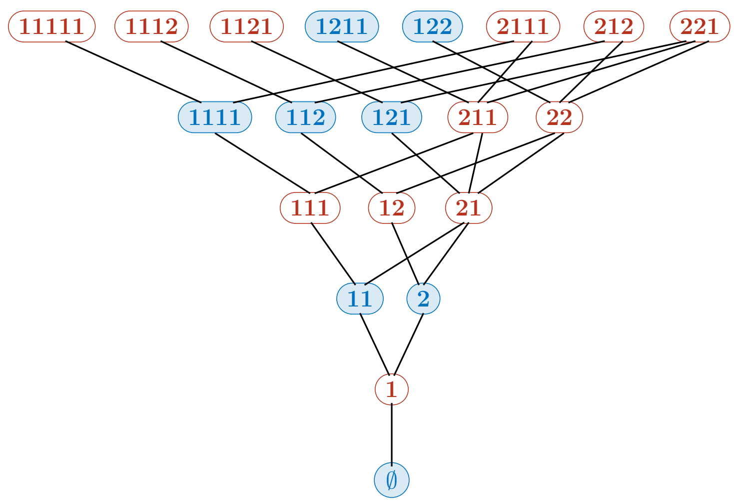

Figure 2. The lattice $\mathbb{YF}$ up to rank 5 with Atniss wins unshaded and colored red and Eeta wins shaded and colored blue.

3.2. **Proof of Theorem 1.1.** Now, we prove Theorem 1.1.

Proof of Theorem 1.1. We induct on the rank r. The statement holds when $r \in \{2, 3\}$. Now, suppose that $r \geq 4$ and that the statement holds for $r-1$ and $r-2$. First, suppose that $v \in \mathbb{YF}_r$ and that $v_1 = 1$. Then $\mathrm{Ung}(v) \setminus \{v\} = \{v_{2:|v|}\}$, because the only element that v covers is $v_{2:|v|}$. Thus, by Lemma 2.1, $v \in \mathbf{E}_r(\mathbb{YF})$ if and only if $v_{2:|v|} \in \mathbf{A}_{r-1}(\mathbb{YF})$ and does not satisfy the theorem statement. Therefore, by the induction hypothesis, $v \in \mathbf{E}_r(\mathbb{YF})$ if and only if $v_{1:|v|-1} = 11\cdots1$ and the number of 1s in v is even, or $v_{1:|v|-1} \neq 11\cdots1$ and the number of 1s to the left of the leftmost 2 in v is odd.

Next, suppose that $v \in \mathbb{YF}_r$ and that $v_1 = 2$. Then $\{1v_{2:|v|}\} \subseteq \mathrm{Ung}(v)$, because $1v_{2:|v|} \lessdot v$. Next, $\{v_{2:|v|}\} \subseteq \mathrm{Ung}(v)$, because if $v_2 = 1$, then $v_{2:|v|} = v_1 v_{3:|v|} \wedge 1v_{2:|v|}$, and if $v_2 = 2$, then $v_{2:|v|} = v_1 1 v_{3:|v|} \wedge 1v_{2:|v|}$. By the induction hypothesis, either $v_{2:|v|}$ or $1v_{2:|v|}$ is an Eeta win. Thus, $v \in \mathbf{A}_r(\mathbb{YF})$ by Lemma 2.1. □

Corollary 1.2 now follows.

Proof of Corollary 1.2. We induct on the rank r. The statement holds for $r = 2$. Now, suppose that $r \geq 3$ and that the statement holds for $r-1$. By Theorem 1.1, $v \in \mathbf{E}_r(\mathbb{YF})$ if and only if $v_1 = 1$ and $v_{2:|v|} \in \mathbf{A}_{r-1}(\mathbb{YF})$. Thus, $|\mathbf{E}_r(\mathbb{YF})| = f_{r-1} - |\mathbf{E}_{r-1}(\mathbb{YF})| = f_{r-2} + (-1)^r$ by the induction hypothesis. □

