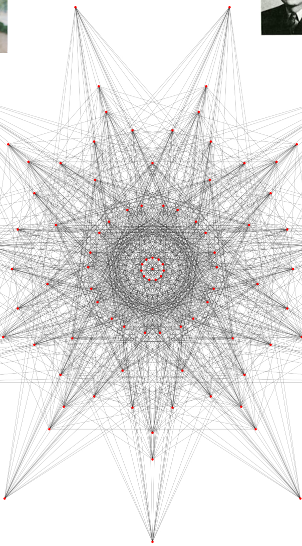
Nearly impossible

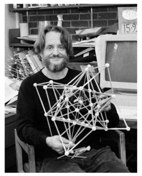
Leech lattice

Richard Borcherds
constructed
„monster Lie algebra"
in 1992

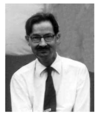
Marcel Golay
constructed
binary Golay code
in 1949



John Conway
disovered thee
sporadic groups
and made
„Monsterous moonshine"
conjecture



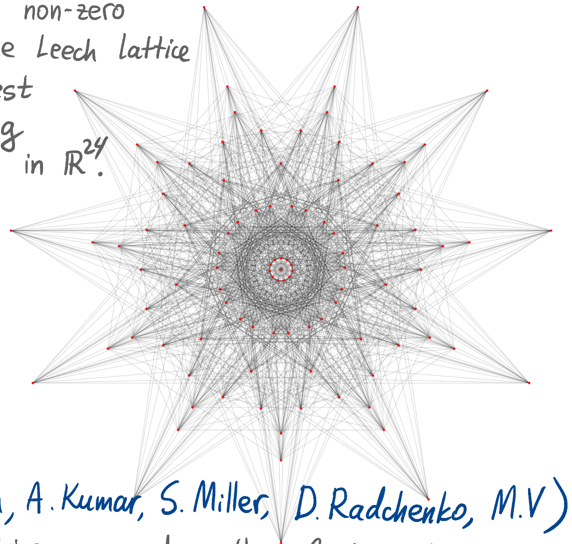
1968, 1969, 1979



John Leech
discovered
Leech lattice
in 1965

1979 (V. Levenstein / A. Odlyzko & N. Sloan)

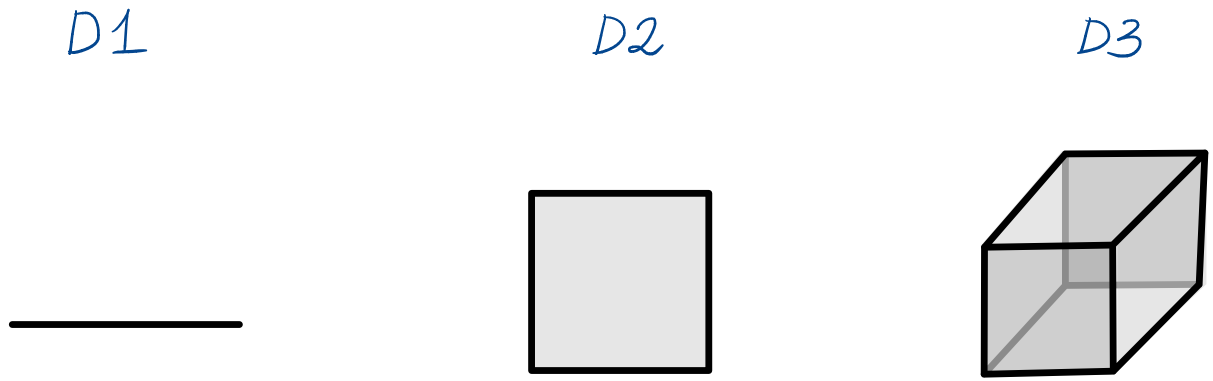
The shortest non-zero vectors of the Leech lattice form the best sphere kissing configuration in $\mathbb{R}^{24}$.



