Overview

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   - Groups
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2. Semisubtractive Semidomains

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Background: Groups

\[(\mathbb{Z}, +) = \{\ldots, -2, -1, 0, 1, 2, \ldots\}\]

A group \(G\) is a set under one binary operator such that:

- \(G\) is closed;
- the operator is associative;
  - \((a + b) + c = a + (b + c)\);
- there is an identity element;
- each element has an inverse.

Example

- \((\mathbb{Q} \setminus \{0\}, \cdot)\);
- \((\{1, -1, i, -i\}, \cdot)\).
Background: Monoids

$(\mathbb{N}_0, +) = \{0, 1, 2, \ldots\}$

A monoid $M$ is a set under one binary operator such that:

- $M$ is closed;
- the operator is associative;
  - $(a + b) + c = a + (b + c)$;
- there is an identity element;
- each element has an inverse.

Example

- $(\mathbb{Z}, +)$;
- $(\mathbb{N}, \cdot)$. 
(\mathbb{Z}, +, \cdot)

An integral domain \( D \) is a set under two binary operators such that:

- \((D, +)\) is a group;
- \((D \setminus \{0\}, \cdot)\) is a monoid;
- Multiplication distributes over addition;
- The only zero-divisor is 0.

An element \( d \in D \) is a zero-divisor if there exists a nonzero element \( d' \) such that \( dd' = 0 \).

Example

- \((\mathbb{Z}[x], +, \cdot)\).

NOT an integral domain: \( \mathbb{Z}/4\mathbb{Z} = \{0, 1, 2, 3\} \) ("integers mod 4").
A semidomain $S$ is a subset of an integral domain such that:

- $(S, +)$ is a monoid;
- $(S \setminus \{0\}, \cdot)$ is a monoid.

Example

- $(\mathbb{N}_0[x], +, \cdot)$;
- $(\mathbb{Z}[x], +, \cdot)$.

Every integral domain is a semidomain.
Given a semidomain $S$, we will define its group of differences, denoted by $G(S)$. The object $G(S)$ is also called the Grothendieck group of $S$.

$G(S)$ consists of pairs of elements $(a, b)$ for $a, b \in S$, representing the value $a - b$. We define $a - b$ to be equal to $c - d$ if $a + d = b + c$.

$G(S)$ is not only a semidomain, but an integral domain. In fact, it is the least integral domain containing $S$. Thus, $S$ is a subset of the integral domain $G(S)$.

**Examples**

- $G(\mathbb{N}_0) = \mathbb{Z}$;
- $G(\mathbb{N}_0[x]) = \mathbb{Z}[x]$. 
Background: Semisubtractive Semidomains

We say a semidomain $S$ is semisubtractive if for all $a, b \in S$, either $a - b$ or $b - a$ is in $S$. More formally, there must exist some $x \in S$ such that $a + x = b$ or $b + x = a$.

Example

$\mathbb{N}_0$ is a semisubtractive semidomain. For $a, b \in \mathbb{N}_0$, $a - b \in \mathbb{N}_0$ if $a \geq b$ and $b - a \in \mathbb{N}_0$ if $b \geq a$. However, it is not an integral domain because none of its positive elements have additive inverses.

Example

$S = \mathbb{N}_0 + x\mathbb{Z}[x]$ also forms a semisubtractive semidomain. For polynomials $P, Q \in S$, at least one of $P - Q, Q - P$ is in $S$ depending on which has a greater constant term. However, it is not an integral domain because 1, for example, does not have an additive inverse.
Equivalently, we may say that a semidomain $S$ is semisubtractive if for all $s \in \mathcal{G}(S)$, either $s$ or $-s$ is in $S$. Since all $s \in \mathcal{G}(S)$ can be written as $a - b$ for $a, b \in S$, this is equivalent to our previous example.

**Example**

Consider the semidomain $S$ of integer polynomials whose lowest degree term is positive (in addition to 0).