4. The Lattices of the Order Ideals of Shifted Staircases

4.1. **Definitions.** The n^{th} *shifted staircase* SS_n consists of all pairs $(i, j) \in \mathbb{N}^2$ such that $1 \leq i \leq j \leq n$ (see for example, Figure 3 for SS_5). There is a natural bijection between the order ideals of SS_n and the length n binary strings $\{0,1\}^n$ defined by taking the path lying directly above an order ideal of SS_n and sending the up steps (i.e., $(1, 0)$) to 1 and the down steps (i.e., $(0, -1)$) to 0. The *binary representation* $s \in \{0,1\}^n$ of an order ideal v of SS_n is the binary string that the order ideal corresponds to under the natural bijection (see for example, Figure 3 for an order ideal of SS_5 represented by 10100). For a path in SS_n with binary representation $s \in \{0,1\}^n$ such that $s_i = 1$, say that $(\sum_{j=1}^{i} s_j, |s| - i + \sum_{j=1}^{i} s_j)$ is *directly below* the i^{th} step of the path. For example, $(2, 4)$ is directly below the third step of the path in Figure 3.

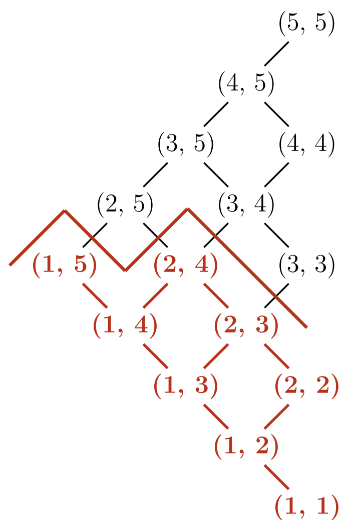

Figure 3. The path above the order ideal of SS_5 with binary representation 10100.

For $s \in \{0,1\}^n$, let $F(s)$ be the set of i that satisfies

- $s_i s_{i+1} = 10$ if $i \leq |s| - 1$, or
- $s_i = 1$ if $i = |s|$.

For example, $F(110101) = \{2, 4, 6\}$. For $A \subseteq F(s)$, let $G(s, A) \in \{0,1\}^n$ be obtained from $s \in \{0,1\}^n$ by replacing s_i with 0 and s_{i+1} (if it exists) with 1 for all $i \in A$. For example, $G(110101, \{2, 6\}) = 101100$. If $|A| = 1$, then we omit the brackets around A.

Now, let $J(\text{SS}_n)$ be the lattice of the order ideals of SS_n ordered by containment. The rank of $v \in J(\text{SS}_n)$ with binary representation $s \in \{0,1\}^n$ is $\rho(s) := 1 \cdot s_{|s|} + 2 \cdot s_{|s|-1} + \cdots + |s| \cdot s_1$. For example, the rank of an order ideal of SS_n with binary representation 1010 is 6. Let $v \in J(\text{SS}_n)$ have binary representation $t \in \{0,1\}^n$. The lattice $J(\text{SS}_n)$ (see for example, Figure 4 for $J(\text{SS}_4)$) is defined such that $u \lessdot v$ if and only if $G(t, i) = s$ for some $i \in F(t)$.

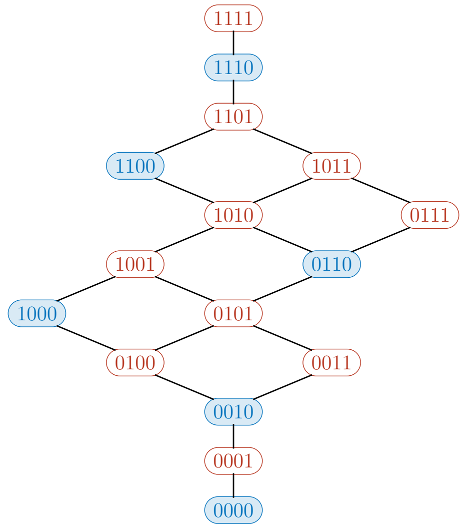

FIGURE 4. The lattice $J(\mathrm{SS}_4)$ with Atniss wins unshaded and colored red and Eeta wins shaded and colored blue.

