Existence of Circle Packings on Certain Translation Surfaces

Anton Levonian

Mentored by Professor Sergiy Merenkov

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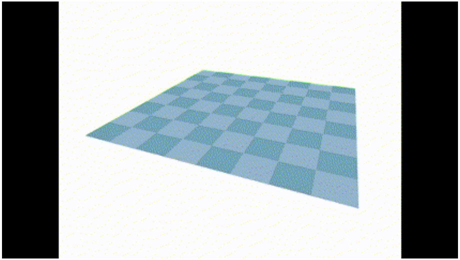
Overview

Translation Surfaces: Demonstration on a Torus



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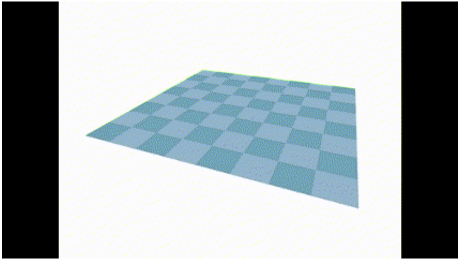

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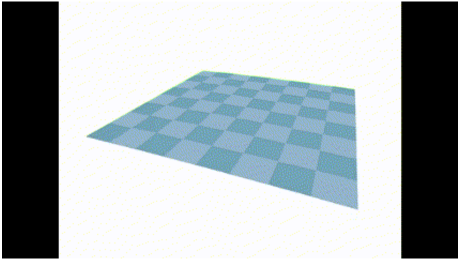

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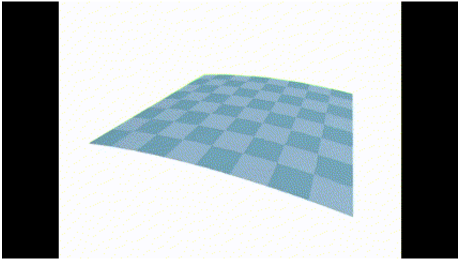

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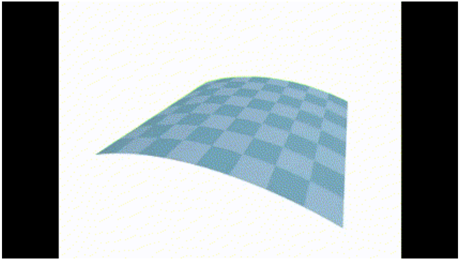

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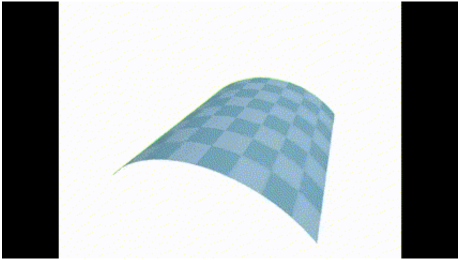

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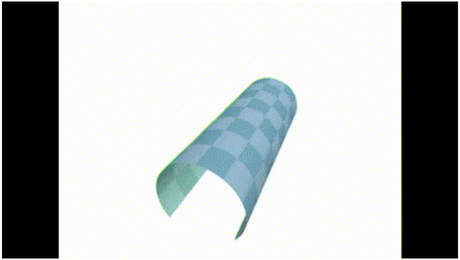

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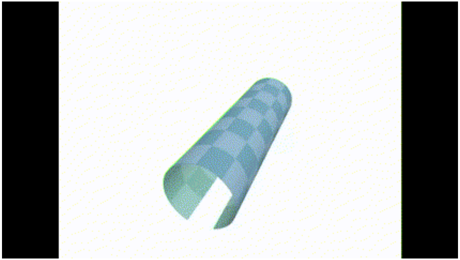

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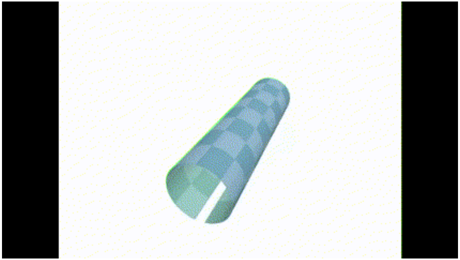

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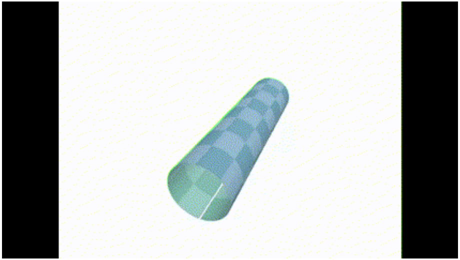

