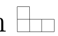
PRIMES 2023: ENTRANCE PROBLEM SET

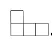
Notation. We let $\mathbb{Z}$ and $\mathbb{R}$ denote the set of integers and the set of real numbers, respectively.

General Math Problems

Problem G1. Can we tile a 4×2023 grid using pieces of the form  ?

Solution. The answer is no. Consider the elements of the additive group $\mathbb{Z}_2 \times \mathbb{Z}_2$ under addition: namely, $0 = (0,0)$, $e_1 = (1,0)$, $e_2 = (0,1)$, and $s = (1,1)$. Now fill out the first (top) row of the grid with the sequence $e_1, e_2, \dots, e_2, e_1$, which contains 2023 terms, starts with e_1, and alternates between copies of e_1 and e_2. Similarly, fill out the second row of the given grid with the sequence $0, s, \dots, s, 0$, which contains 2023 terms, starts with 0, and alternates between copies of 0 and s. Then fill out the third row of the grid with the sequence $e_2, e_1, \dots, e_1, e_2$, which contains 2023 terms, starts with e_2, and alternates between copies of e_2 and e_1. Finally, fill out the fourth (last) row of the grid with the sequence $s, 0, \dots, 0, s$, which contains 2023 terms, starts with s, and alternates between copies of s and 0. We will obtain the following filled-out grid.

e_1	e_2	e_1	...	e_1	e_2	e_1
0	s	0	...	0	s	0
e_2	e_1	e_2	...	e_2	e_1	e_2
s	0	s	...	s	0	s

Observe now that all the elements in the grid add to $\mathbf{0}$ since the sum in each column is $\mathbf{0} + e_1 + e_2 + s = \mathbf{0}$. On the other hand, observe that matter how do we place an L-shaped piece L on the grid, L will cover two copies of the same element and two additional elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$ placed in consecutive rows. Since $2x = \mathbf{0}$ for all $x \in \mathbb{Z}_2 \times \mathbb{Z}_2$ and two elements in consecutive rows add either to e_1 or e_2, the elements covered by L add to either e_1 or e_2. Now suppose, by way of contradiction, that one could tile the grid with (rotated) copies of the L-shaped piece , namely, $L_1, \dots, L_{2023}$. For each $i \in \{1, 2, \dots, 2023\}$, let S_i be the sum of the sum in $\mathbb{Z}_2 \times \mathbb{Z}_2$ of the elements covered by L_i. Since $S_i \in \{e_1, e_2\}$ for all i, we see that

$$\mathbf{0} = \sum_{i=1}^{2023} S_i = me_1 + (2023 - m)e_2$$

for some nonnegative integer m. However, this would imply that both m and $2023 - m$ are even, which is a contradiction. $\square$

Problem G2. Prove that there are infinitely many pairs $(a, b) \in \mathbb{Z}^2$ with $\gcd(a, b) = 1$ such that $\sqrt{-4a^3 - 27b^2}$ is an integer.

Solution. We want $d^2 = -4a^3 - 27b^2$ or, equivalently, $a^3 = (d^2 + 27b^2)/4$. We note that the right hand side is the norm of $(d + 3b\sqrt{-3})/2$ in the field $\mathbb{Q}(\omega)$, where ω is a primitive third root of unity. For any $\alpha \in \mathbb{Z}[\omega]$, the norm of α^3 is the cube of the norm of α. Since the norm of α is integral, we can take any α and set $\alpha^3 = (d + 3b\sqrt{-3})/2$ and then a is the norm of α. To make a and b relatively prime, we can take for example $\alpha = (1 + 3p\sqrt{-3})/2$ with p prime. Then b and d are relatively prime, and hence so are a and b. $\square$

Problem G3. Suppose that α and β are distinct solutions of $x^{2023} - 1 = 0$ in the complex plane, which have been randomly selected. What is the probability that the following inequality $|\alpha + \beta|^2 \geq 2 + \sqrt{3}$ holds?

Solution. Because the 2023-th roots of unity are symmetrically distributed in the unit circle, we can assume, without loss of generality, that $\beta = 1$. After writing $\alpha = \cos\theta + i \sin\theta$, we see that we are looking for the probability that

$$|1 + \alpha|^2 = |1 + \cos\theta + i \sin\theta|^2 = 2 + 2\cos\theta \geq 2 + \sqrt{3}.$$

Therefore we want to find the probability that $\cos\theta \geq \sqrt{3}/2$ or, equivalently, $|\theta| \leq \pi/6$. Since $\alpha \neq 1$, we see that $\theta \in \left\{ \pm \frac{2\pi}{2023} k : 1 \leq k \leq \left\lfloor \frac{2023}{12} \right\rfloor \right\}$.

As there are $2 \cdot 168 = 336$ such angles, the desired probability is $336/2022 \approx 0.166$. $\square$

Problem G4. Let $0 < p < 1$, and let $(a_n)_{n\geq 0}$ be a sequence of nonnegative numbers such that $a_{n+2} \leq (1 - p)a_{n+1} + pa_n$. Prove that $(a_n)_{n\geq 0}$ has a limit.