Next, a 0-*block* (respectively, 1-block) in a binary string is a maximal substring of consecutive 0s (respectively, 1s). For example, in 110001, there are two 1-blocks and one 0-block. Let $H(s)$ be the largest $i \in F(s)$ such that s_i is the rightmost 1 of an odd-length 1-block that is followed by an odd-length 0-block in s, provided such an s_i exists. If such s_i does not exist, then let $H(s) = \infty$, and we call s *odd avoiding*. Otherwise, s is *odd containing*. For example, $H(110100) = \infty$, and 110100 is odd avoiding. However, $H(11010) = 4$, and 11010 is odd containing.

4.2. **Proof of Theorem 1.3.** We now prove auxiliary lemmas towards Theorem 1.3.

Lemma 4.1. *For any $s \in \{0,1\}^n$ and $T \subseteq F(s)$, the meet of all order ideals of SS_n with binary representations $G(s,i)$ for all $i \in T$ is the order ideal with the binary representation $G(s,T)$.*

Proof. Let $s \in \{0,1\}^n$ be the binary representation of an order ideal $v \in J(\mathrm{SS}_n)$. For $i \in F(s)$, let $(x_i, y_i) \in v$ be the lattice point directly below the i^{th} step of the path in $J(\mathrm{SS}_n)$ with binary representation s. In the bijection between the order ideals of SS_n and binary strings, $G(s,i)$ is the binary representation of $v \setminus \{(x_i, y_i)\}$. For $T \subseteq F(s)$, it holds that

$$\wedge\{v \setminus \{(x_i, y_i)\} \mid i \in T\} = \cap\{v \setminus \{(x_i, y_i)\} \mid i \in T\},$$

which has binary representation $G(s,T)$. □

Now, we prove that for any $s \in \{0,1\}^n$, there is a subset of indices $I(s)$ in s such that $G(s, I(s))$ is odd avoiding.

Lemma 4.2. *For any $s \in \{0,1\}^n$, there exists some $I(s) \subseteq F(s)$ such that $G(s, I(s))$ is odd avoiding.*

Proof. We recursively construct $I(s)$ such that

- $|s| \notin I(s)$,
- $|s| - 1 \notin I(s)$ if $s_{|s|-1:|s|} = 00$, and
- $I(s) = I(s_{1:|s|-1})$ if $s_{|s|} = 1$.

When $|s| \leq 2$, let $I(s) = \{1\}$ if $s = 10$; otherwise, let $I(s) := \emptyset$. Now, assume $|s| \geq 3$. First, suppose that $s_{|s|} = 1$. By construction, $|s| - 1 \notin I(s_{1:|s|-1})$. By the definition of odd avoiding, $G(s_{1:|s|-1}, I(s_{1:|s|-1}))1$ is odd avoiding. Thus, $I(s) := I(s_{1:|s|-1})$ suffices.

Next, suppose that $s_{|s|-1:|s|} = 00$. By construction, $|s|-2 \notin I(s_{1:|s|-2})$. By the definition of odd avoiding, $G(s_{1:|s|-2}, I(s_{1:|s|-2}))00$ is also odd avoiding. Thus, $I(s) := I(s_{1:|s|-2})$ suffices.

Now, suppose that $s_{|s|-1:|s|} = 10$ and that $G(s_{1:|s|-1}, I(s_{1:|s|-1}))0$ is odd avoiding. By construction, $|s| - 1 \notin I(s_{1:|s|-1})$. Thus, $I(s) := I(s_{1:|s|-1})$ suffices.

Next, suppose that $s_{|s|-2:|s|} = 010$ and that $G(s_{1:|s|-1}, I(s_{1:|s|-1}))0$ is odd containing. By construction, $|s| - 2 \notin I(s_{1:|s|-2}0)$. By construction and the definition of odd avoiding, $G(s_{1:|s|-2}0, I(s_{1:|s|-2}0))1$ is odd avoiding. Thus, $I(s) := I(s_{1:|s|-2}0) \cup \{|s| - 1\}$ suffices.

