PRIMES Math Problem Set

PRIMES 2023

Due November 30, 2022

Dear PRIMES applicant:

This is the PRIMES 2023 Math Problem Set. Please send us your solutions as part of your PRIMES application by **November 30, 2022**. For complete rules, see `http://math.mit.edu/research/highschool/primes/apply.php`

- Note that this set contains two parts: "General Math problems" and "Advanced Math." Please solve as many problems as you can in both parts.
- You can type the solutions or write them up by hand and then scan them. Please attach your solutions to the application as a PDF file. The name of the attached file must start with your last name, for example, "`etingof-solutions.pdf`" or similar. Include your full name in the heading of the file.
- Please write not only answers, but also proofs (and partial solutions/results/ideas if you cannot completely solve the problem). Besides the admission process, your solutions will be used to decide which projects would be most suitable for you if you are accepted to PRIMES.
- Submissions in LaTeX are preferred, but handwritten submissions are also accepted.
- You are allowed to use any resources to solve these problems, *except other people's help.* This means that you can use calculators, computers, books, and the Internet. However, if you consult books or Internet sites, please give us a reference.
- **Note that posting these problems on problem-solving websites before the application deadline is strictly forbidden!** Applicants who do so will be disqualified, and their parents and recommenders will be notified.

Note that some of these problems are tricky. We recommend that you do not leave them for the last day. Instead, think about them, on and off, over some time, perhaps several days. We encourage you to apply if you can solve at least 50% of the problems.

Enjoy!

Why it makes no sense to cheat

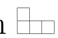
PRIMES expects its participants to adhere to MIT rules and standards for honesty and integrity in academic studies. As a result, **any cases of plagiarism, unauthorized collaboration, cheating, or facilitating academic dishonesty during the application process or during the work at PRIMES may result in immediate disqualification from the program, at the sole discretion of PRIMES.** In addition, PRIMES reserves the right to notify a participant's parents, schools, and/or recommenders in the event it determines that a participant did not adhere to these expectations. For explanation of these expectations, see *What is Academic Integrity?*, `integrity.mit.edu`.

Moreover, even if someone gets into PRIMES by cheating, it would immediately become apparent that their background is weaker than expected, and they are not ready for research. This would prompt an additional investigation with serious consequences. By trying to get into PRIMES by cheating, students run very serious risks of exposing their weak background and damaging their college admissions prospects.

PRIMES 2023: ENTRANCE PROBLEM SET

Notation. We let $\mathbb{Z}$ and $\mathbb{R}$ denote the set of integers and the set of real numbers, respectively.

General Math Problems

Problem G1. Can we tile a 4×2023 grid using pieces of the form  ?

Problem G2. Prove that there are infinitely many pairs of integers (a, b) with $\gcd(a, b) = 1$ such that $\sqrt{-4a^3 - 27b^2}$ is an integer.

Problem G3. Suppose that α and β are distinct solutions of $x^{2023} - 1 = 0$ in the complex plane, and that α and β have been randomly selected. What is the probability that the following inequality $|\alpha + \beta|^2 \geq 2 + \sqrt{3}$ holds?

Problem G4. Let $0 < p < 1$, and let $(a_n)_{n \geq 0}$ be a sequence of nonnegative numbers such that $a_{n+2} \leq (1-p)a_{n+1} + pa_n$. Prove that $(a_n)_{n \geq 0}$ has a limit.

Problem G5. Determine the maximum value of $m^2 + n^2$ if m and n are positive integers less than 2022 such that $(n^2 - mn - m^2)^2 = 1$.
Hint: The pair (F_{n+1}, F_n) satisfies the given equation, where F_n is the n-th Fibonacci number.

Problem G6. Let T be a tree on the set of vertices $\{1, 2, \ldots, m\}$. For a positive integer n with $n > m$, in how many ways can we extend T to a tree on $[n]$? (A tree T' on $\{1, 2, \ldots, n\}$ *extends* T if every edge of T is also an edge of T'.)
Hint: Read about Prüfer codes.

Problem M1. For a prime p and an integer $m \in [2, p+1]$, consider the set $G(p,m)$ of polynomials

$$f(x) = x + c_2x^2 + \cdots + c_mx^m,$$

where $c_2, \ldots, c_m \in \mathbb{Z}/p\mathbb{Z}$. One can check (see part (1) below) that $G(p,m)$ is a group under the following operation of substitution: $(f * g)(x) = f(g(x)) \pmod{x^{m+1}}$.

(1) Check that $G(p,m)$ is a group.
(2) Find a representative of each conjugacy class of $G(p,m)$.
(3) Find the number of conjugacy classes of $G(p,m)$.
(4) Find the size of each conjugacy class of $G(p,m)$.

Problem M2. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a monotonic function. Suppose that $k, \ell, m, n \in \mathbb{R}$ with $km \neq 0$ satisfy that for all $y \in \mathbb{R}$

$$\int_y^{y+1} f(x)\,dx = ky + \ell \quad \text{and} \quad \int_y^{y+\sqrt{2}} f(x)\,dx = my + n.$$

Prove that f is linear (that is, there exist $a, b \in \mathbb{R}$ such that $f(y) = ay + b$ for all $y \in \mathbb{R}$).

Problem M3. There is an $m \times n$ table with both m and n positive integers. In the first column there are m real numbers. We want to fill in the rest of the columns with real numbers such that the value of each entry that is not in the first column equals the average of the values of its neighbors (the neighbors of a given cell are the cells sharing an edge with the given cell).

(1) Prove that we can always do this in a unique way.
(2) Prove that the sums of the entries in each column are the same.
(3) Prove that there is a constant c such that each entry in the last column will approach c as $n \to \infty$.

Problem M4. For a positive integer k, let A be an alphabet of k letters, and let $s = s_1 \ldots s_m$ be a string of length m over A. A string a of length n is called a *freak subchain* of s if the following two properties hold:

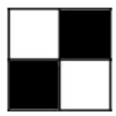
- a is a subsequence of s (i.e., $a_j = s_{i_j}$ for every integer $j \in [1, n]$ and some indices $i_1, i_2, \ldots, i_n$ with $i_1 < i_2 < \cdots < i_n$), and
- there exists an integer $\ell \in [1, n-1]$ such that $i_{\ell+1} - i_\ell > 1$.

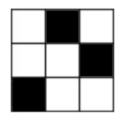
For instance, in our alphabet, "tics" is a substring of the string "mathema**TICS**" that is also a freak subchain, as emphasized in "ma**T**hemat**ICS**". Given a positive integer N, create an efficient algorithm to find the number of strings over the alphabet A such that the length of its largest-length substring that is also its freak subchain is N.

Problem M5. For $n > 2$, each cell in an $n \times n$ grid is either colored black or white. A flip operation consists of choosing a 2×2 square that has at least two black cells and swapping the colors of each of the four cells in the chosen square. Call a given coloring *irreducible* if there exists no sequence of flips that will reduce the number of black cells in it.

(1) Prove that an irreducible coloring with maximum number of black squares cannot have two adjacent black squares.

(2) Prove that there exists an irreducible coloring with maximum number of black squares that avoids the following two patterns.





(3) Let B be the number of black squares in an irreducible coloring with maximum number of black squares. Prove that $B \leq (n+1)^2/4$.

(4) Prove that each irreducible coloring with maximum number of black squares has $\lfloor \frac{(n+1)^2}{4} \rfloor$ black squares.