Betti Graphs of Puiseux Monoids

2023 PRIMES October conference

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(Mentored by Prof. Scott Chapman and Dr. Felix Gotti)

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Terminology

- We let \( \mathbb{N} \) be the set of positive integers and let \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \).
- We let \( \mathbb{P} \) denote the set of primes.
- We set \([b, c] := \{n \in \mathbb{Z} | b \leq n \leq c\}\).
- For a positive rational \( q \), we let \( n(q) \) and \( d(q) \) be the unique pair of relatively prime positive integers such that \( q = \frac{n(q)}{d(q)} \).
- For \( p \in \mathbb{P} \) and \( n \in \mathbb{N} \), let \( \nu_p(n) \) denote the exponent of the largest power of \( p \) dividing \( n \). For \( q \in \mathbb{Q}_{>0} \), we set \( \nu_p(q) = \nu_p(n(q)) - \nu_p(d(q)) \).
What is a Puiseux monoid?

Let $M \subseteq \mathbb{Q}_{\geq 0}$. We say $(M, +)$ is a Puiseux monoid if these two conditions hold:

- The set $M$ contains the identity element 0.
- The set $M$ is closed under addition; that is, for all rationals $a$ and $b$ contained in $M$, their sum $a + b$ is also contained in $M$. 
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Under the operation $+$, the following are examples of Puiseux monoids:

- Naturals (including zero): $\mathbb{N}_0$.
- Nonnegative rationals: $\mathbb{Q}_{\geq 0}$.
- Rationals greater than 1, including zero: $\mathbb{Q}_{>1} \cup \{0\}$. 

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Betti Graphs of Puiseux Monoids
Atoms of Monoids

Let \((M, +)\) be a Puiseux monoid. We say that a nonzero element \(m \in M\) is an atom if whenever we can express \(m = a + b\) for \(a, b \in M\), we must have \(a = 0\) or \(b = 0\). Let \(A(M)\) denote the set of atoms.
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**Example.** The set of all atoms of the Puiseux monoid \(M = (\mathbb{N}_0, +)\) is \(A(M) = \{1\}\); observe that 1 is an atom because the only decomposition of 1 is \(1 = 0 + 1\), which has a 0 in it.
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Atoms are like the building blocks of our factorizations: our goal is better understand the decomposition of the elements of \(M\) into atoms.
A factorization of an element \( m \in M \) is a formal addition of (not necessarily distinct) atoms \( a_1, a_2, \ldots a_\ell \) whose sum is \( m \); namely, \( m = a_1 + a_2 + \cdots + a_\ell \). We call \( \ell \in \mathbb{N} \) the length of the factorization.
Factorizations in Monoid

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If every nonzero element of a Puiseux monoid has a (finite) factorization, we say that the monoid is atomic.
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For a set \( S = \{a_1, a_2, \ldots, a_n\} \), we write \( \langle S \rangle \) to denote the monoid consisting of all linear combinations of the elements of \( S \). The same definition applies when \( S \) is an infinite set. For example, \( \langle 2, 3 \rangle := \{2x + 3y \mid x, y \in \mathbb{N}_0\} \). Note that \( \mathcal{A}(M) \subseteq S \).
Examples of Factorization

**Example.** In the Puiseux monoid \((\mathbb{N}_0, +)\), the only atom is 1, so all factorizations are the sum of copies of 1. The element 4 only has one factorization: \(1 + 1 + 1 + 1\).
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Example. In the monoid \((\mathbb{Q}_{>1} \cup \{0\}, +)\) consisting of all rationals greater than 1, including zero, the atoms are all rational numbers in the interval \((1, 2]\).
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- \(4 = 2 + 2,\)
- \(4 = \frac{4}{3} + \frac{4}{3} + \frac{4}{3},\)
- \(4 = 1.132 + 1.434 + 1.434,\)
- and so on...
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- \(4 = 1.132 + 1.434 + 1.434\),
- and so on...

We see that the element 4 in fact has infinitely many factorizations.
A Puiseux monoid $M$ is a **valuation monoid** if for all $a, b \in M$, there exists some $c \in M$ such that either $a = b + c$ or $b = a + c$.
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**Example.** The Puiseux monoid $M = (\mathbb{Q}_{\geq 0}, +)$ is a valuation monoid, since for any $a, b \in \mathbb{Q}_{\geq 0}$, we choose $c = |a - b| \in \mathbb{Q}_{\geq 0}$, for which either $a = b + c$ or $b = a + c$ must hold.
Valuation Monoids

A Puiseux monoid $M$ is a **valuation monoid** if for all $a, b \in M$, there exists some $c \in M$ such that either $a = b + c$ or $b = a + c$.