Lastly, suppose that $s_{|s|-2:|s|} = 110$ and that $G(s_{1:|s|-1}, I(s_{1:|s|-1}))0$ is odd containing. By construction, $I(s_{1:|s|-1}) = I(s_{1:|s|-2})$. Then

$$G(s_{1:|s|-1}, I(s_{1:|s|-1}))_{1:|s|-2} = G(s_{1:|s|-2}, I(s_{1:|s|-2})).$$

Therefore, the rightmost 1-block of $G(s_{1:|s|-2}, I(s_{1:|s|-2}))$ has even length. By the definition of odd avoiding, $G(s_{1:|s|-2}0, I(s_{1:|s|-2}))1$ is odd avoiding. Thus, $I(s) := I(s_{1:|s|-2}) \cup \{|s| - 1\}$ suffices. □

We now prove Theorem 1.3 using Lemmas 4.1 and 4.2. That is, we prove that $v \in \mathbf{E}(J(\mathrm{SS}_n))$ if and only if the binary representation $s \in \{0,1\}^n$ of v satisfies $s_{|s|} = 0$ and $s_{1:|s|-1}$ is odd avoiding.

Proof of Theorem 1.3. Fix n. We induct on the rank r. The statement holds when $r = 0$. Now, suppose that $r \geq 1$ and that the statement holds for $\leq r-1$. Let $v \in J(\mathrm{SS}_n)$ be of rank r and binary representation $s \in \{0,1\}^n$. First, suppose that $s_{1:|s|-1}$ is odd avoiding and that $s_{|s|} = 1$. Let $u \in J(\mathrm{SS}_n)$ have binary representation $s_{1:|s|-1}0$. Since $u \lessdot v$ and $u \in \mathbf{E}(J(\mathrm{SS}_n))$ by the induction hypothesis and Lemma 2.1, $v \in \mathbf{A}(J(\mathrm{SS}_n))$ by Lemma 2.1.

Next, suppose that $s_{1:|s|-1}$ is odd avoiding and that $s_{|s|} = 0$. Let $u \in \mathrm{Ung}(v) \setminus \{v\}$ have binary representation $t \in \{0,1\}^n$. Then $t = G(s,T)$ for some nonempty $T \subseteq F(s)$ by Lemma 4.1. Now, if $|s| - 1 \in T$, then $t_{|s|} = 1$. Thus, $u \in \mathbf{A}(J(\mathrm{SS}_n))$ by the induction hypothesis. Now, suppose that $|s| - 1 \notin T$. If the $0-$block containing $t_{\max(T)+1}$ has even length, then $H(t_{1:|s|-1}) = \max(T) + 1$. If not, then the $1-$block containing $t_{\max(T)}$ in t has even length, because $t_{1:|s|-1}$ is odd avoiding, and so, $H(t_{1:|s|-1}) = \max(T)-1$. Thus, $t_{1:|s|-1}$ is odd containing, and $u \in \mathbf{A}(J(\mathrm{SS}_n))$ by the induction hypothesis. Therefore, $\mathrm{Ung}(v) \setminus \{v\} \subseteq \mathbf{A}(J(\mathrm{SS}_n))$, and $v \in \mathbf{E}(J(\mathrm{SS}_n))$ by Lemma 2.1.

Finally, suppose that $s_{1:|s|-1}$ is odd containing. Then, $G(s_{1:|s|-1}, I(s_{1:|s|-1}))$ is odd avoiding for some $I(s_{1:|s|-1})$ by Lemma 4.2. By Lemma 4.1, the meet of the order ideals of SS_n with binary representations $s_{1:|s|-1}0$ and $G(s,i)$ for all $i \in I(s_{1:|s|-1})$ has binary representation $t \in \{0,1\}^n$ such that $t_{|s|} = 0$ and $t_{1:|s|-1}$ is odd avoiding. The order ideal with binary representation t is in $\mathbf{E}(J(\mathrm{SS}_n))$ by the induction hypothesis. Thus, $v \in \mathbf{A}(J(\mathrm{SS}_n))$ by Lemma 2.1. □

Corollary 1.4 immediately follows from Theorem 1.3.

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