Algorithmically Generated Pants Decompositions of Combinatorial Surfaces

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Overarching Question: How can you cut up a surface?

• What do we mean by surface?

Riemannian 2-Manifolds

• 2-Manifold: A surface that looks “2-dimensional” around each point.

• Riemannian: The surface is smooth and has a geometry: we can define length, angles, and area.

• Orientable.

Just like polygons can be cut up into triangles, Riemannian 2-Manifolds can be cut up into 3-holed spheres (called pairs of pants).
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Definition

A pants decomposition of a topological surface is a set of disjoint, closed, and non-contractible curves that decompose the surface into three-holed spheres.

• Every genus \( g \geq 2 \) surface has a pants decomposition.
• Each such pants decomposition consists of \( 3g - 3 \) curves that cut the surface into \( 2g - 2 \) pairs of pants.

Fact

Any \( 3g - 3 \) curves on a genus \( g \) surface that are disjoint, closed, non-contractible, and not homotopic give a pants decomposition.
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The Bers’ constant of a Riemannian surface $S$, denoted by $\mathcal{B}_S$, is the smallest length of a pants decomposition of $S$.

- Describes how difficult it is to cut a surface $S$ into simpler surfaces.
- Understanding $\mathcal{B}_S$ is one the largest open problems in the geometry of surfaces.
Prior Results

Theorem (Buser, 1981)
A genus $g \geq 2$ hyperbolic surface $S$ with no boundary components satisfies: $g^{1/2} \lesssim \mathcal{B}_S \lesssim g \log(g)$.

Theorem (Buser, 1992)
A genus $g \geq 2$ closed Riemann surface $S$ with no boundary components satisfies: $\mathcal{B}_S \lesssim \left( \frac{\text{Area}(S)}{g} \right)^{1/2}$.

• Uses theoretical algorithm.
• Unknown optimal behavior.

\[ a(S) \lesssim b(S) \implies \text{there exists universal constant } C \text{ such that } a(S) \leq b(S)C. \]
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Motivating Questions

**Question #1**

What pants decompositions can we actually find?
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Question #2
Does Buser’s algorithm give shorter pants decompositions for “average” surfaces?
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Question #3
What’s the length of the $n$th cut in the decomposition?
How do we make a “discrete” surfaces?

1. Glue together \( n \) triangles with side length one into an \( n \)-gon.
2. Identify edges of the polygon.

A combinatorial surface is a type of Riemannian 2-manifold that is amenable to computation.

Gives rise to random surfaces.
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Theorem (H. 2023)
Let $S$ be a genus $g$ combinatorial surface. Algorithm #1 finds a length $\lesssim (g \text{Area}(S))^{1/2}$ pants decomposition of $S$ in $\mathcal{O}(g^3)$ time.
Results of Algorithm #1

Question #2

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No!

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Question #3
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After a certain point, every third cut has length $\frac{4}{3}n$. 
Finding Short Curves: Algorithm #2

New idea:

• Grow a ball around a random point until we find a loop that is in a different homotopy class than previous loops.
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Average Pants Decomposition Length vs. Genus

Pants Decomposition of a Genus 50 Surface
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Average Pants Decomposition Length vs. Genus

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