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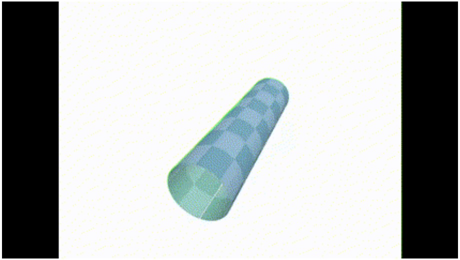

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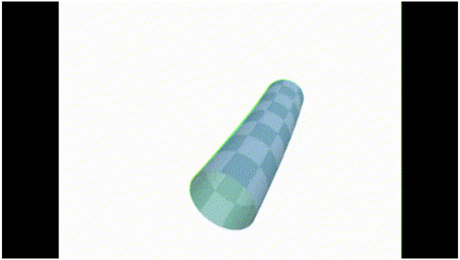

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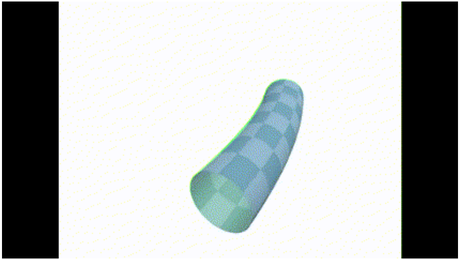

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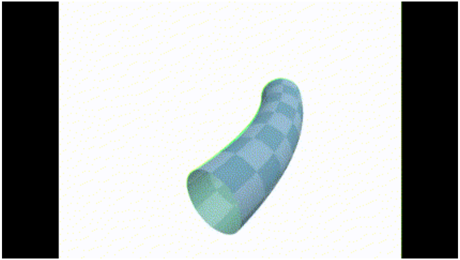

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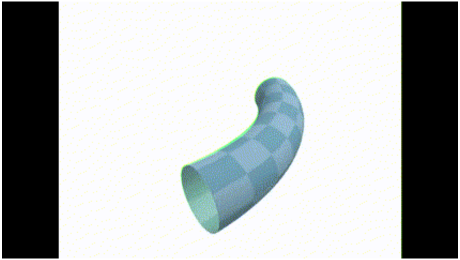

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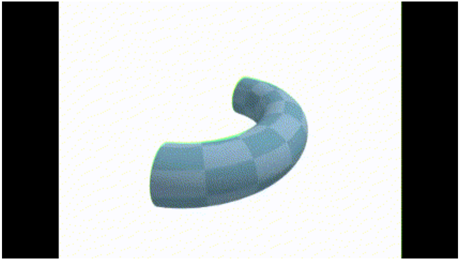

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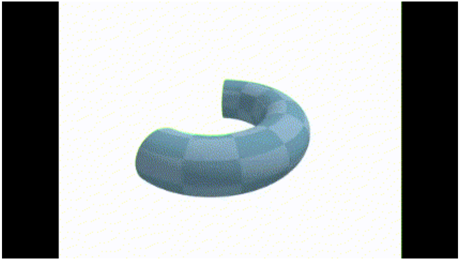

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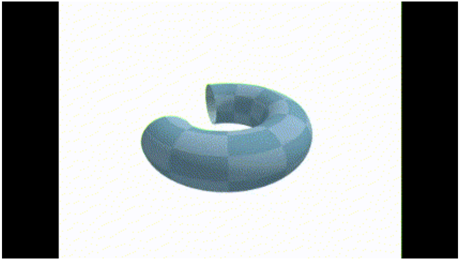

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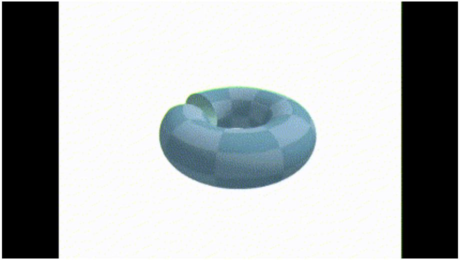

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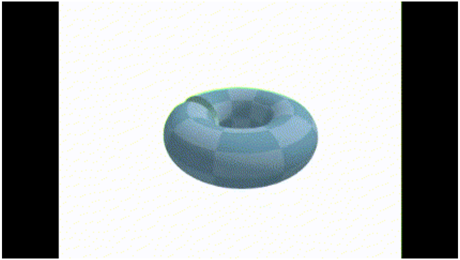