2016 (H. Cohn, A. Kumar, S. Miller, D. Radchenko, M.V)

Leech lattice provides the best sphere packing in dimension 24.

Dimension

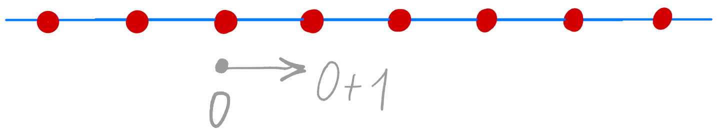


The d-dimensional Euclidean space $\mathbb{R}^d$ consists of the points $x = (x_1, x_2, \dots, x_d)$.

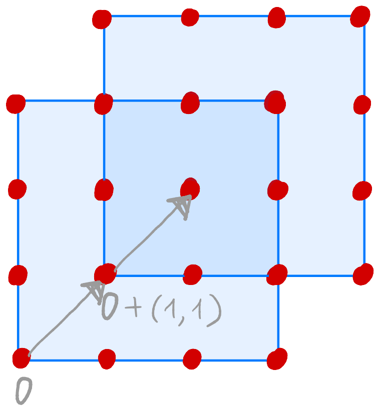
Each coordinate x_i is a real number.

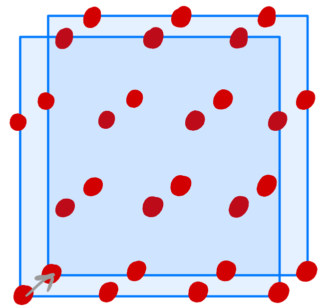
Lattices

D1



D2





periodic

not a lattice

Example: $\mathbb{Z}^2 = \{(m,n) \mid m,n \in \mathbb{Z}\}$

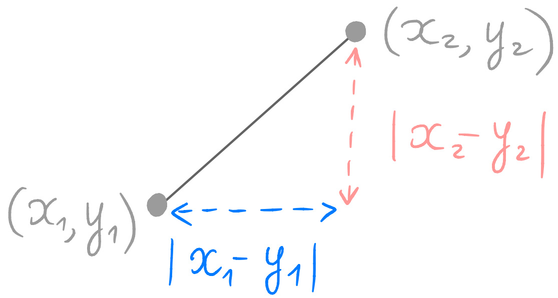
the set of points on the plane with integer coordinates.

Distance between 2 points in $\mathbb{R}^d$

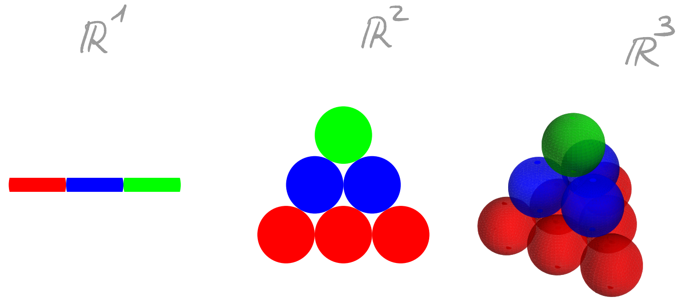
Let $x = (x_1, \dots, x_d)$ and $y = (y_1, \dots, y_d)$ be two points in $\mathbb{R}^d$.

