Classification of Non-degenerate Symmetric Bilinear Forms in the Verlinde Category Ver_4^+

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Why do we care?

- Symmetric tensor categories (STCs) are a home to do commutative algebra, algebraic geometry, Lie theory, etc.
- In characteristic 0, all STCs arise as representation categories of “groups” unless they are very big in some sense.
- In characteristic $p > 0$, this is no longer true. For $\mathrm{p} = 2$, the most basic counterexample is Ver_4^+.
- Studying symmetric bilinear forms in this category will give rise to new geometric objects.

What is Ver_4^+?

- Categories consist of objects and maps between objects.
- An object $U \in \mathrm{Ver}_4^+$ is a vector space over a char 2 field $\mathbb{K}$ characterized by
  - Two integers m, n (determine dimension of basis).
  - A basis $\{v_1, \ldots, v_m, w_1, x_1, \ldots w_n, x_n\}$.
  - A mapping $t : U \to U$ such that $v_i \xrightarrow{t} 0$, $w_i \xrightarrow{t} x_i \xrightarrow{t} 0$.
- $\mathbb{1}$ subobjects each spanned by v_i.
- P subobjects each spanned by w_i, x_i.
- $U = m\mathbb{1} \oplus nP$

Ver_4^+ — Maps and Braiding

- Maps between two objects U, S are linear maps that respect t.
- In Ver_4^+, the braiding on each pair of objects U, S is a map $U \otimes S \to S \otimes U$ sending $u \otimes s$ to $s \otimes u + ts \otimes tu$.
- If the braiding were $u \otimes s \to s \otimes u$, would be a representation category of a group.
- Braiding controls idea of symmetry.

Symmetric bilinear forms

Let V be a vector space over a field $\mathbb{K}$. The map $\beta : V \times V \to \mathbb{K}$ is a bilinear form if

- $\beta(a, b_1) + \beta(a, b_2) = \beta(a, b_1 + b_2)$
- $\beta(a_1, b) + \beta(a_2, b) = \beta(a_1 + a_2, b)$
- $\beta(ka, b) = k\beta(a, b) = \beta(a, kb)$ for $k \in \mathbb{K}$.

β is symmetric if $\beta(a, b) = \beta(b, a)$ $\forall$ vectors a, b.

An example is the dot product.

- $u \cdot v + u \cdot w = u \cdot (v + w)$
- $u \cdot w + v \cdot w = (u + v) \cdot w$
- $u \cdot v = v \cdot u$

For Ver_4^+, β also has to satisfy $\beta(a, tb) = \beta(ta, b)$.

Non-degeneracy of Symmetric Bilinear Forms

Given a basis $\{u_1, u_2, ..., u_n\}$, the associated matrix of β is

$$\begin{array}{c} \\ u_1 \\ u_2 \\ \vdots \\ u_n \end{array} \begin{array}{c} \begin{array}{cccc} u_1 & u_2 & \dots & u_n \end{array} \\ \begin{bmatrix} \beta(u_1, u_1) & \beta(u_1, u_2) & \dots & \beta(u_1, u_n) \\ \beta(u_1, u_2) & \beta(u_2, u_2) & \dots & \beta(u_2, u_n) \\ \vdots & \vdots & \ddots & \vdots \\ \beta(u_n, u_1) & \beta(u_n, u_2) & \dots & \beta(u_n, u_n) \end{bmatrix} \end{array}.$$

β is non-degenerate if this matrix is invertible.

Classification in vector spaces over a characteristic 2 field

We find a basis such that β has one of these associated matrices.
The forms are non-isomorphic.

$$\begin{bmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & 1 & & & \\ & & & & \ddots & & \\ & & & & & 1 & \\ & & & & & & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & & & & & \\ 1 & 0 & & & & & \\ & & 0 & 1 & & & \\ & & 1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & 1 \\ & & & & & 1 & 0 \end{bmatrix}$$

Classification for an object in Ver_4^+

New condition: basis changes must respect t.

- Recall $U = m\mathbb{1} \oplus nP$.
- β restricted to $m\mathbb{1}$ is non-degenerate.

$$\begin{array}{c} \\ v_1 \\ v_2 \\ \vdots \\ v_m \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & \cdots & v_m \end{array} \\ \begin{bmatrix} \beta(v_1, v_1) & \beta(v_1, v_2) & \cdots & \beta(v_1, v_m) \\ \beta(v_1, v_2) & \beta(v_2, v_2) & \cdots & \beta(v_2, v_m) \\ \vdots & \vdots & \ddots & \vdots \\ \beta(v_m, v_1) & \beta(v_m, v_2) & \cdots & \beta(v_m, v_m) \end{bmatrix} \end{array}$$

- Since $t(v_i) = 0$, the condition $\beta(a, tb) = \beta(ta, b)$ is not important.
- Can view $m\mathbb{1}$ as an ordinary vector space in char 2.
- Our classification of β on $m\mathbb{1}$ is already done.

