On Modular Categories With Frobenius-Perron Dimension Congruent to 2 Modulo 4

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Motivation

Modular tensor categories (MTCs) are related to a variety of fields:

1. topological quantum field theory
2. topological quantum computation
3. topological phases of matter
4. quantum groups

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Category I

Definition

A *category* $\mathcal{C}$ consists of

1. objects
2. morphisms (maps from objects to each other)
3. composition of morphisms

such that every object has an identity morphism, and the composition of morphisms is associative.

Definition

A nonzero object X in $\mathcal{C}$ is *simple* if its only subobjects are itself and the zero object.

Category II

Let **k** be a field.

Examples

1. **Set**
    - Objects: sets
    - Morphisms: total functions
2. **Vec**
    - Objects: vector spaces over **k**
    - Morphisms: **k**-linear maps
    - Simple Objects: 1-dimensional vector spaces
3. **Grp**
    - Objects: groups
    - Morphisms: group homomorphisms

Tensor Categories

Let $\mathbf{k}$ be an algebraically closed field of characteristic zero.

Definition (Vague)

A *tensor category* $\mathcal{C}$ is one with

1. an abelian structure for $\oplus$
2. a monoidal structure for $\otimes$
3. $\mathrm{End}_{\mathcal{C}}(\mathbf{1}) \cong \mathbf{k}$

Examples

1. **Vec**
2. $\mathrm{Rep}(G)$ for any group G

Fusion Categories I

Definition (Vague)

A *fusion category* $\mathcal{C}$ is a tensor category such that

1. every object is semisimple
2. there are finitely many simple objects

Examples

1. **Vec**
2. $\mathrm{Rep}(G)$ for a finite group G

Fusion Categories II

Definition

The *Deligne product* of two fusion categories $\mathcal{A}$ and $\mathcal{B}$ is the fusion category $\mathcal{A} \boxtimes \mathcal{B}$ whose simple objects are $X \otimes Y$ for $X \in \mathcal{O}(\mathcal{A})$ and $Y \in \mathcal{O}(\mathcal{B})$.

Notation

The set of isomorphism classes of simple objects in a fusion category $\mathcal{C}$ will be denoted as $\mathcal{O}(\mathcal{C})$, whose size is the *rank* of $\mathcal{C}$.

Frobenius-Perron Dimension

Definition

For each object X in a fusion category $\mathcal{C}$, there is a corresponding real number called the *Frobenius-Perron dimension*, which is denoted by $\text{FPdim}(X)$.

Definition

The *Frobenius-Perron dimension* $\text{FPdim}(\mathcal{C})$ of a fusion category $\mathcal{C}$ is defined as

$$\text{FPdim}(\mathcal{C}) := \sum_{X \in \mathcal{O}(\mathcal{C})} \text{FPdim}(X)^2.$$

The category $\mathcal{C}$ is said to be *weakly-integral* if $\text{FPdim}(\mathcal{C}) \in \mathbb{Z}$ and *integral* if $\text{FPdim}(X) \in \mathbb{Z}$ for all $X \in \mathcal{O}(\mathcal{C})$.

Invertible Objects

Let X be an object in a fusion category $\mathcal{C}$.

Remark

X is invertible if and only if $\mathrm{FPdim}(X) = 1$.

Notation

Isomorphism classes of invertible objects in $\mathcal{C}$ as a group will be denoted as $\mathcal{G}(\mathcal{C})$.

Remark

A fusion category is *pointed* if all of its simple objects are invertible. All fusion pointed categories are classified by group data.

Modular Categories

Definition

A *braiding* on a fusion category $\mathcal{C}$ is a natural isomorphism

$$c_{X,Y} : X \otimes Y \xrightarrow{\cong} Y \otimes X,$$

for all $X, Y \in \mathcal{C}$, satisfying the hexagonal axioms.

Definition

A *modular category* is a braided fusion category equipped with a spherical structure satisfying an additional non-degeneracy condition.

Müger centralizer

Definition

Let $\mathcal{C}$ be a braided fusion category with braiding $c_{X,Y} : X \otimes Y \xrightarrow{\cong} Y \otimes X$. The *Müger centralizer* of a fusion subcategory $\mathcal{K}$ is the fusion subcategory $\mathcal{K}'$ of $\mathcal{C}$ with objects Y in $\mathcal{C}$ satisfying

$$c_{Y,X} \circ c_{X,Y} = \mathrm{id}_{X \otimes Y}, \quad \text{for all } X \in \mathcal{K}.$$

Modular Subcategory

Definition

Let $\mathcal{C}$ be a modular category and $\mathcal{K}$ be a fusion subcategory of $\mathcal{C}$. Then, we say that $\mathcal{K}$ is a *modular subcategory* of $\mathcal{C}$ if and only if $\mathcal{K} \cap \mathcal{K}' = \mathbf{Vec}$.

Remark (Müger)

If $\mathcal{C}$ is a modular category and $\mathcal{K}$ is a modular subcategory of $\mathcal{C}$, then we have the ribbon equivalence

$$\mathcal{C} \simeq \mathcal{K} \boxtimes \mathcal{K}'.$$

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Classification Technique

Many efforts have been made to classify MTCs of a given rank because of the following theorem.

Theorem (Bruillard, Ng, Rowell, Wang)

There are finitely many MTCs of a fixed rank (up to equivalence).

Past Results

Theorem (Bruillard, Plavnik, Rowell)

MTCs of Frobenius-Perron dimension not divisible by 4 are integral.

Theorem (Bruillard, Czenky, Rowell, Gvozdjak, Plavnik)

Odd-dimensional MTCs of rank up to 23 are pointed.

Theorem (Alekseyev, Bruns, Palcoux, Petrov)

MTCs of Frobenius-Perron dimension congruent to 2 modulo 4 and rank up to 10 are pointed.

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Research Question

Question

Can we advance the classification of MTCs of Frobenius-Perron dimension congruent to 2 modulo 4 by rank?

However, we later noticed a connection between odd-dimensional MTCs and those of Frobenius-Perron dimension congruent to 2 modulo 4.

Question

Can we find a relationship between odd-dimensional MTCs and those of Frobenius-Perron dimension congruent to 2 modulo 4?

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Factorization Result I

The following is our main result.

Theorem

Let $\mathcal{C}$ be an MTC with $\text{FPdim}(\mathcal{C}) \equiv 2 \pmod 4$. *Then,* $\mathcal{C} \cong \widetilde{\mathcal{C}} \boxtimes \text{semion}$, *where* $\widetilde{\mathcal{C}}$ *is an odd-dimensional modular category and* semion *is the rank 2 pointed modular category.*

From the low rank classification of odd-dimensional MTCs, we have:

Corollary

MTCs with Frobenius-Perron dimension congruent to 2 modulo 4 and rank up to 46 are pointed.

Factorization Result II

We were able to generalize the previous theorem to the following.

Theorem

Let $\mathcal{C}$ be a weakly-integral MTC and p be an odd prime dividing $|\mathcal{G}(\mathcal{C})|$. If p has multiplicity 1 in $\mathrm{FPdim}(\mathcal{C})$, then $\mathcal{C} \cong \tilde{\mathcal{C}} \boxtimes \mathcal{P}$ for $\tilde{\mathcal{C}}$ an MTC of Frobenius-Perron dimension not divisible by p and $\mathcal{P}$ a pointed MTC of rank p.

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