On Generalized Eulerian Numbers

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Permutations

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- Treat these as functions (bijections) from \{1, 2, ..., n\} to itself.
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- Treat these as functions (bijectons) from \{1, 2, \ldots, n\} to itself.
- There are two main ways to write permutations.

Two-line Notation

Example:

\[ \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 3 & 1 & 4 & 2 \end{pmatrix} \]
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**Two-line Notation**

Example:

\[
\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 3 & 1 & 4 & 2 \end{pmatrix}
\]

Here, \(\sigma(1) = 5, \sigma(2) = 6, \sigma(3) = 3\), etc.

Sometimes, we simplify and write 563142.
Permutations

- Previous Example: $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 3 & 1 & 4 & 2 \end{pmatrix}$

- Reapplying $\sigma$ on any element returns back to itself eventually:
  \[ \sigma(1) = 5, \quad \sigma(\sigma(1)) = 4, \quad \sigma(\sigma(\sigma(1))) = 1. \]
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- Each arrow represents an application of \( \sigma \) to the node.
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  - Each arrow represents an application of \( \sigma \) to the node.
  - We similarly use shorthand and write \( \sigma = (154)(26)(3) \).
  - By convention, we arrange cycles by smallest element, and put smallest element on the left (ensures uniqueness!)
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- As ascent indices are marked in green.
- **Descents** are whenever $\sigma(i) > \sigma(i + 1)$ (indices marked in red).
- Two ascents: ascent of size 1 at $i = 1$, ascent of size 3 at $i = 3$. 
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\((2, 6)\)

\((1, 5)\)

\((3, 3)\)

\((5, 4)\)

\((6, 2)\)

\((4, 1)\)

- Excedances are marked in green.
- **Anti-excedances**, whenever \( \sigma(i) < i \), are marked in red.
- Two excedances: an excedance of size 4 at \( i = 1 \) and \( i = 2 \).
Why are these definitions interesting?

Definition (Foata Transform)

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- Takes a permutation $\sigma$ in two-line notation.
- Splits the permutation into blocks:
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**Definition (Foata Transform)**

The Foata transform:

- Takes a permutation $\sigma$ in two-line notation.
- Splits the permutation into blocks:
  - Stops at every element smaller than all previous elements, and start a new block before that element.
- Creates a new permutation $F(\sigma)$ where every block in $\sigma$ is interpreted as cycle in $F(\sigma)$. 

Example permutation:

\[ \sigma = (1 \ 2 \ 3 \ 4 \ 5 \ 6) \]

Stop at every element smaller than all previous elements, and start a new block before that element. Interpret blocks as cycles in transformed permutation \( F(\sigma) \):

- Number of ascents in \( \sigma \) equal to number of excedances in \( F(\sigma) \).
- Ascents in \( \sigma \) correspond exactly with excedances in \( F(\sigma) \).
- Descents inside blocks also correspond exactly.
- Finally, by convention, there must always be a descent/anti-excedance at the end of blocks.
The Foata Transform

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F(\sigma) = (56)(3)(142) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 1 & 3 & 2 & 6 & 5 \end{pmatrix}.
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The Foata Transform

**Proposition**

After an application of the Foata transform on any permutation \( \sigma \), number of ascents in \( \sigma \) **always** equal to number of excedances in \( F(\sigma) \).

The Foata transform is reversible: write in cycle notation and then interpret as one-line.

\[ F(\sigma) = (56)(3)(142) = \Rightarrow \sigma = 563142. \]

It is therefore a bijection!
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After an application of the Foata transform on any permutation $\sigma$, number of ascents in $\sigma$ \textit{always} equal to number of excedances in $F(\sigma)$.

- The Foata transform is reversible: write in cycle notation and then interpret as one-line.

$$F(\sigma) = (56)(3)(142) \implies \sigma = 563142.$$
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Definition (Eulerian Numbers)

The Eulerian number $E(n, m)$ is the number of permutations on $1, 2, \ldots, n$ with exactly $m$ ascents.
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Eulerian Numbers

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The *Eulerian number* $E(n, m)$ is the number of permutations on $1, 2, \ldots, n$ with exactly $m$ ascents.