Then

$$\mathrm{dist}(x, y) = \sqrt{(x_1 - y_1)^2 + \dots + (x_d - y_d)^2}.$$

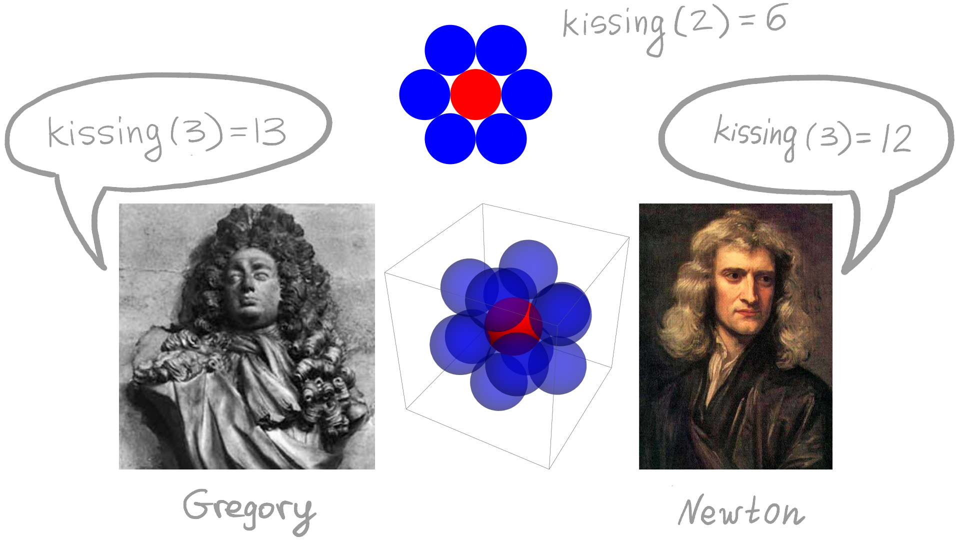


Balls and spheres in $\mathbb{R}^d$



The Ball with center x and radius r is the set of points y such that the distance between x and y is less then r.

Kissing number



What do we know about kissing numbers?

dimension	kissing number	
1	2	
2	6	
3	12	
4	24	2003 O. Musin
5	40 – 44	
6	72 – 78	
7	126 – 134	
8	240	The shortest vectors of E_8 lattice
⋮	⋮	
24	196560	The shortest vectors of Leech lattice

1979

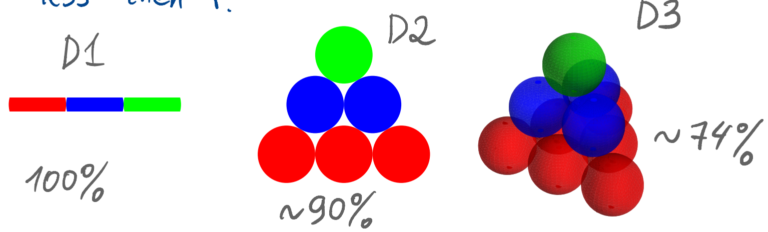
V. Levenstein

independently

A. Odlyzko & N. Sloane

Sphere packing in $\mathbb{R}^d$

The ball with center x and radius r is the set of points y such that the distance between x and y is less then r.