Solution. Suppose $M = \max(a_0, a_1)$. Then by induction $a_n \leq M$ for every nonnegative integer n. Also, after setting $b_n := a_{n+1} - a_n$, we see that $b_{n+1} \leq -pb_n$. Thus, $b_n \leq p^{n-1}M$. Hence $a_{n+m} - a_n = b_n + \cdots + b_{n+m-1} \leq p^n M/(1 - p)$ for all m, n. Now assume that $x \leq y$ are subsequential limits of $(a_n)_{n\geq 0}$. Then there are subsequences $(a_{n_k})_{k\geq 0}$ and $(a_{n_k+m_k})_{k\geq 0}$ converging to x and y, respectively. However, $a_{n_k+m_k} - a_{n_k} \leq p^{n_k} M/(1-p)$. Taking the limit when k goes to infinity, we get that $y - x \leq 0$. Hence $y = x$, and so we can conclude that the sequence $(a_n)_{n\geq 0}$ converges. $\square$

Problem G5. Determine the maximum value of $m^2 + n^2$ if m and n are positive integers less than 2022 such that $(n^2 - mn - m^2)^2 = 1$.
Hint: The pair (F_{n+1}, F_n) satisfies the given equation, where F_n is the n-th Fibonacci number.

Solution. First note that if $x, y \in \mathbb{Z}^+$ satisfy the given equation then

$$x^2 - xy - y^2 \geq -1 \Rightarrow x^2 \geq xy + y^2 - 1 \geq y^2$$

The last inequality holds because $xy \geq 1$. It follows that $x \geq y$. Now, suppose that $(x, y) = (n, m)$ is a solution of the given equation. Take a nonnegative integer a such that $n = m + a$, and observe that

$$1 = (n^2 - mn - m^2)^2 = ((m+a)^2 - m(m+a) - m^2)^2 = (a^2 + ma - m^2)^2 = (m^2 - ma - a^2)^2.$$

Thus, if $n > m$, we obtain that $(m, a) = (m, n - m)$ is also a solution of the given equation, where m and $n - m$ are positive integers. Hence, if we suppose $n > m$, then we can create a sequence of solutions $S_{-1}, S_0, S_1, \ldots, S_k, \ldots$ for the given equation, where

$$S_{-1} = (n, m), S_k = ((-1)^{k+1} F_{k+1} m + (-1)^k F_k n, (-1)^{k+1} F_{k+1} + (-1)^{k+2} F_{k+2} n),$$

for $k \in \mathbb{Z}, k \geq 0$, where $(F_n)_{n \geq 0}$ is the Fibonacci sequence.

Suppose, by way of contradiction, that $\{S_n \mid n \in \mathbb{Z}_{\geq 0}\}$ is infinite. This means that $a_n > b_n$ for every $n \in \mathbb{Z}_{\geq 0}$. Therefore $b_n = a_{n-1} - b_{n-1}$ and $a_n = b_{n-1}$, which implies that

$$a_n + b_n = b_{n-1} < a_{n-1} + b_{n-1}.$$

As a result, if we set $c_n = a_n + b_n$, the sequence $(c_n)_{n \geq 0}$ would be a strictly decreasing sequence of positive integers, which is a contradiction. Hence $\{S_n \mid n \in \mathbb{Z}_{\geq 0}\}$ must be finite, and so there exists $l \in \mathbb{Z}_{>0}$ such that $a_l = b_l$. Then

$$(-1)^{l+1} F_{l+1} m + (-1)^l F_l n = (-1)^{l+1} F_{l+1} + (-1)^{l+2} F_{l+2} n$$

or, equivalently,

$$(F_l + F_{l+1}) n = (F_{l+1} + F_{l+2}) m.$$

Since $(F_n)_{n \in \mathbb{Z}_{\geq 0}}$ is the Fibonacci sequence, the previous equality is the same as

$$F_{l+2} n = F_{l+3} m.$$

From the initial equation, it is not hard to see that $\gcd(m, n) = 1$. Also, $\gcd(F_{l+2}, F_{l+3}) = 1$. Then $(n, m) = (F_{l+3}, F_{l+2})$ and, therefore, $(a_l, b_l) = (1, 1)$.

Conversely, since $(1, 1)$ is a trivial solution for the given equation, (F_{n+1}, F_n) is also a solution of the same equation for each $n \in \mathbb{Z}, n \geq 0$. Also, $(1, 1)$ is the only solution of the given equation of the form (a, a). Thus, $\{(F_{n+1}, F_n), n \in \mathbb{Z}_{\geq 0}\}$ is the set of solutions of the given equation.

Finally, the answer is $F_{m+1}^2 + F_m^2$, where m is the greatest positive integer such that $F_{m+1} < 2022$. $\square$

Problem G6. Let T be a tree on the set of vertices $\{1, 2, \ldots, m\}$. For a positive integer n with $n > m$, in how many ways can we extend T to a tree on $[n]$?
Hint: Read about Prüfer codes.

Solution. Let S be a tree on $[n]$ that extends T. We will construct a list of $n-m$ numbers repeating the following process $n-m$ times: take the leaf with the largest value, delete it, add the value of its parent to the list. Clearly, at each step, we are left with a tree. We claim that at each step there is a leaf with value greater than m. Indeed, if there are $m+k$ vertices and no leaves outside T then there are $m-1$ edge inside T and at least $2k$ edges outside T, giving at least $m+2k-1 > m+k-1$ edges, a contradiction. Hence this process deletes vertices from $[n] \setminus [m]$ in some order and gives back the initial tree T. Thus, at each step, we are left with a tree that is obtained as an extension of T. We also note that at the last step the parent of the corresponding leaf belongs to $[m]$, whence the last number in the obtained list belongs to $[m]$. Then we have constructed a map φ from the set $\mathscr{T}$ of trees on $[n]$ extending T to the set $\mathscr{L}$ of lists $d_1, \ldots, d_{n-m}$ with $d_1, \ldots, d_{n-m+1} \in [n] \setminus [m]$ and $d_{n-m} \in [m]$.