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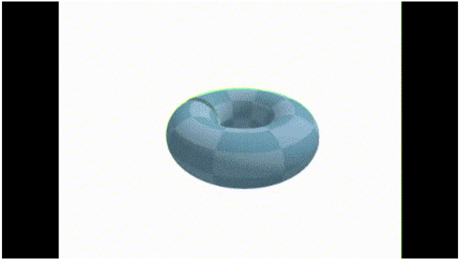

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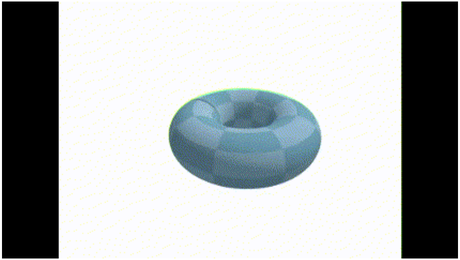

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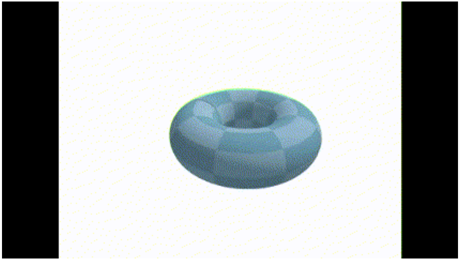

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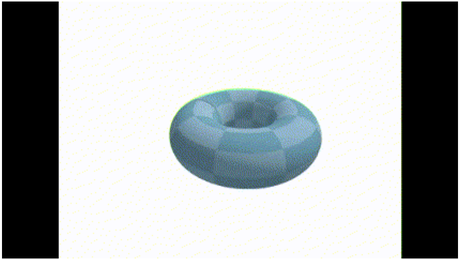

- The torus has *genus* 1, where the genus of a surface is the number of holes in the surface.
- A torus is an example of translation surface.

Translation Surfaces: Definition

- A *translation surface* is formed by identifying opposite sides of $\mathscr{P}$, where $\mathscr{P}$ is a collection of several polygon in the plane satisfying the following conditions:
  - $\mathscr{P}$ has an even number of sides.
  - Opposite sides of $\mathscr{P}$ are parallel and equal in length.

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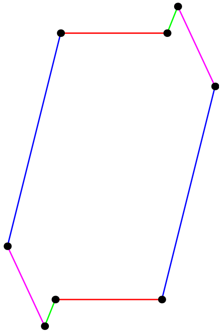
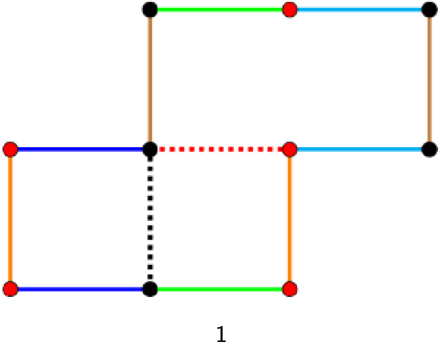


Square Tiled Surfaces

- A *square-tiled surface* is a translation surface for which $\mathscr{P}$ is formed by joining opposite sides of congruent squares together.
- A torus is an example of a square-tiled surface.



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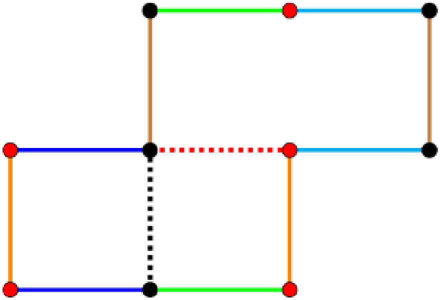

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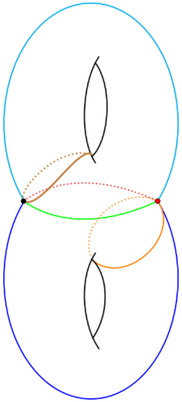

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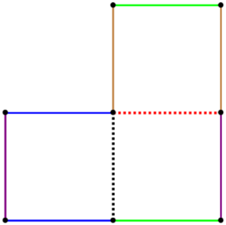
Singular Points



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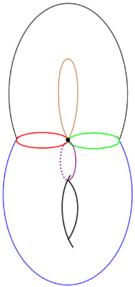
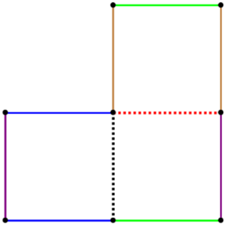


2

- A *singular point* of a translation surface is a point to which multiple vertices of the polygon are identified.
- The angle at a singular point, or *cone angle*, is $2p(d+1)$, where d is the *order* of the singular point.
- The above singular point has order $\left(5 \cdot \frac{p}{2} + \frac{3p}{2} + 2p\right) \cdot \frac{1}{2p} - 1 = 2$.