**Example.** The Puiseux monoid $M = (\mathbb{Q}_{\geq 0}, +)$ is a valuation monoid, since for any $a, b \in \mathbb{Q}_{\geq 0}$, we choose $c = |a - b| \in \mathbb{Q}_{\geq 0}$, for which either $a = b + c$ or $b = a + c$ must hold.

**Example.** The Puiseux monoid $M = (\mathbb{Q}_{> 1} \cup \{0\}, +)$ is not a valuation monoid, since $\frac{4}{3}$ and $\frac{5}{3}$ are elements of $M$, yet $c = \frac{5}{3} - \frac{4}{3}$ is not an element of $M$. 
A Puiseux monoid $M$ is an antimatter monoid if $\mathcal{A}(M) = \emptyset$; that is, $M$ has no atoms.

Example. The Puiseux monoid $M = \langle 1^2^k | k \in \mathbb{N}_0 \rangle$ is antimatter. This is because $1^2^k = 1^2^{k+1} + 1^2^{k+1}$, so $1^2^k$ is not an atom for all $k \in \mathbb{N}_0$. However, $\mathcal{A}(M) \subseteq \{ 1^2^k | k \in \mathbb{N}_0 \}$, so it follows that $\mathcal{A}(M)$ must be empty.
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**Example.** The Puiseux monoid $M = \langle \frac{1}{2^k} \mid k \in \mathbb{N}_0 \rangle$ is antimatter. This is because $\frac{1}{2^k} = \frac{1}{2^{k+1}} + \frac{1}{2^{k+1}}$, so $\frac{1}{2^k}$ is not an atom for all $k \in \mathbb{N}_0$. However, $\mathcal{A}(M) \subseteq \{ \frac{1}{2^k} \mid k \in \mathbb{N}_0 \}$, so it follows that $\mathcal{A}(M)$ must be empty.
Motivating Results around Puiseux Monoids

1. Grams used Puiseux monoids to disprove Cohn’s conjecture that any atomic domain must satisfy the ACCP.

2. Anderson, Anderson, and Zafrullah used Puiseux monoids to find a BFD whose integral closure is not a BFD.

3. Gotti and Li used Puiseux monoids to construct an atomic integral domain whose polynomial ring is not atomic.
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Betti Graphs and Betti Elements

The Betti graph of an element $m \in M$, denoted by $\nabla_m$, is the graph whose vertices are factorizations of $m$, where factorizations are connected by an edge if they share a common atom.
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Given a Puiseux monoid $(M, +)$, we say that an element $m \in M$ is a Betti element if its Betti graph $\nabla_m$ is disconnected; that is, there exist two vertices in $\nabla_m$ not connected by a path of edges.
Examples of Betti Elements

**Example.** Consider the Puiseux monoid $M = \langle 2, 3 \rangle$, the set of all nonnegative integers excluding 1.
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Note that \( \mathcal{A}(M) = \{2, 3\} \). Therefore, the only factorizations of 6 are \( 2 + 2 + 2 \) and \( 3 + 3 \).
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These factorizations do not share an atom, so the graph of $\nabla_6$ consists of two vertices with no edge between them.
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Note that $A(M) = \{2, 3\}$. Therefore, the only factorizations of 6 are $2 + 2 + 2$ and $3 + 3$.

These factorizations do not share an atom, so the graph of $\nabla_6$ consists of two vertices with no edge between them.

Since $\nabla_6$ is a disconnected graph, we see that 6 is a Betti element.
Examples of Betti Elements

Example. For Puiseux monoid $N = \langle 5, 7, 17, 23 \rangle$, we have that 40 is not Betti element, whereas 46 is. The notation $(a, b, c, d)$ represents the factorization $a \cdot 5 + b \cdot 7 + c \cdot 17 + d \cdot 23$.

Figure: The figure shows the Betti graph of $40 \notin \text{Betti}(N)$ on the left and that of $46 \in \text{Betti}(N)$ on the right.
Let \((p_n)_{n \geq 0}\) be the strictly increasing sequence of odd primes. We define the monoid

\[
M := \left\langle \frac{1}{2^n p_n} \mid n \in \mathbb{N}_0 \right\rangle
\]

to be the *Grams' monoid*, used in the construction of an atomic ring that does not satisfy the ascending chain condition of principal ideals. Its Betti elements are \(\left\{ \frac{1}{2^n} \mid n \in \mathbb{N}_0 \right\}\).
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Atomization

Let \((q_n)_{n \geq 1}\) be a sequence of rationals, and let \((p_n)_{n \geq 1}\) be a sequence of pairwise distinct primes such that

\[
gcd(p_i, n(q_i)) = gcd(p_i, d(q_j)) = 1
\]

for all \(i, j \in \mathbb{N}\). We say that

\[
M := \left\langle \frac{q_n}{p_n} \mid n \in \mathbb{N} \right\rangle
\]

is the Puiseux monoid of \((q_n)_{n \geq 1}\) atomized at \((p_n)_{n \geq 1}\).