The Grothendieck group of $S$ will be $\mathbb{Z}[x]$.

Finally, this set is closed under multiplication and addition.
Motivation

Why should we care about these objects?

- Semisubtractive semidomains generalize the properties of integral domains.

- What properties are shared between integral domains and semisubtractive semidomains?

- In future research, we only have to prove that an object is a semisubtractive semidomain to understand its properties.

- Semisubtractive semidomains correspond closely to the natural numbers.
Factorization Properties

Let $S$ be a semisubtractive semidomain.

- An element $u \in S$ is invertible if there exists $u' \in S$ such that $uu' = 1$.
- An element $a \in S$ is an atom if $a$ cannot be expressed as a product of two non-invertible elements of $S$. The set of atoms of $S$ is denoted $\mathbb{A}(S)$.
- $S$ is atomic if every element of $S$ can be expressed as a product of atoms.
- Two factorizations are considered the same if the atoms in the two factorizations only differ by invertible elements. (For example, in $\mathbb{Z}$, $14 = 2 \cdot 7$ and $14 = (-2)(-7)$ would be considered the same factorization, because $-1$ is invertible.)

Example

- $\mathbb{N}_0$;
- $\mathbb{N}_0[x]$. 
Factorization Properties

There are several properties that an atomic semisubtractive semidomain can have that describe the factorizations of elements.

- Bounded Factorization
- Finite Factorization
- Half-Factorial
- Unique Factorization

**Question**

How do the factorization properties of $S$ relate to the factorization properties of $\mathcal{G}(S)$?
Bounded Factorization

Definition
An atomic semisubtractive semidomain $S$ is a bounded factorization semidomain (BFS) if for each $s \in S$, there are finitely many lengths that a factorization of $s$ can have.

Example
- The Gaussian integers $\mathbb{Z}[i]$.

Theorem (Fox-Goel-Liao, 2023)
Let $S$ be a semisubtractive semidomain. Then, $S$ is a bounded factorization semidomain iff $\mathcal{G}(S)$ is a bounded factorization domain.
Finite Factorization

**Definition**
An atomic semisubtractive semidomain $S$ is a **finite factorization semidomain (FFS)** if for each $s \in S$, there are finitely many factorizations of $s$.

Every FFS is a BFS.

**Example**
- $\mathbb{N}_0 + x^2\mathbb{N}_0[x]$;
- $\mathbb{N}_0 + x\mathbb{Z}[x]$.

**Theorem (Fox-Goel-Liao, 2023)**
Let $S$ be a semisubtractive semidomain. Then, $S$ is a finite factorization semidomain iff $\mathcal{G}(S)$ is a finite factorization domain.
Half-Factorial

Definition

An atomic semisubtractive semidomain $S$ is a **half-factorial semidomain** if for each $s \in S$, there is one possible length of factorization of $s$.

Every HFS is a BFS.

Example

- $\mathbb{N}_0 + \mathbb{Z}\sqrt{2}$;
- $\sum_{p \in \mathbb{N}_0 + x\mathbb{Z}[x]} \mathbb{N}_0 y^p$.

Theorem (Fox-Goel-Liao, 2023)

Let $S$ be a semisubtractive semidomain. Then, $S$ is a half-factorial semidomain iff $G(S)$ is a half-factorial domain and $A(S) = S \cap A(G(S))$. 
Unique Factorization

**Definition**

An atomic semisubtractive semidomain \( S \) is a **unique factorization semidomain (UFS)** if for each \( s \in S \), there is one factorizations of \( s \).

Every UFS is an HFS and FFS.

**Example**

- \( \mathbb{N}_0 \);
- \( \mathbb{N}_0 + x\mathbb{Z}[x] \).

**Theorem (Fox-Goel-Liao, 2023)**

Let \( S \) be a semisubtractive semidomain. If \( S \) is a unique factorization semidomain, then \( G(S) \) is a unique factorization domain.
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Questions?