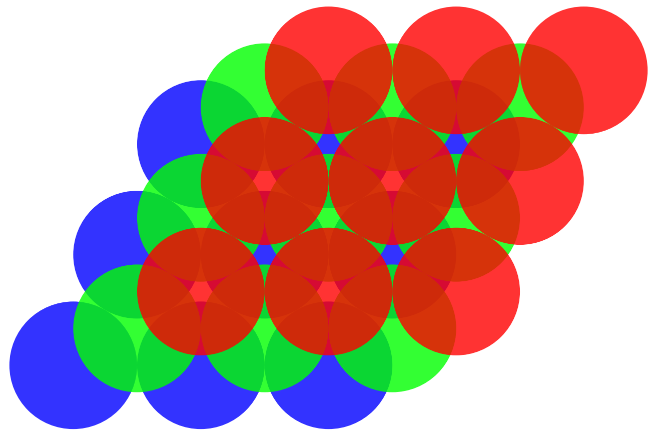


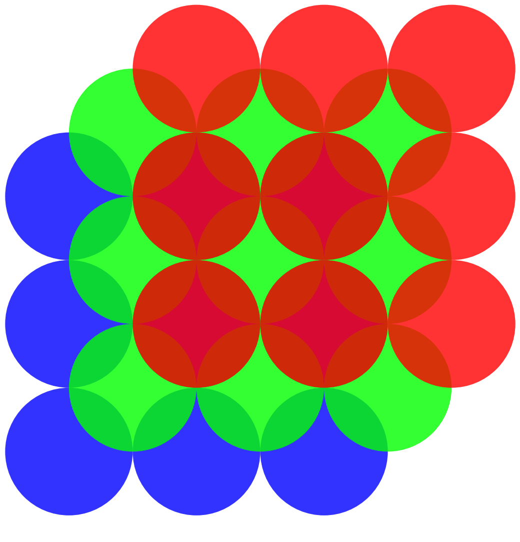
What is the densest possible configuration of balls in $\mathbb{R}^d$?

Keplers conjecture

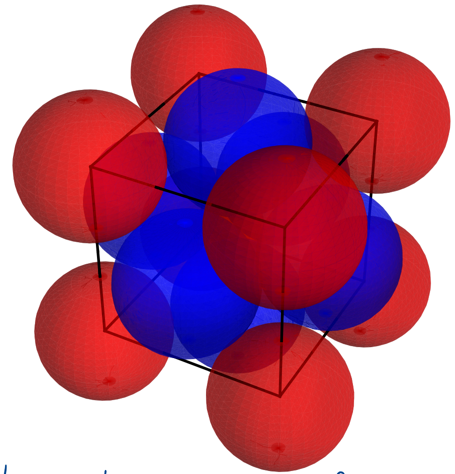
Johan Kepler

Thomas Harriot

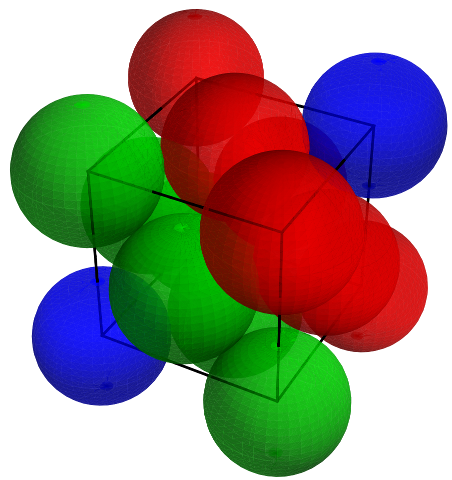


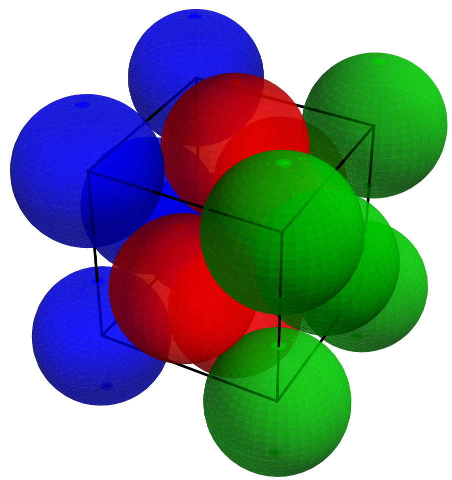


face centered cubic lattice

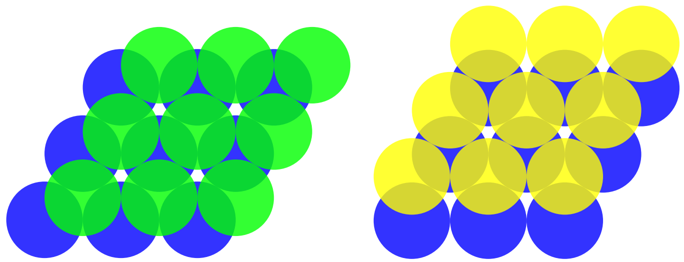


$\mathbb{Z}^3 \cup$ all centers of the faces of the cubes





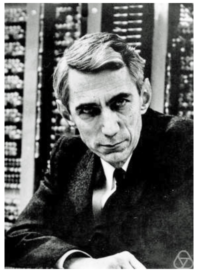
Why is Kepler conjecture so difficult?