Now we show that $\varphi\colon \mathscr{T} \to \mathscr{L}$ is injective. Let D be a list in $\mathscr{L}$; that is $D = d_1, \ldots, d_{n-m}$ with $d_1, \ldots, d_{n-m+1} \in [n] \setminus [m]$ and $d_{n-m} \in [m]$. By construction, the degree of vertex $i > m$ is the number of times it appears in D plus one. Hence we know what are the leaves: vertices that do not appear in D. So we know what was the first deleted edge: this was an edge between d_1 and the maximum number $l_1 \in [n] \setminus [m]$ that is not in D. Repeating this gives the second deleted edge: this is an edge between d_2 and maximal number not in $\{l_1, d_2, \ldots, d_{n-m}\}$. Repeating this process we get back the sequence of deleted edges. Hence there is a unique tree S extending T such that $\varphi(S) = D$.

Let us proceed to argue that φ is surjective. To do so, fix $D \in \mathscr{L}$. We define the list of leaves $l_1, \ldots, l_{n-m}$ as before: l_1 is the maximum element of $[n] \setminus [m]$ that is not in D, then l_2 is the maximum element of $[n] \setminus [m]$ that is not in $\{l_1, d_2, \ldots, d_{n-m}\}$, and so on until we get the last leaf l_{n-m} of our list, which is the unique element of $[n] \setminus [m]$ that is not in $\{l_1, \ldots, l_{n-m-1}, d_{n-m}\}$. We will add edges to T one by one starting with $d_{n-m}l_{n-m}$. By construction $\{l_1, \ldots, l_k\}$ does not intersect with $\{d_k, \ldots, d_{n-m}\}$ for any $1 \le k \le n-m$. Hence when we add a new edge $l_k d_k$ to $T \cup \{l_{n-m}d_{n-m}\} \cup \cdots \cup \{l_{k+1}d_{k+1}\}$ we use a new vertex l_k and do not create a cycle. It follows that $S = T \cup \{l_{n-m}d_{n-m}\} \cup \cdots \cup \{l_1 d_1\}$ is a tree.

Hence we conclude that φ is a bijection and, therefore, we can extend T to a tree on $[n]$ in $|\mathscr{T}| = |\mathscr{L}| = mn^{n-m-1}$ ways. $\square$

Advanced Math Problems

Problem M1. For a prime p and an integer $m \in [2, p+1]$, consider the set $G(p,m)$ of polynomials

$$f(x) = x + c_2 x^2 + \cdots + c_m x^m,$$

where $c_2, \ldots, c_m \in \mathbb{Z}/p\mathbb{Z}$. One can check (see part (1) below) that $G(p,m)$ is a group under the following operation of substitution: $(f * g)(x) = f(g(x)) \pmod{x^{m+1}}$.

(1) Check that $G(p,m)$ is a group.
(2) Find a representative of each conjugacy class of $G(p,m)$.
(3) Find the number of conjugacy classes of $G(p,m)$.
(4) Find the size of each conjugacy class of $G(p,m)$.

Solution. Conjugacy classes are represented by polynomials of the form $x + cx^r + dx^{2r-1}$ with $c \neq 0$. If $r \geq 2$ and $2r - 1 \leq m$, then there are $(p-1)p$ conjugacy classes. When $2r - 1 > m$ but $r \leq m$, we see that the coefficient d disappears, and so in this case there are $p - 1$ conjugacy classes. Also we have the identity. Adding this up gives a result. $\square$

Problem M2. Let $f \colon \mathbb{R} \to \mathbb{R}$ be a monotonic function. Suppose that we can pick $a, b, c, d \in \mathbb{R}$ with $ac \neq 0$ such that for all $y \in \mathbb{R}$

$$\int_y^{y+1} f(x)\,dx = ay + b \quad \text{and} \quad \int_y^{y+\sqrt{2}} f(x)\,dx = cy + d.$$

Prove that f is linear.

Solution. Define the function $g \colon \mathbb{R} \to \mathbb{R}$ as $g(y) = \int_0^y f(x)\,dx$. Since f is monotonic, it is integrable and, therefore, g is a continuous function. In addition, g satisfying the equalities