Singular Points



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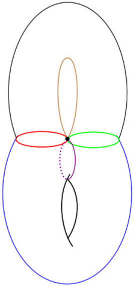

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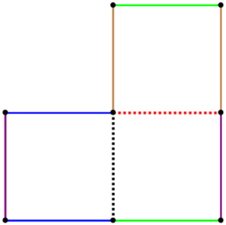

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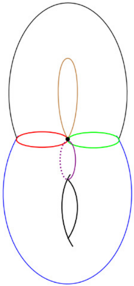

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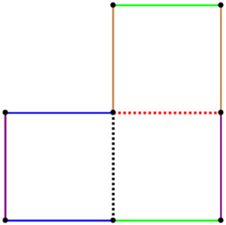

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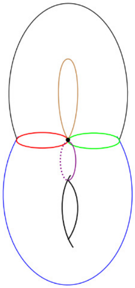

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Strata

Theorem (Gauss-Bonnet)

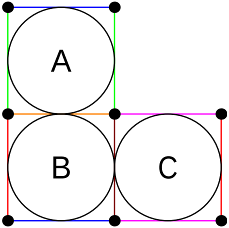
Let X be a translation surface with k singular points v_i, each with order $d(v_i)$, and let $c(X)$ be the Euler characteristic of X. Then

$$\sum_{i=1}^{k} d(v_i) + c(X) = 0.$$

- $c(X) = 2 - 2g$.
- A *stratum*, denoted by $\mathscr{H}(k)$, is determined by a partition of $2g - 2$.

Circle Packings

- A *circle packing* is defined as a collection of interiorwise disjoint disks.



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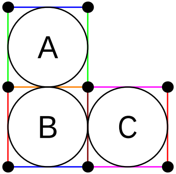
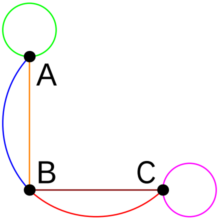


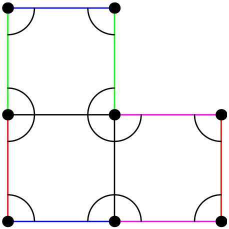
The circle packing C_3 on a surface in $\mathscr{H}(2)$

- A *contacts graph* G: circles corresponds to vertices of G and tangencies correspond to edges of G.

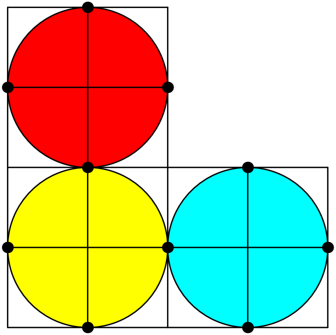


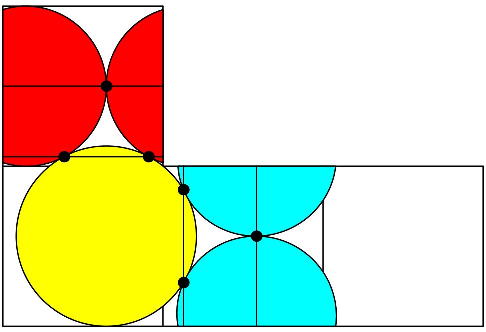
Circles Centered at Singular Points

- Below is a circle with radius less than $\frac{1}{2}$ centered at a singular point.



Equivalence of Circle Packings

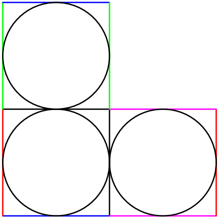




Questions

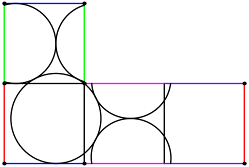
- Given a circle packing on a square tiled surface $X \in \mathscr{H}(k)$, is it generally possible to realize an equivalent circle packing on a square tiled surface $Y \in \mathscr{H}(k)$ with a different number of squares from X? If not, can an equivalent packing be realized on an affine transformation of Y?
- What are the "simplest" contacts graphs that cannot be realized on any surface in a certain stratum?

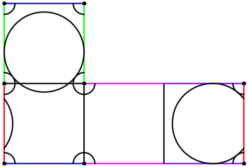
Realizability of C_3



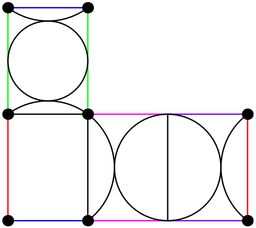
Theorem

An equivalent packing to C_3 cannot be realized on any four-squared translation surface in $\mathscr{H}(2)$ without applying an affine transformation.





