Betti Graphs of Puiseux Monoids

2023 PRIMES October conference

Joshua Jang, Jason Mao, Skyler Mao

(Mentored by Prof. Scott Chapman and Dr. Felix Gotti)

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Betti Graphs of Puiseux Monoids  
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Terminology

- We let \( \mathbb{N} \) be the set of positive integers and let \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \).
- We let \( \mathbb{P} \) denote the set of primes.
- We set \( [b, c] := \{n \in \mathbb{Z} \mid b \leq n \leq c\} \).
- For a positive rational \( q \), we let \( n(q) \) and \( d(q) \) be the unique pair of relatively prime positive integers such that \( q = \frac{n(q)}{d(q)} \).
- For \( p \in \mathbb{P} \) and \( n \in \mathbb{N} \), let \( \nu_p(n) \) denote the exponent of the largest power of \( p \) dividing \( n \). For \( q \in \mathbb{Q}_{>0} \), we set \( \nu_p(q) = \nu_p(n(q)) - \nu_p(d(q)) \).
What is a Puiseux monoid?

Let $M \subseteq \mathbb{Q}_{\geq 0}$. We say $(M, +)$ is a Puiseux monoid if these two conditions hold:

- The set $M$ contains the identity element $0$.
- The set $M$ is closed under addition; that is, for all rationals $a$ and $b$ contained in $M$, their sum $a + b$ is also contained in $M$. 
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Under the operation $+$, the following are examples of Puiseux monoids:

- Naturals (including zero): $\mathbb{N}_0$. 

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- Nonnegative rationals: \( \mathbb{Q}_{\geq 0} \).
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Under the operation $+$, the following are examples of Puiseux monoids:

- Naturals (including zero): $\mathbb{N}_0$.
- Nonnegative rationals: $\mathbb{Q}_{\geq 0}$.
- Rationals greater than 1, including zero: $\mathbb{Q}_{>1} \cup \{0\}$.
Atoms of Monoids

Let \((M, +)\) be a Puiseux monoid. We say that a nonzero element \(m \in M\) is an atom if whenever we can express \(m = a + b\) for \(a, b \in M\), we must have \(a = 0\) or \(b = 0\). Let \(A(M)\) denote the set of atoms.

Example. The set of all atoms of the Puiseux monoid \(M = (\mathbb{N}_0, +)\) is \(A(M) = \{1\}\); observe that 1 is an atom because the only decomposition of 1 is \(1 = 0 + 1\), which has a 0 in it.
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Atoms are like the building blocks of our factorizations: our goal is better understand the decomposition of the elements of \(M\) into atoms.
A factorization of an element $m \in M$ is a formal addition of (not necessarily distinct) atoms $a_1, a_2, \ldots a_\ell$ whose sum is $m$; namely, $m = a_1 + a_2 + \cdots + a_\ell$. We call $\ell \in \mathbb{N}$ the length of the factorization.
Factorizations in Monoid

A **factorization** of an element \( m \in M \) is a formal addition of (not necessarily distinct) atoms \( a_1, a_2, \ldots a_\ell \) whose sum is \( m \); namely, 
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If every nonzero element of a Puiseux monoid has a (finite) factorization, we say that the monoid is **atomic**.
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For a set $S = \{a_1, a_2, \ldots, a_n\}$, we write $\langle S \rangle$ to denote the monoid consisting of all linear combinations of the elements of $S$. The same definition applies when $S$ is an infinite set. For example, $\langle 2, 3 \rangle := \{2x + 3y \mid x, y \in \mathbb{N}_0\}$. Note that $\mathcal{A}(M) \subseteq S$. 
Examples of Factorization

**Example.** In the Puiseux monoid \((\mathbb{N}_0, +)\), the only atom is 1, so all factorizations are the sum of copies of 1. The element 4 only has one factorization: \(1 + 1 + 1 + 1\).
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- \(4 = 2 + 2\),
- \(4 = \frac{4}{3} + \frac{4}{3} + \frac{4}{3}\),
- \(4 = 1.132 + 1.434 + 1.434\),
- and so on...
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We see that the element 4 in fact has infinitely many factorizations.
Valuation Monoids

A Puiseux monoid \( M \) is a valuation monoid if for all \( a, b \in M \), there exists some \( c \in M \) such that either \( a = b + c \) or \( b = a + c \).
Valuation Monoids

A Puiseux monoid $M$ is a **valuation monoid** if for all $a, b \in M$, there exists some $c \in M$ such that either $a = b + c$ or $b = a + c$.