- By the Foata transform, this is **ALSO** the number of permutations with exactly $m$ excedances.
- Example: $E(3, 1) = 4$. Four with exactly one ascent:
  
  \[ 132, 213, 231, 312. \]

  Four with exactly one excedance:
  
  \[ 132, 213, 312, 321. \]
Definition (\(r\)-Ascent)

Let \(\sigma\) be a permutation of \(1, 2, \ldots, n\). An \(r\)-ascent is any position \(i\) where \(\sigma(i) + r \leq \sigma(i + 1)\).
Generalized Eulerian Numbers

Definition ($r$-Ascent)

Let $\sigma$ be a permutation of $1, 2, \ldots, n$. An $r$-ascent is any position $i$ where $\sigma(i) + r \leq \sigma(i + 1)$.

- 1-ascent are equivalent to regular ascents.
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Definition ($r$-Excedance)
Let $\sigma$ be a permutation of $1, 2, \ldots, n$. An $r$-excedance is any position $i$ where $\sigma(i) \geq i + r$.

- Similarly, 1-excedances are equivalent to regular excedances.
A generalized Eulerian number $E_r(n, m)$ counts the number of permutations on $1, 2, \ldots, n$ with exactly $m$ $r$-ascents.
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- We claim $E_r(n, m)$ also counts the number of permutations with exactly $m$ $r$-excedances.
Generalized Eulerian Numbers

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A *generalized Eulerian number* $E_r(n, m)$ counts the number of permutations on $1, 2, \ldots, n$ with exactly $m$ $r$-ascents.

- We claim $E_r(n, m)$ also counts the number of permutations with exactly $m$ $r$-excedances.
- Consider our old examples:

  $$
  \sigma = \begin{pmatrix}
  1 & 2 & 3 & 4 & 5 & 6 \\
  5 & 6 & 3 & 1 & 4 & 2
  \end{pmatrix}, \quad
  F(\sigma) = \begin{pmatrix}
  1 & 2 & 3 & 4 & 5 & 6 \\
  4 & 1 & 3 & 2 & 6 & 5
  \end{pmatrix}.
  $$

- Power of Foata transform: ascent size in $\sigma$ matched exactly with excedance size in $F(\sigma)$. 

Inspired by past projects, we defined:

**Definition**

The number $E_r(n, m, k)$ counts the number of permutations $1, 2, \ldots, n$ with exactly $m$ $r$-excedances, **and** ends with $k$ (i.e., $\sigma(n) = k$).
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Main theorem proven:

**Theorem (Dong 2023)**

The number $E_r(n, m, k)$ also counts the number of permutations $1, 2, \ldots, n$ with exactly $m$ $r$-ascents and ends with $n + 1 - k$. 
A Further Generalization

- Inspired by past projects, we defined:

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- Main theorem proven:

**Theorem (Dong 2023)**
The number $E_r(n, m, k)$ also counts the number of permutations $1, 2, \ldots, n$ with exactly $m$ $r$-ascents and ends with $n + 1 - k$.

- We can show that $E_r(n, m, k)$ also counts the permutations with $m$ $r$-descents and ends with $k$ (somewhat nicer, though in either case symmetry is broken).
A Further Generalization

We also proved several other properties of these numbers, including:

- The following generalization of Worpitzky’s identity holds:
  \[(x + 1)^{n-k+1}x^{k-1} = \sum_{i=0}^{n} E_1(n, i, k) \binom{x + i}{n - 1}.\]

- It is possible to convert this generating function into an explicit formula for \(E_1(n, m, k)\).
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\]

- It is possible to convert this generating function into an explicit formula for \( E_1(n, m, k) \).

- For all integers \( n, m, k \) with \( k \geq 2 \), we have the equality:

\[
E_{r+1}(n, m, k) = E_r(n, m + 1, k - 1) + (r - 1) E_r(n - 1, m, k - 1) \\
- (r - 1) E_r(n - 1, m + 1, k - 1).
\]

Furthermore, \( E_{r+1}(n, m, 1) = E_r(n, m, n) \).

- This allows us to compute and potentially derive an explicit formula for \( E_r(n, m, k) \).
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