$$g(y+1) - g(y) = ay + b \quad \text{and} \quad g(y+\sqrt{2}) - g(y) = cy + d$$

for all $y \in \mathbb{R}$. Fix a polynomial p_1 (of degree two) such that $p_1(y+1) - p_1(y) = ay + b$ for all $y \in \mathbb{R}$. Then after setting $g_1 := g - p_1$, we see that $g_1(y+1) - g_1(y) = 0$, and so g_1 has period 1. Similarly, we can fix a polynomial p_2 (of degree two) such that $p_2(y+\sqrt{2}) - p_2(y) = cy + d$, and we can see that the function $g_2 := g - p_2$ has period $\sqrt{2}$. Since $p_1(y) - p_2(y) = g_2(y) - g_1(y)$ for all $y \in \mathbb{R}$, the fact that g_1 and g_2 are continuous and periodic functions guarantees that $p_1 - p_2$ is a bounded polynomial, and so a constant function. Thus, there exists $C \in \mathbb{R}$ such that $g_1(y) = g_2(y) + C$ for all $y \in \mathbb{R}$. Now the fact that g_2 has period $\sqrt{2}$ implies that g_1 also has $\sqrt{2}$ as a period. Hence g_1 have both periods 1 and $\sqrt{2}$, which implies that g_1 has period $m + n\sqrt{2}$ for all $m, n \in \mathbb{Z}$. Since the set $\{m + n\sqrt{2} : m, n \in \mathbb{Z}\}$ is dense in $\mathbb{R}$, the continuity of g_1 guarantees that g_1 is a constant function. As a result, $g = p_1 + g_1$ is a polynomial of degree 2. Write $g(y) = my^2 + py + q$ for some $m, p, q \in \mathbb{R}$. Then it follows from the

Fundamental Theorem of Calculus that $f(y) = g'(y) = 2my + p$. As a result, f is a linear function, as desired. $\square$

Problem M3. There is an $m \times n$ table. In the first column there are m real numbers $a_1, a_2, \dots, a_m$. You want to fill in the rest of the columns with real numbers such that the value of each entry that is not in the first column equals the average of the values of its neighbors.

(1) Show that you can always do this in a unique way.

(2) You can assume that part 1 is true. Show that the summations of each column are the same.

(3) You can assume that part 1 and 2 are true. Now fix m and $a_1, a_2, \dots, a_m$. Show that there is a constant c such that each entry in the last column will approach c as $n \to \infty$.

Solution. (1) For the first part, we use the following lemma.

Lemma 0.1. *If $a_i = 0$ for all $i \in [\![1, m]\!]$, then there is a unique solution (namely, everything 0).*

Proof. Everything 0 is obviously a solution. Now suppose it has another solution, then we can assume there's an element > 0 (otherwise we swap the sign of each element).

We pick the largest number, call it $a > 0$, then because $a =$ average, all neighbors of a is still a, following this process we see all of the entries are a, but the entries in first column are 0, a contradiction. $\square$

Basically, we want to solve a linear system. There are $m(n-1)$ unknown variables, and $m(n-1)$ equations ($a_{ij} =$ average for $1 \le i \le m, 1 < j \le n$). Hence we can write this linear system as $AX = Y$, and we want to show that there's an unique solution X. However, $AX = Y$ has an unique solution $\Leftrightarrow \det A \neq 0 \Leftrightarrow AX = \mathbf{0}$ has an unique solution.

Now the system $AX = \mathbf{0}$ correspond to the case that all entries in the first column are 0, and we already proved that there's a unique solution in this case.

(2) Note that if we only pick the last few columns, then it is also satisfactory (ignore the average problem of the first column). We call an entry good if it is equal to the average of its neighbors. We introduce two models.

Model 1: (red entries are good)

x_1	y_1
x_2	y_2
$\vdots$	$\vdots$
x_m	y_m

Model 2: (red entries are good)

x_1	y_1	z_1
x_2	y_2	z_2
$\vdots$	$\vdots$	$\vdots$
x_m	y_m	z_m

We make 2 claims.

Claim 1: In model 1, we have $\sum x_i = \sum y_i$.

Proof. We have the following

$$\begin{aligned} 2y_1 &= x_1 + y_2. \\ 3y_2 &= x_2 + y_1 + y_3. \\ &\vdots \\ 2y_m &= x_m + y_{m-1}. \end{aligned}$$

Adding them up we get our conclusion. □

Claim 2: In model 2, we have $2\sum y_i = \sum x_i + \sum z_i$.

Proof. We have the following

$$\begin{aligned} 3y_1 &= x_1 + z_1 + y_2. \\ 4y_2 &= x_2 + z_2 + y_1 + y_3. \\ &\vdots \\ 3y_m &= x_m + z_m + y_{m-1}. \end{aligned}$$

Adding them up we get our conclusion. □

Now we can finish our proof by easy induction.

(3) First we need a result about positive definite matrix.

Lemma 0.2. *$\{A \in \mathscr{M} : A$ is positive definite$\}$ is open (when view as a sub-topological space of $\mathscr{M} \subset \mathbb{R}^{t^2}$), where $\mathscr{M} = \{A \in Mat_{t\times t}(\mathbb{R}) : A^T = A\}$.*

Proof. Use Sylvester's criterion and note that det is continuous. □

Here's another proof without using Sylvester's criterion.

Proof. Instead we will prove that its complement is closed. Suppose $A_k \to A$ where A_k is not positive definite for all k, we want to show that nor is A. There exists $v_k \in \mathbb{R}^t$ such that $v_k^T A_k v_k \leq 0$. Furthermore, after scaling, we can select $v_k \in \{x \in \mathbb{R}^t : \|x\| = 1\}$. Now because v_k are in a compact set, we can suppose $v_k \to v \neq 0$ (otherwise just pick a sub-sequence). We claim that $v^T Av \leq 0$. This is because as $k \to \infty$, we have

$$|v^T Av - v_k^T A_k v_k| \leq |v^T Av - v^T A_k v| + |v^T A_k v - v_k^T A_k v_k|$$

$$\leq |v^T(A-A_k)v| + 2|v^T A_k(v-v_k)| + |(v-v_k)^T A_k(v-v_k)| \to 0.$$

And this shows that A is also not positive definite. ☐

Note that we can we can assume $\sum a_i = 0$ because we can add the same amount to every element.