**Example.** The Puiseux monoid $M = (\mathbb{Q}_{\geq 0}, +)$ is a valuation monoid, since for any $a, b \in \mathbb{Q}_{\geq 0}$, we choose $c = |a - b| \in \mathbb{Q}_{\geq 0}$, for which either $a = b + c$ or $b = a + c$ must hold.
A Puiseux monoid $M$ is a \textit{valuation monoid} if for all $a, b \in M$, there exists some $c \in M$ such that either $a = b + c$ or $b = a + c$.

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\textbf{Example.} The Puiseux monoid $M = (\mathbb{Q}_{> 1} \cup \{0\}, +)$ is not a valuation monoid, since $\frac{4}{3}$ and $\frac{5}{3}$ are elements of $M$, yet $c = \frac{5}{3} - \frac{4}{3}$ is not an element of $M$. 

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Betti Graphs of Puiseux Monoids
Antimatter Monoids

A Puiseux monoid $M$ is an antimatter monoid if $\mathcal{A}(M) = \emptyset$; that is, $M$ has no atoms.

Example. The Puiseux monoid $M = \langle 1^{2^k} | k \in \mathbb{N}_0 \rangle$ is antimatter. This is because $1^{2^k} = 1^{2^k+1} + 1^{2^{k+1}}$, so $1^{2^k}$ is not an atom for all $k \in \mathbb{N}_0$. However, $\mathcal{A}(M) \subseteq \{1^{2^k} | k \in \mathbb{N}_0\}$, so it follows that $\mathcal{A}(M)$ must be empty.
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Motivating Results around Puiseux Monoids

1. Grams used Puiseux monoids to disprove Cohn’s conjecture that any atomic domain must satisfy the ACCP.

2. Anderson, Anderson, and Zafrullah used Puiseux monoids to find a BFD whose integral closure is not a BFD.

3. Gotti and Li used Puiseux monoids to construct an atomic integral domain whose polynomial ring is not atomic.
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Given a Puiseux monoid $(M, +)$, we say that an element $m \in M$ is a **Betti element** if its Betti graph $\nabla_m$ is disconnected; that is, there exist two vertices in $\nabla_m$ not connected by a path of edges.
Example. Consider the Puiseux monoid $M = \langle 2, 3 \rangle$, the set of all nonnegative integers excluding 1.
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Note that $A(M) = \{2, 3\}$. Therefore, the only factorizations of 6 are $2 + 2 + 2$ and $3 + 3$. 
Examples of Betti Elements

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These factorizations do not share an atom, so the graph of $\nabla_6$ consists of two vertices with no edge between them.
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These factorizations do not share an atom, so the graph of $\nabla_6$ consists of two vertices with no edge between them.

Since $\nabla_6$ is a disconnected graph, we see that 6 is a Betti element.
Examples of Betti Elements

Example. For Puiseux monoid $N = \langle 5, 7, 17, 23 \rangle$, we have that 40 is not Betti element, whereas 46 is. The notation $(a, b, c, d)$ represents the factorization $a \cdot 5 + b \cdot 7 + c \cdot 17 + d \cdot 23$.

