Cyclic Base Orderings and Equitability of Matroids

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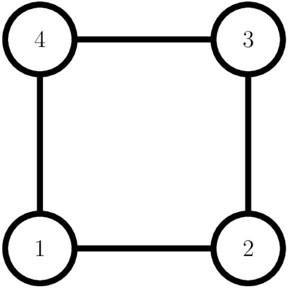
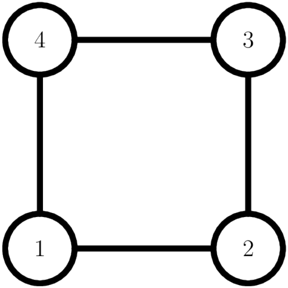
Opening Question

There are 4 train stations connected by 4 train tracks.



Opening Question

Can you order the 4 train tracks in a circle such that every 3 consecutive train tracks "connects" the 4 train stations (i.e. with only the 3 train tracks, any two stations are connected)?



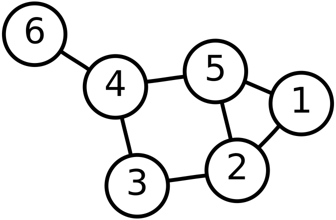
Quick Review of Graph Theory...

A graph G is an ordered pair (V, E).

- V is called the *vertex set*, whose elements are called vertices.
- E is called the *edge set* and is comprised of paired vertices, which are called edges.

Example

Let G be the graph shown below.



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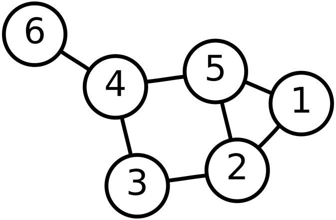

- $V = \{1, 2, 3, 4, 5, 6\}$
- $E = \{(4, 6), (4, 3), (4, 5), (3, 2), (5, 2), (5, 1), (2, 1)\}$.

Cycles

A cycle C is a path (sequence of consecutive edges) that starts and ends at the same vertex.

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A graph with no cycles is said to be *acyclic*.

Example

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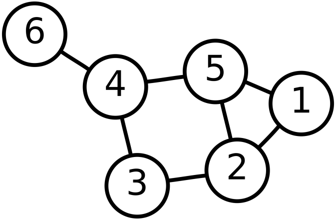

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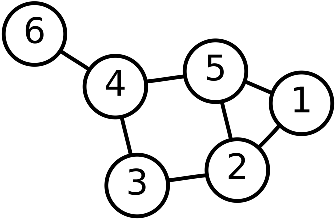


Examples of cycles in G

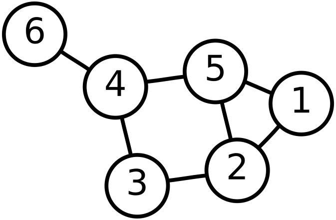
- $(5, 1), (1, 2), (2, 5)$.
- $(4, 3), (3, 2), (2, 5), (5, 4)$.

Subgraphs

A *subgraph* of a graph $G = (V, E)$ is another graph formed from a subset of V and all of the edges from G that connect vertices in the subset.

Example

Let G be the graph shown below.



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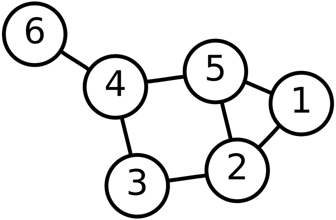

Examples of subgraphs of G

- $V = \{1, 2, 5\}, E = \{(1, 2), (2, 5), (5, 1)\}$.
- $V = \{2, 3, 4, 5\}, E = \{(2, 5), (5, 4), (4, 3), (3, 2)\}$.

Spanning Trees

A spanning tree of a graph G is an acyclic subgraph with $|V|$ vertices and $|V| - 1$ edges.

Example

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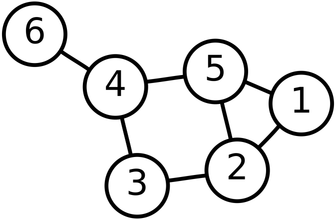

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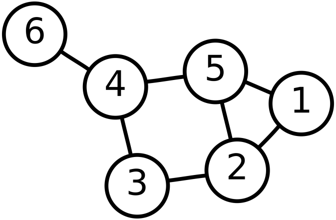

Examples of spanning trees of G

- $V = \{1, 2, 3, 4, 5, 6\}, E = \{(6, 4), (4, 3), (3, 2), (2, 5), (5, 1)\}$
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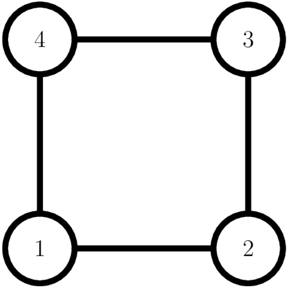
Opening Question Reformulated

Can you create a cyclic ordering of the edges such that each three consecutive edges forms a spanning tree?