Claim 1: In model 2, we have $\sum y_i^2 \leq 1/2\,(\sum x_i^2 + \sum z_i^2)$.

Proof. We have (by Cauchy-Schwarz)

$$3y_1^2 = \frac{1}{3}(x_1+z_1+y_2)^2 \leq x_1^2 + z_1^2 + y_2^2.$$

$$4y_2^2 = \frac{1}{4}(x_2+z_2+y_1+y_3)^2 \leq x_2^2 + z_2^2 + y_1^2 + y_3^2.$$

$$\vdots$$

$$3y_m^2 = \frac{1}{3}(x_m+z_m+y_{m-1})^2 \leq x_m^2 + z_m^2 + y_{m-1}^2.$$

Adding them up we get our conclusion. ☐

Claim 2: In model 1, we have $\sum y_i^2 \leq \sum x_i^2$, and the equality can only be achieved when $x_i = x_j (= y_i)$ for all $i, j \in [\![1, m]\!]$.

Proof. We have (by Cauchy-Schwarz)

$$2y_1^2 = \frac{1}{2}(x_1+y_2)^2 \leq x_1^2 + y_2^2.$$

$$3y_2^2 = \frac{1}{3}(x_2+y_1+y_3)^2 \leq x_2^2 + y_1^2 + y_3^2.$$

$$\vdots$$

$$2y_m^2 = \frac{1}{2}(x_m+y_{m-1})^2 \leq x_m^2 + y_{m-1}^2.$$

Adding them up we get our conclusion. Then by carefully examine the inequality (with the knowledge of Cauchy-Schwarz inequality) we get the equality condition. ☐

For the k^{th} column $(c_1\ c_2\ \dots\ c_m)^T$, we let $s_k = \sum c_i^2$ $(1 \leq k \leq n)$. Then by easy induction, we can show that $s_1 \geq s_2 \geq \cdots \geq s_n$, which tells us that the entries of the last column "most likely" converges to 0 (remember we always assume $\sum a_i = 0$). But this is not enough, so we are seeking to improve our last claim, we want to show that $\sum x_i^2 \geq \alpha \sum y_i^2$ where α is a constant > 1 (of course, we assume $\sum y_i = 0$). Note that once we know y_i's, we can uniquely figure x_i's out, by a linear transformation: $AY = X$ where $Y = (y_1\ y_2\ \cdots\ y_m)^T, X = (x_1\ x_2\ \cdots\ x_m)^T$. All we need to do is prove $Y^T A^T A Y - \alpha Y^T Y \geq 0$ if $Y \in L$. Where L is the kernel of the function $\mathbb{R}^m \to \mathbb{R} : Y \mapsto (1\ 1\ \cdots\ 1)Y$. Hence it suffices to show that $A^T A - \alpha I$ is positive definite in L. When we select a basis for L and restrict $A^T A - \alpha I$ to L, we form a new matrix, call it A_α. It's easy to see that $\alpha \mapsto A_\alpha$ is continuous, and we already know that A_1 is positive definite. Thus by Lemma 2 there exists $\alpha > 1$ such that $A^T A - \alpha I$ is positive definite in L. What's good is that the α doesn't depend on n. Note that we can assume $\alpha < 2$. Thus with these facts, we can give a list of inequalities. WLOG $s_1 = 1$ (because we can multiply a constant to every element). We have

$$s_2 \leq \frac{1+s_3}{2}.$$
$$s_3 \leq \frac{s_2+s_4}{2}.$$
$$\vdots$$
$$s_{n-1} \leq \frac{s_{n-2}+s_n}{2}.$$
$$\alpha s_n \leq s_{n-1}.$$

(And clearly we have $s_k \geq 0$). We want to maximize s_n. To do this, let E be the collection of all $(s_2, s_3, \dots, s_n) \in \mathbb{R}^{n-1}$ that satisfy such conditions. E is compact because E is the intersection of closed sets and $E \in [0,1]^{n-1}$, and E is non-empty because $\mathbf{0} \in E$. Consider the projection $p : \mathbb{R}^{n-1} \to \mathbb{R}, (c_1, c_2, \dots, c_{n-1}) \mapsto c_{n-1}$, it is continuous. Thus pE is also compact, so it can attain its maximum. Suppose at the point $(s_2, s_3, \dots, s_n)$, it attains its maximum, that is, s_n. Now we claim that all of the above inequalities are actually equality. This is because if one is strictly less, say $s_k < \frac{s_{k-1}+s_{k+1}}{2}$, then we can slightly increase s_k, then slightly increase s_{k+1}, $\cdots$, finally we can slightly increase s_n, a contradiction. Thus if we set $s_2 = 1 - \delta$, then $s_k = 1 - (k-1)\delta$. We have $\alpha(1-(n-1)\delta) = 1-(n-2)\delta$, hence $\delta = 1/\left(n + \frac{2-\alpha}{\alpha-1}\right)$, and $s_n = 1-(n-1)\delta = 1/\left(\frac{2-\alpha}{n} + \alpha - 1\right) \cdot 1/n \leq 1/(\alpha-1) \cdot 1/n = O(n^{-1})$.