There exist uncountably many sphere packings of maximum density.

Thomas Hales resolved Kepler's conjecture in 1998.

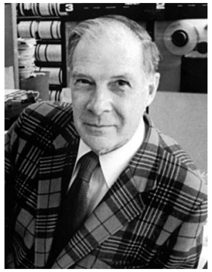
Packings and error correcting codes



Claud Shannon

„A mathematical theory of communication"

1948



Richard Hamming

Error correcting codes

1947

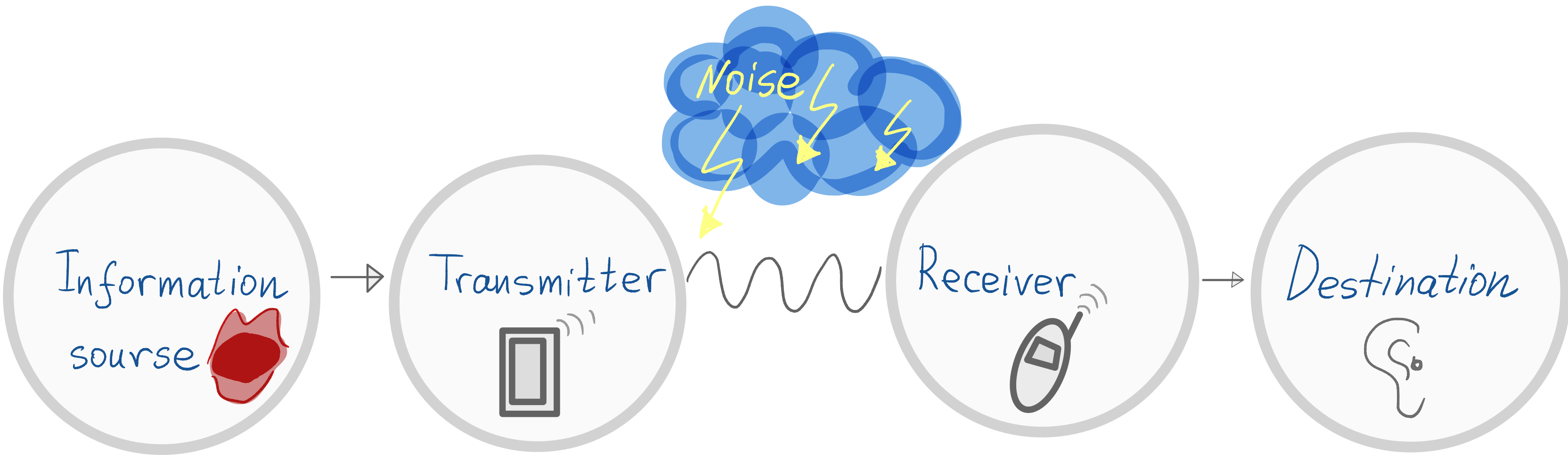
Marcel Golay

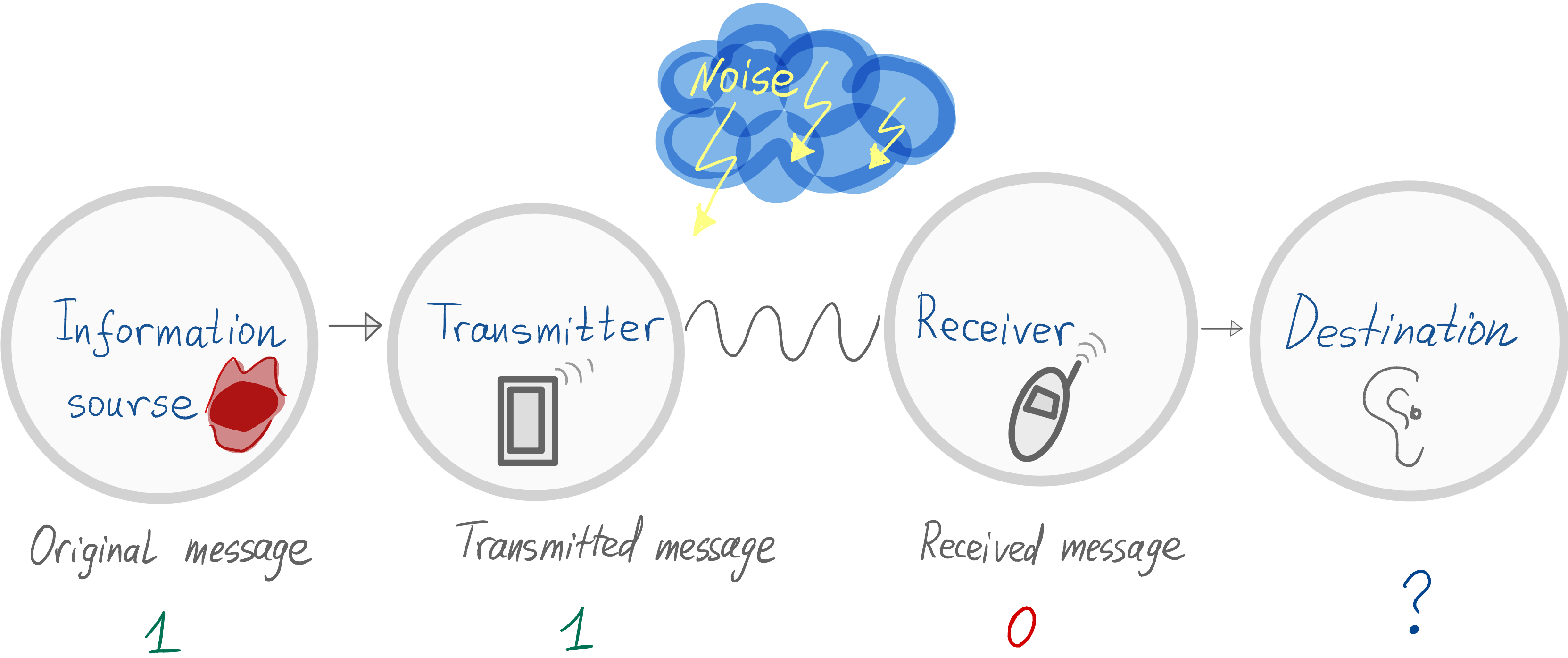
Binary Golay code

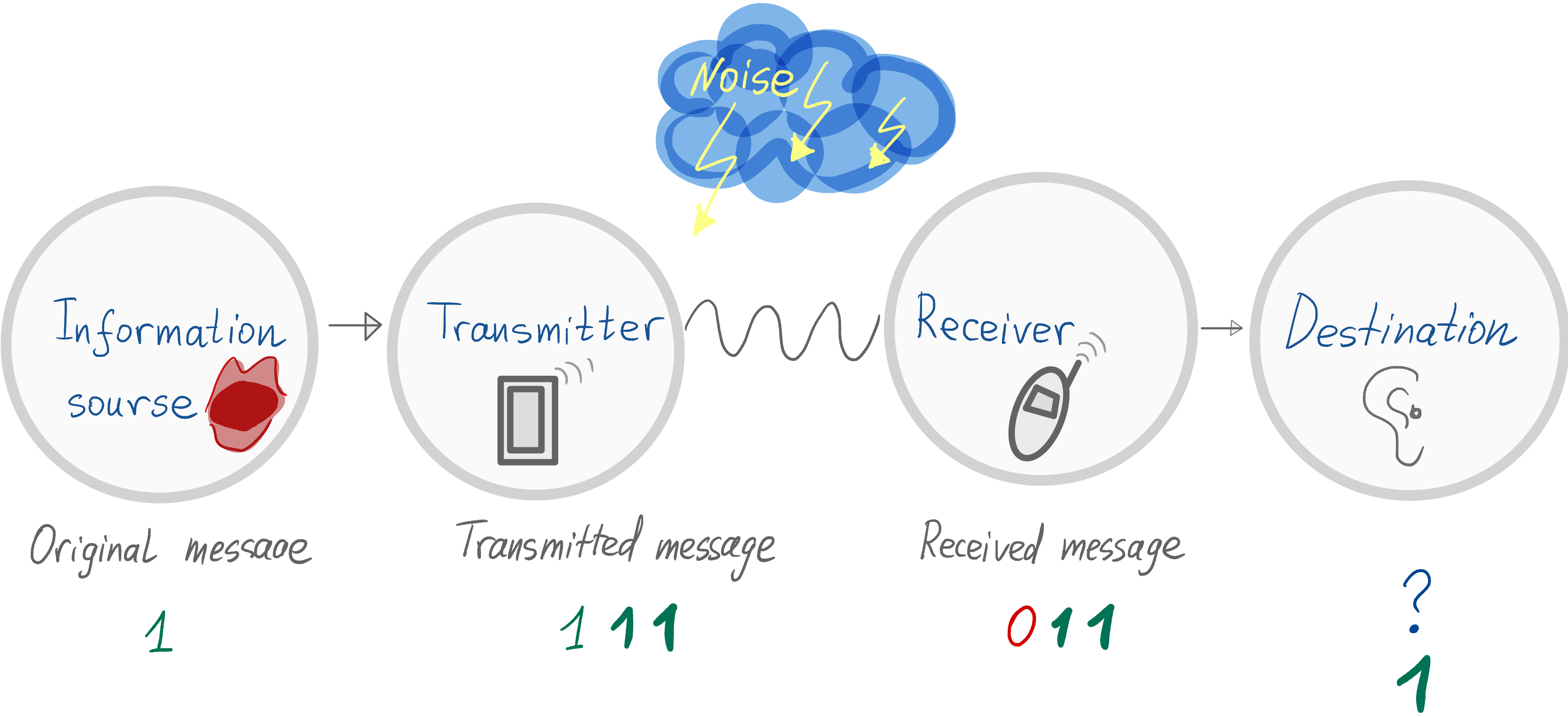
1949

A general communication system

