Existence of Circle Packings on Certain Translation Surfaces

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MIT PRIMES Conference

October 15, 2023
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The torus has genus 1, where the genus of a surface is the number of holes in the surface.

A torus is an example of translation surface.
A translation surface is formed by identifying opposite sides of $\mathcal{P}$, where $\mathcal{P}$ is a collection of several polygon in the plane satisfying the following conditions:

- $\mathcal{P}$ has an even number of sides.
- Opposite sides of $\mathcal{P}$ are parallel and equal in length.
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A *translation surface* is formed by identifying opposite sides of \( \mathcal{P} \), where \( \mathcal{P} \) is a collection of several polygon in the plane satisfying the following conditions:

- \( \mathcal{P} \) has an even number of sides.
- Opposite sides of \( \mathcal{P} \) are parallel and equal in length.
A square-tiled surface is a translation surface for which \( \mathcal{P} \) is formed by joining opposite sides of congruent squares together.

A torus is an example of a square-tiled surface.
A square-tiled surface is a translation surface for which $P$ is formed by joining opposite sides of congruent squares together.

A torus is an example of a square-tiled surface.
A singular point of a translation surface is a point to which multiple vertices of the polygon are identified.

The angle at a singular point, or cone angle, is $2\pi(\delta + 1)$, where $\delta$ is the order of the singular point.

The above singular point has order \((5 \cdot \frac{\pi}{2} + 3\frac{\pi}{2} + 2\pi) \cdot \frac{1}{2\pi} - 1 = 2\).
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Theorem (Gauss-Bonnet)

Let $X$ be a translation surface with $k$ singular points $v_i$, each with order $\delta(v_i)$, and let $\chi(X)$ be the Euler characteristic of $X$. Then

$$\sum_{i=1}^{k} \delta(v_i) + \chi(X) = 0.$$ 

- $\chi(X) = 2 - 2g$.
- A *stratum*, denoted by $\mathcal{H}(\kappa)$, is determined by a partition of $2g - 2$. 
A circle packing is defined as a collection of interiorwise disjoint disks.
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The circle packing $C_3$ on a surface in $H(2)$

A contacts graph $G$: circles corresponds to vertices of $G$ and tangencies correspond to edges of $G$. 
Below is a circle with radius less than $\frac{1}{2}$ centered at a singular point.
Equivalence of Circle Packings

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Questions

- Given a circle packing on a square tiled surface $X \in \mathcal{H}(\kappa)$, is it generally possible to realize an equivalent circle packing on a square tiled surface $Y \in \mathcal{H}(\kappa)$ with a different number of squares from $X$? If not, can an equivalent packing be realized on an affine transformation of $Y$?

- What are the "simplest" contacts graphs that cannot be realized on any surface in a certain stratum?
Realizability of $C_3$

An equivalent packing to $C_3$ cannot be realized on any four-squared translation surface in $\mathcal{H}(2)$ without applying an affine transformation.
$C_3$ realized on a four-squared surface stretched vertically by a factor of $\frac{4}{3}$. 
Theorem

A maximum of 8 multi-loops and 8 multi-edges are realizable on any contacts graph in $\mathcal{H}(2)$.

8 multi-loops in $\mathcal{H}(2)$

8 multi-edges in $\mathcal{H}(2)$

Demonstration of theorem in $\mathcal{H}(2)$. 
Theorem

Given a genus $g$ stratum $\mathcal{H}(2g-2)$, $4g$ multi-loops and $4g$ multi-edges can be realized on at least one surface of $\mathcal{H}(2g-2)$.

Demonstration of theorem in $\mathcal{H}(4)$.

12 multi-loops in $\mathcal{H}(4)$

12 multi-edges in $\mathcal{H}(4)$
Realizable Contacts Graphs in $\mathcal{H}(1,1)$

Theorem

Up to 5 multi-loops and 6 multi-edges are realizable on any contacts graph in $\mathcal{H}(1,1)$.

5 multi-loops in $\mathcal{H}(1,1)$

6 multi-edges in $\mathcal{H}(1,1)$

Demonstration of theorem in $\mathcal{H}(1,1)$. 
Theorem

Given a genus $g$ stratum $\mathcal{H}(g-1, g-1)$, $2g+1$ multi-loops and $2g+2$ multi-edges can be realized on at least one surface of $\mathcal{H}(g-1, g-1)$.

7 multi-loops in $\mathcal{H}(2,2)$

8 multi-edges in $\mathcal{H}(2,2)$

Demonstration of theorem in $\mathcal{H}(2,2)$. 
I would like to thank...

- My mentor, Professor Sergiy Merenkov, for his guidance during the research
- Dr. Tanya Khovanova, Dr. Slava Gerovitch, Professor Pavel Etingof and the PRIMES-USA program for this invaluable research opportunity
- My family, for their support