Therefore, each entry of the last column is $O(n^{-1/2})$. □

Problem M4. For a positive integer k, let A be an alphabet of k letters, and let $s = s_1 \dots s_m$ be a string of length m over A. A string a of length n is called a *freak subchain* of s if the following two properties hold:

- a is a subsequence of s (i.e., $a_j = s_{i_j}$ for every integer $j \in [1,n]$ and some indices $i_1, i_2, \dots, i_n$ with $i_1 < i_2 < \cdots < i_n$), and
- there exists an integer $\ell \in [1, n-1]$ such that $i_{\ell+1} - i_\ell > 1$.

For instance, in our alphabet, "tics" is a substring of the string "mathema**TICS**" that is also a freak subchain, as emphasized in "ma**T**hemat**ICS**". Given a positive integer N, create an efficient algorithm to find the number of strings over the alphabet A such that the length of its largest-length substring that is also its freak subchain is N.

Solution. Let s be a string over A. A string a is called a *freak substring* of s provided that a is both a substring and a freak subsequence of s. For a length-n freak substring a of s, the set $\{i_1, \dots, i_n\}$, where $i_1 < \cdots < i_n$, is a *good set of indices* of a in s if $a = s_{i_1} s_{i_2} \dots s_{i_n}$ and $i_{\ell+1} - i_\ell > 1$ for some $\ell \in [1, n-1]$.

Claim 1: If a freak substring a of s has the largest length possible, then a must be a prefix or a suffix of s.

Proof of Claim 1: Let a be a freak substring of s having the largest length possible n, and assume by contradiction that a is neither a prefix nor a suffix of s. Assume that $a = s_{b+1}s_{b+2}\dots s_{b+n}$ for some integer $b \in [1, m-n)$. Let $\{i_1, i_2, \dots, i_n\}$ with $i_1 < \cdots < i_n$ be a good set of indexes of a in s, and take an integer $k \in [1, n-1]$ such that $i_{k+1} - i_k > 1$. Observe that $i_n - i_1 > n-1$ and, therefore, either $i_1 < b+1$ or $b+n < i_n$. If $i_1 < b+1$, then $(b+2) - i_1 > 1$ and so the fact that $b+n < m$ ensures that $\{i_1, b+2, \dots, b+n, b+n+1\}$ is a good set of indices of the length-$(n+1)$ substring $s_{i_1}s_{b+2}\dots s_{b+n}s_{b+n+1}$ in s. However, this contradicts the maximality of n. We can proceed similarly to obtain a contradiction when $b+n < i_n$. Hence the claim follows.

We let $s[: k]$ denote the prefix of s that ends at position k. Similarly, we let $s[k :]$ denote the suffix of s that starts at position k.

Claim 2: Let k the the largest subindex in $[1, m-1]$ such that there exists a subindex $\ell \in [k+1, m]$ such that $s_\ell = s_k$. Then $s[: k]$ is the largest prefix of s that is a freak substring.

Proof of Claim 2: Consider the set $\{i_1, \dots, i_k\}$, where $i_j = j$ for every $j \in [1, k-1]$ and $i_k = \ell$. It is clear that $\{i_1, \dots, i_k\}$ is a good set of indices of $s[: k]$, which implies that $s[: k]$ is a freak substring of s. On the other hand, the maximality of k guarantees that there are no k' and ℓ' such that $k < k' < \ell'$ and $s_{k'} = s_{\ell'}$, and this implies that among all prefixes of s that are also freak substrings, $s[: r]$ is the largest one. Hence the claim follows.

Claim 3: Let k the the smallest subindex in $[2, m]$ such that there exists a subindex $\ell \in [1, k-1]$ such that $s_\ell = s_k$. Then $s[k :]$ is the largest suffix of s that is a freak substring.

Proof of Claim 3: The proof is similar to that of Claim 2.

Now suppose that s is a string over A having a largest freak substring of length N. Then it follows from Claim 1 that s has either a largest prefix or a largest suffix that is also a freak substring and that has length N. In the first case, it follows from Claim 2 that the string $s[N+1 :]$ cannot repeat any letter and, therefore, the length of s is at most $N+k$. In the second case, we can similarly arrive to the conclusion that the length of s is at most $N+k$.

Let us further use the characterizations in Claims 2 and 3 of the largest prefix and suffix of s that are freak substrings to count in how many strings s the largest between those prefixes and suffixes has length N. We separate our work into the following two cases.

Case 1: The largest freak substring is achieved as a suffix of s. Fix an integer i such that $s[i+1 :]$ is a largest freak substring of s. Then the following three conditions hold:

- $s[: i]$ is conformed by different characters only,

- there exists an index $j \in [1, i]$ such that $s_j = s_{i+1}$,
- all the letters of $s[N+1:]$ must be different.

