A weaker notion of the finite factorization property

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Outline

General Notation

General notation we will use throughout this talk:

- $\mathbb{N} := \{1, 2, 3, \ldots\}$,
- $\mathbb{N}_0 := \{0\} \cup \mathbb{N} = \{0, 1, 2, \ldots\}$,
- For $S \subseteq \mathbb{R}$ and $r \in \mathbb{R}$, we set $S_{\geq r} = \{s \in S \mid s \geq r\}$,
- $3 = p_1 < p_2 < \cdots$ is the sequence of odd primes.

Positive Monoids

A positive monoid M is a subset of $\mathbb{R}_{\geq 0}$ (nonnegative real numbers) under the addition operation that satisfies the following two conditions:

- If $a \in M$ and $b \in M$, then $a + b \in M$.
- An identity element (0) is an element of M.

Examples:

- The set of nonnegative integers, $\mathbb{N}_0$, is a positive monoid since it is closed under addition and contains 0.
- The set $\{0\} \cup \mathbb{Q}_{\geq 1}$ is a positive monoid since it is closed under addition and contains 0.

Generating Sets

For a positive monoid M, a submonoid of M is a subset of M that is closed under addition and contains 0.

Example:

- The set $\{0\} \cup \mathbb{Q}_{\geq 1}$ is a submonoid of $\mathbb{Q}_{\geq 0}$.

If S is a subset of M, then $\langle S \rangle$ stands for the smallest submonoid (under inclusion) of M that contains S. We say that S generates M.

Examples:

- The positive monoid $\mathbb{N}_0$ can be expressed as $\langle 1 \rangle$.
- The monoid $M = \left\langle \frac{1}{2^n p_n} \mid n \in \mathbb{N} \right\rangle$ is called the Grams' monoid.

Atomicity

Definitions: Let M be a positive monoid.

- $a \in M \setminus \{0\}$ is an atom (or an irreducible) if for any $b, c \in M$ the equality $a = b + c$ implies that either $b = 0$ or $c = 0$.
- We let $\mathcal{A}(M)$ denote the set of atoms of M.
- An element of M is atomic if it is 0 or can be expressed as a sum of atoms.

Definition

A monoid is called atomic if every element of M is atomic.

Atomicity

Example 1:

- Consider the monoid $M = \{0\} \cup \mathbb{Q}_{\geq 1}$.
- The set of atoms $\mathcal{A}(M) = \mathbb{Q} \cap [1, 2)$ since if $a + b \in \mathbb{Q} \cap [1, 2)$ for some $a, b \in M$ implies either $a < 1$ or $b < 1$ so one of them must be zero.
- Every $q \in M$ where $q \geq 2$ can be expressed as $q = (\lfloor q \rfloor - 1) + (q - \lfloor q \rfloor + 1)$.
- Hence, $M = \{0\} \cup \mathbb{Q}_{\geq 1}$ is atomic.

Example 2:

- Consider the Grams' monoid $M = \left\langle \frac{1}{2^n p_n} \mid n \in \mathbb{N} \right\rangle$.
- We see that $\mathcal{A}(M)$ is precisely the generating set $\{\frac{1}{2^n p_n} \mid n \in \mathbb{N}\}$ since each element of the set has a unique prime number in the denominator.
- All elements of M can be written as a sum of elements from the generating set, so M is atomic.

Finite Factorization Property

Let M be a positive monoid.

- All formal sums of atoms in M are called factorizations in M.
- For each $b \in M$, all formal sums of atoms that sum to b are called factorizations of b.

Definition (Anderson-Anderson-Zafrullah, 1990)

The monoid M is called an FFM if M is atomic and the number of factorizations of b is finite for every $b \in M$.

Examples:

- The monoid $\mathbb{N}_0$ is an FFM since every element has only one factorization (the trivial one).
- The monoid $M = \left\langle n + \frac{1}{p_n} \mid n \in \mathbb{N} \right\rangle$ is an FFM since for each $b \in M$, there are a finite number of atoms less than b.

Remark: The finite factorization property is considered one of the most important properties in factorization theory.

Bounded Factorization Property

- If $z := a_1 + \cdots + a_\ell$ is a factorization in M for some $a_1, \dots, a_\ell \in \mathcal{A}(M)$, then we call ℓ the length of z, and we often denote ℓ by $|z|$.
- For $b \in M$, the set of the lengths of all factorizations of b is called the set of lengths of b.

Definition (Anderson-Anderson-Zafrullah, 1990)

We say that M is a BFM if M is atomic and the set of lengths of b is finite for every $b \in M$.

Examples:

- The monoid $M = \{0\} \cup \mathbb{Q}_{\geq 1}$ is a BFM since all atoms are at least 1, so for all factorizations z of $b \in M$, the length $|z| \leq b$.
- The Grams' monoid $M = \left\langle \frac{1}{2^n p_n} \mid n \in \mathbb{N} \right\rangle$ is not a BFM since the length of factorizations of the element 1 is not bounded.

Length-Finite Factorization Property (LFFM)

Definition (Geroldinger-Zhong, 2021)

We say that M is a length-finite factorization monoid (LFFM) if M is atomic and for all $b \in M$ and $\ell \in \mathbb{N}$, the number of factorizations of b with length ℓ is finite.

Remarks:

- From the definitions, we obtain that every FFM is an LFFM.
- A monoid satisfies the FFM property if and only if it satisfies both the LFFM property and the BFM property.
- Investigating the LFFM property should lead to a better understanding of the finite factorization property.