(by C. Shannon)



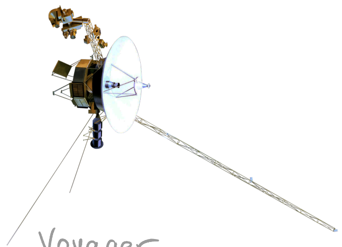

Noise
Information sourse
Transmitter
Receiver
Destination
Original message
1
Transmitted message
1
Received message
0
?



Repeat each bit 3 times:

We can detect 2 errors and correct 1 error.

Binary Golay code



Voyager

1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	1	1	1	1	0	0	0	1
0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	1	1	1	1	0	1	0
0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	1	1	1	1	0	1
0	0	0	1	0	0	0	0	0	0	0	0	1	0	0	1	0	0	1	1	1	1	1	0
0	0	0	0	1	0	0	0	0	0	0	0	1	1	0	0	1	0	0	1	1	1	0	1
0	0	0	0	0	1	0	0	0	0	0	0	1	1	1	0	0	1	0	0	1	1	1	0
0	0	0	0	0	0	1	0	0	0	0	0	1	1	1	1	0	0	1	0	0	1	0	1
0	0	0	0	0	0	0	1	0	0	0	0	1	1	1	1	1	0	0	1	0	0	1	0
0	0	0	0	0	0	0	0	1	0	0	0	0	1	1	1	1	1	0	0	1	0	0	1
0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	1	1	1	1	0	0	1	1	0
0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	1	0	1	0	1	0	1	1	1
0	0	0	0	0	0	0	0	0	0	0	1	1	0	1	0	1	0	1	0	1	0	1	1