Packings on Distinct Surfaces in $\mathscr{H}(2)$

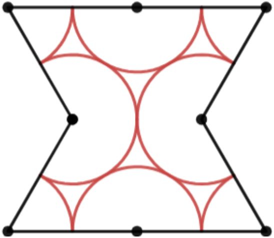


C_3 realized on a four-squared surface stretched vertically by a factor of $\frac{4}{3}$.

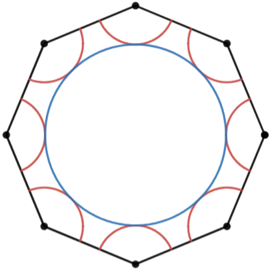
Realizable Contacts Graphs in $\mathscr{H}(2)$

Theorem

A maximum of 9 multi-loops and 8 multi-edges are realizable on any contacts graph in $\mathscr{H}(2)$.



9 multi-loops in $\mathscr{H}(2)$



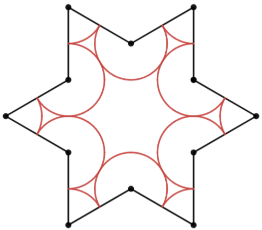
8 multi-edges in $\mathscr{H}(2)$

Demonstration of theorem in $\mathscr{H}(2)$.

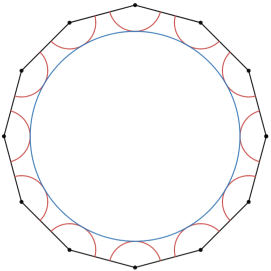
One Singular Point Theorem

Theorem

Given a genus g stratum $\mathscr{H}(2g-2)$, $4g$ multi-loops and $4g$ multi-edges can be realized on at least one surface of $\mathscr{H}(2g-2)$.



12 multi-loops in $\mathscr{H}(4)$



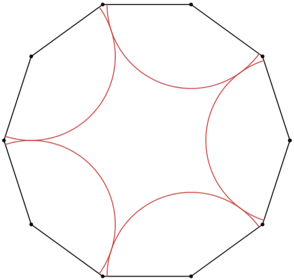
12 multi-edges in $\mathscr{H}(4)$

Demonstration of theorem in $\mathscr{H}(4)$.

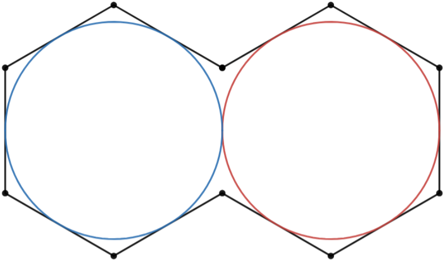
Realizable Contacts Graphs in $\mathscr{H}(1,1)$

Theorem

Up to 5 multi-loops and 6 multi-edges are realizable on any contacts graph in $\mathscr{H}(1,1)$.



5 multi-loops in $\mathscr{H}(1,1)$



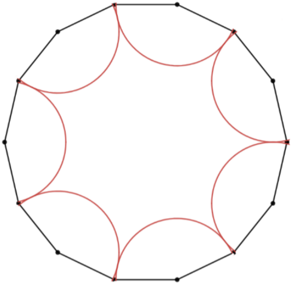
6 multi-edges in $\mathscr{H}(1,1)$

Demonstration of theorem in $\mathscr{H}(1,1)$.

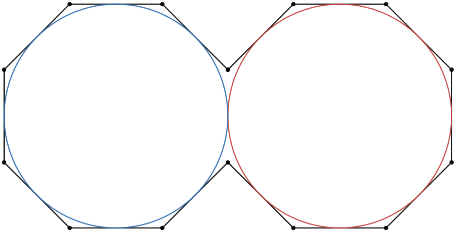
Two Singular Points Theorem

Theorem

Given a genus g stratum $\mathscr{H}(g-1,g-1)$, $2g+1$ multi-loops and $2g+2$ multi-edges can be realized on at least one surface of $\mathscr{H}(g-1,g-1)$.



7 multi-loops in $\mathscr{H}(2,2)$



8 multi-edges in $\mathscr{H}(2,2)$

Demonstration of theorem in $\mathscr{H}(2,2)$.

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- My mentor, Professor Sergiy Merenkov, for his guidance during the research
- Dr. Tanya Khovanova, Dr. Slava Gerovitch, Professor Pavel Etingof and the PRIMES-USA program for this invaluable research opportunity
- My family, for their support

References

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