The first two conditions follow directly from Claim 3, while the third one follows from the fact that there is no prefix of s with length strictly greater than N that is a freak substring. Also, notice that these three conditions are sufficient to ensure that $s[i+1:]$ is the largest freak substring of s. Our task is to count all the words s, of length $N+i$ that fulfill the previous three conditions for a fix positive integer i. We can distinguish the following two subcases:

Case 1.1: $i \geq N$. Since the the last i characters of s must be different the number of way of choose them and its positions is $k!/(k-i)!$. It only remains to pick the first w characters of s. One of this characters must match to s_{i+1}, and we can choose which one is in w ways. The others must be all different and different from the $i-w+1$ that are already fixed in the prefix of length i. This give us $(k-i+w-1)!/(k-i)!$ ways of choosing them. Hence the total number of words in this case is

$$\frac{k!(k-i+w-1)!w}{(k-i)!(k-i)!}.$$

Case 1.2: $i < N$. In the second case we can fix as well the last i characters of the word in $k!/(k-i)!$ ways. As $i < w$ the rest of characters of the suffix of length w can be chosen without restrictions. This give us k^{w-i} ways of doing it. Now between the first i characters we must choose the one that would match s_{i+1} in i different ways. Finally the rest of characters of the prefix of length i must be all different and different from the previously fixed one, which give us $(k-1)!/(k-i)!$ ways of picking them. Hence the total number of words in this case is

$$\frac{ik!(k-1)!k^{w-i}}{(k-i)!(k-i)!}.$$

Case 2: The largest freak substring is a prefix. This case is almost similar to the previous one, with the only difference being that all the characters in the prefix of length $i+1$ must be different. This guarantees that the suffix will not be also a largest freak substring of length N, which is necessary since the words with both prefix and suffix been maximal freak substring of length N were counted in the previous case. This time we split into the following two subcases.

Case 2.1: $i + 1 \geq N$.

In the first case we get that the amount of words is

$$\frac{k!(k-i+N-2)!(N-1)}{(k-i-1)!(k-i)!}.$$

Case 2.2: $i + 1 < N$. In this case, we can see that

$$\frac{ik!(k-1)!k^{N-i-1}}{(k-i)!(k-i-1)!}.$$

Finally note that the fixed i must be smaller than $k + 1$ as otherwise we could not find i different letters to fulfill the first condition. Below we show the desired algorithm.

Input: $k =$ number of letters of the alphabet A; $N =$ length of the largest freak substring
Output: Number of strings with a largest freak substring of length N

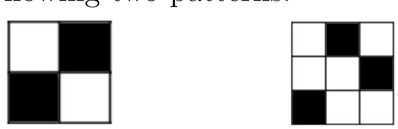
$answer = 0$
for $i \in [1, k]$ **do**
    if $i \geq N$ **then**
        $answer+ = k!(k-i+N-1)! * N/(k-i)!/(k-i)!$
    else
        $answer+ = i * k!(k-1)! * pow(k, N-i)/(k-i)!/(k-i)!$
    if $i < k$ **then**
        if $i + 1 \geq N$ **then**
            $answer+ = k!(k-i+N-2)! * (N-1)/(k-i)!/(k-i-1)!$
        else
            $answer+ = i * k!(k-1)! * pow(k, N-i-1)/(k-i)!/(k-i-1)!$
return $answer$

Final Remark: We have assumed that the computation of factorials and powers of k is constant since we can precalculate them in a linear array in a total cost of $O(k)$. $\square$

Problem M5. For $n > 2$, each cell in an $n \times n$ grid is either colored black or white. A flip operation consists of choosing a 2×2 square that has at least two black cells and swapping the colors of each of the four cells in the chosen square. Call a given coloring *irreducible* if there exists no sequence of flips that will reduce the number of black cells in it.

(1) Prove that an irreducible coloring with maximum number of black squares cannot have two adjacent black squares.

(2) Prove that there exists an irreducible coloring with maximum number of black squares that avoids the following two patterns.



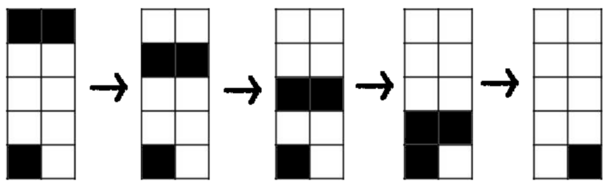
(3) Let B be the number of black squares in an irreducible coloring with maximum number of black squares. Prove that $B \leq (n+1)^2/4$.

(4) Prove that each irreducible coloring with maximum number of black squares has $\lfloor \frac{(n+1)^2}{4} \rfloor$ black squares.

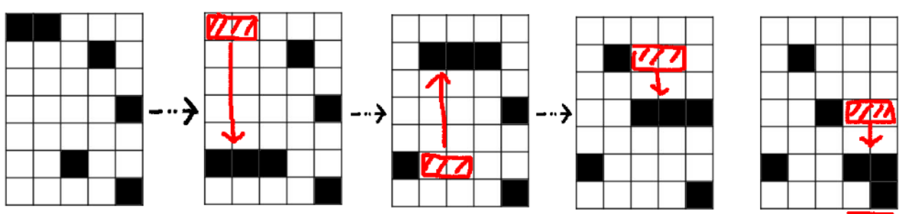
NOTE: This problem was kindly proposed by the MIT student Khalid Ajran.

Solution. (1) Assume, towards a contradiction, that there are two adjacent black squares, say in positions (x, y) and $(x+1, y)$.

Suppose first that there is another black square in one of the columns x and $x+1$, and suppose that the nearest one to the two given adjacent squares is in position (x, y'). Then, we can perform the sequence of moves as shown in the picture below to reduce the number of black cells, contradicting that the coloring is irreducible. Thus, there can be no other black cells in neither column x nor $x+1$.