Atomization can be used to construct monoids with desired properties.
Let $M$ be the Puiseux monoid of $(q_n)_{n \geq 1}$ atomized at $(p_n)_{n \geq 1}$, for suitable rationals $(q_n)_{n \geq 1}$ and primes $(p_n)_{n \geq 1}$. Every element $q \in M$ has a unique canonical decomposition

$$q = n_q + \sum_{n \in \mathbb{N}} c_n \frac{q_n}{p_n},$$

where $n_q \in \langle q_n \mid n \in \mathbb{N} \rangle$ and $c_n \in \mathbb{Z}$.

This is an interesting property of atomized monoids that is a key driver behind our results.
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Theorem (Chapman, Gotti, Jang, Mao, Mao, 2023)

Let $M$ be the Puiseux monoid resulting from atomizing $(q_n)_{n \geq 1}$ at $(p_n)_{n \geq 1}$. Then the following hold:

1. For every $j \in \mathbb{N}$, the factorization $p^j q^j p^j$ of $q^j$ is an isolated vertex in the Betti graph $\nabla q^j$.
2. $\text{Betti}(M) \subseteq \langle q^n | n \in \mathbb{N} \rangle$.
3. $\{q^n | n \in \mathbb{N}\} \subseteq \text{Betti}(M)$ if $\langle q^n | n \in \mathbb{N} \rangle$ is an antimatter monoid.
4. $\text{Betti}(M) \subseteq \{q^n | n \in \mathbb{N}\}$ if $\langle q^n | n \in \mathbb{N} \rangle$ is a valuation monoid.
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Let $M$ be the Puiseux monoid resulting from atomizing $(q_n)_{n \geq 1}$ at $(p_n)_{n \geq 1}$. Then the following hold:

1. For every $j \in \mathbb{N}$, the factorization $p_j \frac{q_j}{p_j}$ of $q_j$ is an isolated vertex in the Betti graph $\nabla_{q_j}$.

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Betti Graphs of Puiseux Monoids
Theorem (Chapman, Gotti, Jang, Mao, Mao, 2023)

Let \( M \) be the Puiseux monoid resulting from atomizing \((q_n)_{n \geq 1}\) at \((p_n)_{n \geq 1}\). Then the following hold:

1. For every \( j \in \mathbb{N} \), the factorization \( p_j^{q_j/p_j} \) of \( q_j \) is an isolated vertex in the Betti graph \( \nabla_{q_j} \).
2. \( \text{Betti}(M) \subseteq \langle q_n \mid n \in \mathbb{N} \rangle. \)
3. \( \{q_n \mid n \in \mathbb{N}\} \subseteq \text{Betti}(M) \) if \( \langle q_n \mid n \in \mathbb{N} \rangle \) is an antimatter monoid.
Results on Betti elements

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3. $\{q_n \mid n \in \mathbb{N}\} \subseteq \text{Betti}(M)$ if $\langle q_n \mid n \in \mathbb{N} \rangle$ is an antimatter monoid.

4. $\text{Betti}(M) \subseteq \{q_n \mid n \in \mathbb{N}\}$ if $\langle q_n \mid n \in \mathbb{N} \rangle$ is a valuation monoid.
Monoids with any number of Betti elements

Here is an application of our results:

Theorem (Chapman, Gotti, Jang, Mao, Mao, 2023)

For each $k \in \mathbb{N}$, we can construct a Puiseux monoid with exactly $k$ Betti elements.
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**Theorem (Chapman, Gotti, Jang, Mao, Mao, 2023)**

*For each $k \in \mathbb{N}$, we can construct a Puiseux monoid with exactly $k$ Betti elements.*

**Sketch of proof.** Consider the Puiseux monoid

\[ M := \left\langle \frac{1}{p_1}, \frac{2}{p_2}, \ldots, \frac{k}{p_k}, \frac{1}{p_{k+1}}, \ldots, \frac{k}{p_{2k}}, \ldots \right\rangle, \]

where $(p_n)_{n \geq 1}$ is an increasing sequence of primes with $p_1 > k$. From the previous result, we can conclude that \( \text{Betti}(M) = [1, k] \).
Open Research Questions

1. Suppose \( M \) is an atomic Puiseux monoid that does not satisfy the \textbf{ACCP}; that is, there exists an infinite sequence of elements \( a_1, a_2, a_3, \ldots \) of \( M \) such that for all integers \( i \geq 1 \), there exists some nonzero \( d_i \in M \) satisfying \( a_i = a_{i+1} + d_i \).

Must \( M \) necessarily have infinitely many Betti elements?
Acknowledgements

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References

Thank you!