2^{12} codewords of length 24

each 2 distinct codewords have at least 8 distinct coordinates

Original message of length 12

Transmitted message of length 24

Detects 7 errors

Corrects 3 errors

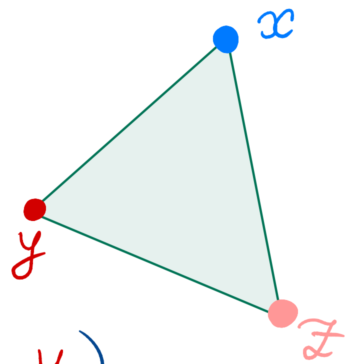
Geometric interpretation of error correcting codes

Hamming space: all words of fixed length

$$x = (x_1, \dots, x_d) \qquad x_i \in \{0, 1\}$$

Hamming distance: the number of distinct symbols on corresponding positions.

Example: $\mathrm{dist}\,((0,1,0),(0,0,0)) = 1$



Triangle inequality:

$$\mathrm{dist}\,(x, y) \le \mathrm{dist}\,(x, z) + \mathrm{dist}\,(z, y)$$

Recall:

The Ball with center x and radius r is the set of points y such that the distance between x and y is less then r.

$$B(x,r) := \{ y \mid dist(x,y) < 1 \}$$

A code that corrects r errors

is

a packing of balls of radius r in Hamming space.

From Golay code to Leech lattice

Leech lattice is the set of vectors of the form:

$$2^{-3/2}\,(\overset{1}{a_1},\ \overset{0}{a_2},\ \overset{0}{,}\ \overset{1}{.}\overset{1}{.}\,..,\ \overset{1}{a_{24}})$$

a_i are integers such that

$$a_1 + a_2 + \dots + a_{24} \equiv 4a_1 \equiv 4a_2 \equiv \dots \equiv 4a_{24} \pmod 8$$

and

for each fixed residue class modulo 4

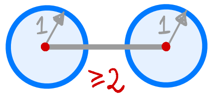
the 24 bit word, whose 1-s correspond to the coordinates

i such that a_i belongs to this residue class

is a word in a binary Golay code.

- Leech lattice has 1 point per unit of volume in $\mathbb{R}^{24}$
- distance between two distinct points is ≥ 2

$$\mathcal{P}_{\Lambda_{24}} := \bigcup_{\ell \in \Lambda_{24}} B(\ell, 1)$$



$\ell_1, \ell_2 \in \Lambda_{24}$

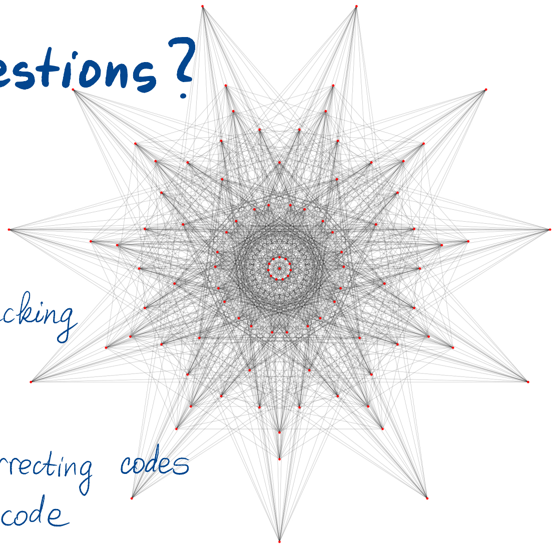
$|\ell_1 - \ell_2| = \sqrt{2n}$

$n \in \mathbb{Z}_{\geq 0}$

Theorem (2016) No packing of equal balls in 24-dimensional Euclidean space has density greater then the density of the Leech lattice packing.

Thank you!

Questions?



1) Sphere packing
2) Dimensions
3) Lattices
4) Error correcting codes
5) Golay code
6) Leech lattice
7) Mathematical proofs