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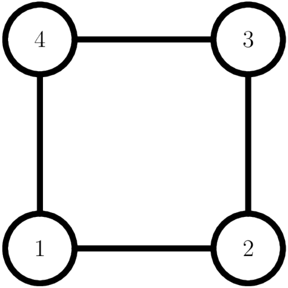

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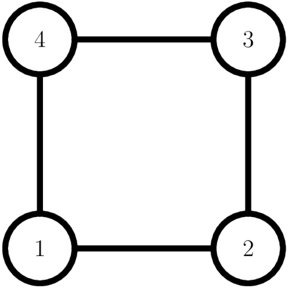

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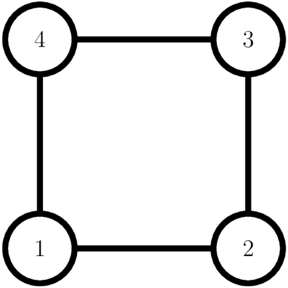

Which graphs have this property?

That is, for which graphs $G = (V, E)$ can we order the edges of G in a circle such that every $|V| - 1$ consecutive edges induce a spanning tree?

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M must satisfy the following axioms as well:

- If $A \in \mathcal{I}$, then any subset of A is in $\mathcal{I}$ as well. That is, if $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$.
- If $A, B \in \mathcal{I}$ and $|A| > |B|$, then there exists an element $e \in A \backslash B$ such that $B \cup \{e\} \in \mathcal{I}$ as well.

Note: If $A \in \mathcal{I}$, then A is said to be *independent*.

Bases

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It can be shown from the matroid axioms that all bases have equal cardinality, call it r.

Examples of Matroids

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- Linear Matroid: Let A be a $m \times n$ matrix, $E = \{1, \ldots, n\}$. Define
$$\mathcal{I} = \{S \subseteq E : \text{the columns indexed by } S \text{ are linearly independent}\}.$$
Then, $(E, \mathcal{I})$ is a matroid.

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- Kajitani et al. proved the conjecture for the $k = 2$ case of graphic matroids.
- The graphic matroids of 2-trees, 3-trees, complete bipartite graphs, and other graph classes have been shown to exhibit cyclic base orderings.
- Unsolved for graphic matroids when $k \geq 3$ and linear matroids when $k \geq 2$

Extension of $k = 2$ case for Graphic Matroids

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Theorem (L., Pan)

Suppose a graph can be decomposed into two edge-disjoint spanning trees T_1 and T_2. Then, its graphic matroid contains a cyclic base ordering where r consecutive elements are the edges of T_1 and the other r consecutive elements are the edges of T_2.

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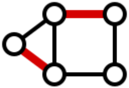

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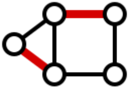

- A *maximum matching* is a matching that contains the largest number of edges. The *matching number*, denoted $\nu(G)$, is the size of a maximum matching.

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- The above matching is maximum.

Two necessary conditions for cyclic base orderings

Let $G = (V, E)$ be a graph. Define

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Lemma (L., Pan)

Let G be a bipartite graph with vertex partition A and B. If $|A| \neq |B|$, the matching matroid of G has no cyclic base ordering.

Equitability

Conjecture

Let $M = (E, \mathcal{I})$ *be a matroid. If the ground set* E *can be partitioned into* 2 *bases, then for any set* $X \subseteq E$*, there is a basis* B *such that* $E \backslash B$ *is also a base and* $\lfloor |X|/2 \rfloor \leq |B \cap X| \leq \lceil |X|/2 \rceil$.

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For any set $X \subseteq E$, there exists a base in the cyclic base ordering that satisfies the condition.

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Acknowledgements

- I would like to express my deepest thanks to my PRIMES mentor, Yuchong Pan, for suggesting this research project and providing invaluable guidance throughout the past year.
- I would also like to thank Dr. Tanya Khovanava, the MIT mathematics department, and the MIT PRIMES-USA Program for providing me with this research opportunity.

References

[1] Y. Kajitani, S. Ueno, and H. Miyano, "Ordering of the elements of a matroid such that its consecutive w elements are independent," *Discrete Mathematics*, vol. 72, no. 1-3, pp. 187–194, 1988.

Thank you!