Now let $[a, b]$ be the largest range such that $x, x+1 \in [a, b]$ and each column $c \in [a, b]$ has at least one black cell in it (so columns $a-1$ and $b+1$ have no black cells, or do not exist). We claim that all columns in this range must have exactly one black cell. Suppose that one of them has at least two black cells. Then, we can perform the following sequence of moves to reduce the number of black cells. To shorten the sequence, simply note that whenever we have two adjacent black cells, we can shift the two of them up or down by any amount.



In the last step of the sequence, we have three cells in the same 2×2, which can be swapped to contradict the irreducibility of the coloring. Hence each column $c \in [a, b]$ has exactly one black cell. So, we now have a $(b-a+1) \times n$ region of the grid that is only occupied by $b-a+1$ black cells. Notably, this region is surrounded on both side either by edges of the grid, or by columns that have no black cells in them. If $n > 4$, then we can replace the $b-a+1$ cells in this region by black cells in a pattern of 1 black cell per 2×2 square. This increases the number of black cells in the construction but does not affect that it is irreducible. We can check by that when $n \leq 4$, we can also find more optimal ways to fill the region, contradicting that maximality of the coloring.

(2) Now assign values to each cell in the grid as follows two different ways.

2^{2n-2}	2^{2n-3}	$\cdots$	$\cdots$	$\cdots$
2^{2n-3}	$\cdots$	$\cdots$	$\cdots$	8
$\cdots$	$\cdots$	$\cdots$	8	4
$\cdots$	$\cdots$	8	4	2
$\cdots$	8	4	2	1

Labelling 1

$\cdots$	$\cdots$	$\cdots$	2^{2n-3}	2^{2n-2}
8	$\cdots$	$\cdots$	$\cdots$	2^{2n-3}
4	8	$\cdots$	$\cdots$	$\cdots$
2	4	8	$\cdots$	$\cdots$
1	2	4	8	$\cdots$

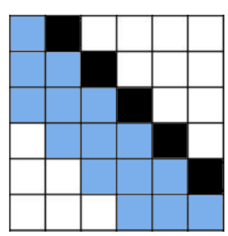
Labelling 2

For a coloring C of the grid, define $f_1(C)$ to be the sum of the values assigned to the black cells of the coloring by labelling 1, and define $f_2(C)$ similarly. Say that an irreducible coloring with a maximum number of black cells has exactly B black cells. Now, take all irreducible colorings with B black cells, and sort them lexicographically by the pair $(f_1(C), f_2(C))$ (i.e., sort them by the $f_1(C)$ values, and then sort ties by the $f_2(C)$ values). Let C^* be the first coloring with respect to this order.

We claim that none of the given two patterns in part (2) can occur in C^*. This is indeed the case as flipping the first pattern results in increasing the f_1 value of C^*, while performing two flips on the second pattern results in the f_1 value staying the same and the f_2 value increasing.

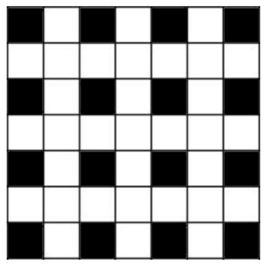
(3) Let C^* as in the previous part, and look at all connected blocks of black cells in C^* (we say that two cells are connected if they share a vertex). It follows from parts (1) and (2) that all connected shapes in C^* are diagonal strings covering the positions $(x, y), (x+1, y-1), (x+2, y-2), \ldots, (x+k, y-k)$.

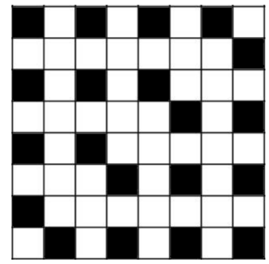
Now we extend the given grid to an $(n+1)\times(n+1)$ grid by adding a phantom leftmost column and a phantom bottom-most row consisting both of white cells. We now turn blue three white cells in the new grid for each of the black cells by going through each of the diagonal strings of black cells as the following picture illustrates.



Using the previous parts, we can confirm that the blue colored cells can not be already black, and also that the corresponding 3-sets of blue cells of two distinct black cells are disjoint. Thus, $4B \leq (n+1)^2$, and so the desired inequality follows.

(4) First, here are constructions for even and odd n. The constructions below for $n = 7$ and $n = 8$ are easy to generalize.





Observe that no flips are possible in the odd construction, so clearly it is irreducible. Finally, it remains to show that the construction provided for even n is valid. Divide the grid into $(n^2/4)$ 2×2 square regions. We need the following claim.

Claim 3. The following properties of the construction remain invariant as we do flips.

(1) $n/2$ of the regions have two black cells, and the remaining regions have exactly one black cell.
(2) No two regions containing two black cells are in the same row or column.
(3) Any black cell that is the lone black cell in its region is not adjacent (even diagonally) to any other black cells.

Proof of Claim 3. Whenever these properties hold, we can check that the only possible times we may perform a flip are either within a region, or between diagonally adjacent regions that each contain two black cells. We can check that either move preserves all three properties. Then Claim 3 follows.

Since these properties hold for the provided construction, and we can show that they do not hold whenever there exist two orthogonally adjacent black cells, then it's not possible to reduce the number of black cells in this construction. □