Figure: The figure shows the Betti graph of $40 \notin \text{Betti}(N)$ on the left and that of $46 \in \text{Betti}(N)$ on the right.
Grams Monoid Example

Let \((p_n)_{n \geq 0}\) be the strictly increasing sequence of odd primes. We define the monoid

\[
M := \langle \frac{1}{2^n p_n} \mid n \in \mathbb{N}_0 \rangle
\]

to be the **Grams' monoid**, used in the construction of an atomic ring that does not satisfy the ascending chain condition of principal ideals. Its Betti elements are \(\{\frac{1}{2^n} \mid n \in \mathbb{N}_0\}\).
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Atomization

Let \((q_n)_{n \geq 1}\) be a sequence of rationals, and let \((p_n)_{n \geq 1}\) be a sequence of pairwise distinct primes such that

\[
gcd(p_i, n(q_i)) = gcd(p_i, d(q_j)) = 1
\]

for all \(i, j \in \mathbb{N}\). We say that

\[
M := \left\langle \frac{q_n}{p_n} \mid n \in \mathbb{N} \right\rangle
\]

is the Puiseux monoid of \((q_n)_{n \geq 1}\) atomized at \((p_n)_{n \geq 1}\).

Atomization can be used to construct monoids with desired properties.
Canonical Decomposition

Let $M$ be the Puiseux monoid of $(q_n)_{n \geq 1}$ atomized at $(p_n)_{n \geq 1}$, for suitable rationals $(q_n)_{n \geq 1}$ and primes $(p_n)_{n \geq 1}$. Every element $q \in M$ has a unique canonical decomposition

$$q = n_q + \sum_{n \in \mathbb{N}} c_n \frac{q_n}{p_n},$$

where $n_q \in \langle q_n \mid n \in \mathbb{N} \rangle$ and $c_n \in [0, p - 1]$.

This is an interesting property of atomized monoids that is a key driver behind our results.
Theorem (Chapman, Gotti, Jang, Mao, Mao, 2023)

Let $M$ be the Puiseux monoid resulting from atomizing $(q_n)_{n \geq 1}$ at $(p_n)_{n \geq 1}$. Then the following hold:

1. For every $j \in \mathbb{N}$, the factorization $p^j q^j p^j$ of $q^j$ is an isolated vertex in the Betti graph $\nabla q^j$.
2. $\text{Betti}(M) \subseteq \langle q^n | n \in \mathbb{N} \rangle$.
3. $\{q^n | n \in \mathbb{N}\} \subseteq \text{Betti}(M)$ if $\langle q^n | n \in \mathbb{N} \rangle$ is an antimatter monoid.
4. $\text{Betti}(M) \subseteq \{q^n | n \in \mathbb{N}\}$ if $\langle q^n | n \in \mathbb{N} \rangle$ is a valuation monoid.
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4. $\text{Betti}(M) \subseteq \{q_n \mid n \in \mathbb{N}\}$ if $\langle q_n \mid n \in \mathbb{N} \rangle$ is a valuation monoid.
Monoids with any number of Betti elements

Here is an application of our results:

**Theorem (Chapman, Gotti, Jang, Mao, Mao, 2023)**

*For each* $k \in \mathbb{N}$, *we can construct a Puiseux monoid with exactly* $k$ *Betti elements.*
Monoids with any number of Betti elements

Here is an application of our results:

Theorem (Chapman, Gotti, Jang, Mao, Mao, 2023)

For each \( k \in \mathbb{N} \), we can construct a Puiseux monoid with exactly \( k \) Betti elements.

Sketch of proof. Consider the Puiseux monoid

\[
M := \left\langle \frac{1}{p_1}, \frac{2}{p_2}, \ldots \frac{k}{p_k}, \frac{1}{p_{k+1}}, \ldots, \frac{k}{p_{2k}}, \ldots \right\rangle,
\]

where \((p_n)_{n \geq 1}\) is an increasing sequence of primes with \( p_1 > k \).

From the previous result, we can conclude that \( \text{Betti}(M) = [1, k] \).
Suppose $M$ is an atomic Puiseux monoid that does not satisfy the ACCP; that is, there exists an infinite sequence of elements $a_1, a_2, a_3, \ldots$ of $M$ such that for all integers $i \geq 1$, there exists some nonzero $d_i \in M$ satisfying $a_i = a_{i+1} + d_i$. Must $M$ necessarily have infinitely many Betti elements?
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Thank you!