Strategy

- Orthogonal: $\beta(a, b) = 0$.
- Orthogonal spaces: $\beta(a, b) = 0 \ \forall a \in S_1, b \in S_2$
- In U, let S be the subspace orthogonal to $m\mathbb{1}$. $S \cong nP$.
- β on S is also non-degenerate.
- Plan: Reduce to classifying on S.

$$\begin{array}{cc} m\mathbb{1} & S \end{array}$$

$$\left[\begin{array}{c|c} \checkmark & 0 \\ \hline 0 & ? \end{array}\right]$$

Classification on $m\mathbb{1} \oplus nP$

Recall β on $m\mathbb{1}$ takes one of the following forms.

$$\begin{bmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & 1 & & & \\ & & & & \ddots & & \\ & & & & & 1 & \\ & & & & & & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & & & & & \\ 1 & 0 & & & & & \\ & & 0 & 1 & & & \\ & & 1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & 1 \\ & & & & & 1 & 0 \end{bmatrix}$$

- Decompose $m\mathbb{1}$ either entirely into 1-dimensional or entirely into 2-dimensional subspaces.
- Decompose nP either entirely into P subobjects or entirely into $2P$ subobjects.

Classification on $m\mathbb{1} \oplus nP$

$$\left[\begin{array}{c|c} 1 & \\ \hline & \begin{matrix} \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & & & \\ & \ddots & & \\ & & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & \\ & & & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \end{matrix} \end{array}\right]$$

$$\left[\begin{array}{c|c} 1 & \\ \hline & \begin{matrix} \begin{matrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{matrix} & & & \\ & \begin{matrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{matrix} & & \\ & & \ddots & \\ & & & \begin{matrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{matrix} \end{matrix} \end{array}\right]$$

$$\left[\begin{array}{c|c} \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & \\ \hline & \begin{matrix} \begin{matrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{matrix} & & & \\ & \begin{matrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{matrix} & & \\ & & \ddots & \\ & & & \begin{matrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{matrix} \end{matrix} \end{array}\right]$$

$$\left[\begin{array}{c|c} \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & \\ \hline & \begin{matrix} \begin{matrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{matrix} & & & \\ & \begin{matrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{matrix} & & \\ & & \ddots & \\ & & & \begin{matrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{matrix} \end{matrix} \end{array}\right]$$

$$\left[\begin{array}{c|c} \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & \\ \hline & \begin{matrix} \begin{matrix} a & 1 \\ 1 & 0 \end{matrix} & & & \\ & \ddots & & \\ & & \begin{matrix} a & 1 \\ 1 & 0 \end{matrix} & \\ & & & \begin{matrix} a & 1 \\ 1 & 0 \end{matrix} \end{matrix} \end{array}\right]$$

$$\left[\begin{array}{c|c} \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & \\ \hline & \begin{matrix} \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & & & \\ & \ddots & & \\ & & \begin{matrix} 1 & 1 \\ 1 & 0 \end{matrix} & \\ & & & \begin{matrix} a-1 & 1 \\ 1 & 0 \end{matrix} \end{matrix} \end{array}\right]$$

Summary

- Ver_4^+ is a special STC, with possibly new algebra.
- Non-degenerate symmetric bilinear forms let us study the symmetries of geometric objects.
- In char 2 vector spaces, there are 2.
- For a fixed object in Ver_4^+, there are 4 forms + 2 families of forms.

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Bibliography I

D. Benson, P. Etingof, and V. Ostrik.
New incompressible symmetric tensor categories in positive characteristic, 2021.

K. Conrad.
Bilinear forms.
2008.

P. Deligne.
Catégories tensorielles.
Moscow Mathematical Journal, 2(2):227–248, Feb. 2002.

P. Etingof and A. S. Kannan.
Lectures on symmetric tensor categories, 2021.

Bibliography II

S. Venkatesh.
Hilbert basis theorem and finite generation of invariants in symmetric tensor categories in positive characteristic.
International Mathematics Research Notices, 2016(16):5106–5133, oct 2015.