Co-well-ordered Monoids

One of the focuses of our research was to identify a large class of LFFM monoids. To do this, we use the concept of a co-well-ordered sequence.

We say that a sequence of real numbers is co-well-ordered if it contains no strictly increasing subsequence.

Definition (Harold Polo, 2022)

We say that a positive monoid M is co-well-ordered if it can be generated by a co-well-ordered sequence.

Examples:

- The Grams' monoid $M = \left\langle \frac{1}{2^n p_n} \mid n \in \mathbb{N} \right\rangle$ is co-well-ordered since it is generated by a decreasing sequence, and therefore a co-well-ordered sequence.
- The monoid $M = \{0\} \cup \mathbb{Q}_{\geq 1}$ is not a co-well-ordered monoid since its set of atoms contain increasing subsequences such as $a_n = 2 - \frac{1}{n}$ for $n \in \mathbb{N}$.

A Class of LFFMs

Theorem (LFFM Theorem, Jiang-Kanungo-Kim, 2023)

Every atomic co-well-ordered positive monoid is an LFFM.

Remark: This theorem allows us to find a large class of LFFMs.

Examples:

- The Grams' monoid $M = \left\langle \frac{1}{2^n p_n} \mid n \in \mathbb{N} \right\rangle$ is co-well-ordered, and hence an LFFM.
- The monoid $M = \left\langle \frac{1}{p_n} \mid n \in \mathbb{N} \right\rangle$ is co-well-ordered since it is generated by a decreasing sequence, and thus an LFFM.

Comparing LFFM to BFM

Example of a BFM that is not an LFFM:

- We have already seen that the positive monoid $M := \{0\} \cup \mathbb{Q}_{\geq 1}$ is a BFM.
- The equality $3 = \left(\frac{3}{2} - \frac{1}{n}\right) + \left(\frac{3}{2} + \frac{1}{n}\right)$ yields a length-2 factorization of 3 for each $n \in \mathbb{N}$ with $n \geq 3$. Thus M is not an LFFM.

Example of an LFFM that is not a BFM:

- Consider the Grams' monoid: $M = \left\langle \frac{1}{2^n p_n} \;\middle|\; n \in \mathbb{N} \right\rangle$.
- We have already seen that the Grams' monoid is an LFFM but not a BFM.

Thus, we can conclude that BFM and LFFM properties are **not comparable**.

Positive Semirings

Let S be a subset of the nonnegative reals.

Definition

We say that S is a positive semiring if

- S contains 0 and 1, and
- S is closed under addition and multiplication.

Note that every positive semiring forms a positive monoid under addition. If the monoid formed by the nonzero elements of S under multiplication is a BFM, we say S is a BFS. Similarly define FFS and LFFS properties.

The $E(M)$ Construction

Let M be a monoid whose elements are algebraic numbers.

Definition

Let $E(M)$ be the positive monoid generated by $\{e^m \mid m \in M\}$.

- $E(M)$ contains 1 and is closed under multiplication, so $E(M)$ is a positive semiring.

A natural question to ask is:

Question

If M satisfies the BFM (resp. FFM, LFFM) property, is $E(M)$ a BFS (resp. FFS, LFFS)?

In our research, we have completely answered this question.

The BFS and FFS properties and the $E(M)$ Construction

For the BFS and FFS properties, the answer to the question posed in the previous slide is "yes". We have the following two propositions:

Proposition (Jiang-Kanungo-Kim, 2023)

If M is a positive monoid satisfying the BFM property, then $E(M)$ also satisfies the BFS property.

Proposition (Jiang-Kanungo-Kim, 2023)

If M is a positive monoid satisfying the FFM property, then $E(M)$ also satisfies the FFS property.

Unfortunately we do not have time to prove these propositions, but you can find a proof in our paper, "A Weaker Notion of the Finite Factorization Property" (`https://arxiv.org/abs/2307.09645`).

LFFMs and the $E(M)$ Construction

The FFM and BFM properties ascend to the $E(M)$ construction, but the LFFM property does not.

Proposition (Jiang-Kanungo-Kim, 2023)

There exists a monoid M of positive rationals such that M is an LFFM, but $E(M)$ is not an LFFS.

Some notable facts:

- Particularly, the monoid we found is

$$M = \left\langle \left\{ \frac{1}{p_i} \mid i \in \mathbb{N}, i \geq 3 \right\} \cup \left\{ \frac{7}{6p_i} - \frac{1}{p_i^2} \mid i \in \mathbb{N}, i \geq 3 \right\} \right\rangle .$$

  (recall $3 = p_1 < p_2 < \cdots$ are the odd primes).

- Notice that M is generated by the union of two decreasing sequences, so it is not hard to show M itself is decreasing.
- Then, we can conveniently directly see M is an LFFM by the LFFM theorem mentioned before.
- Proving that $E(M)$ is not an LFFS is technical and not insightful, so we will skip it.

When LFFM Ascends

However, we can impose some conditions on the monoid M so that $E(M)$ is indeed an LFFS when M is an LFFM.

Theorem (LFFM Ascent Theorem, Jiang-Kanungo-Kim, 2023)

Let M be an atomic positive LFFM of rational numbers. Assume that there exists an integer N such that, for all $n > N$, there is at most one atom of M with denominator divisible by n in simplest form. Then $E(M)$ is an LFFS.

We will give some examples of monoids satisfying the condition:

- $M = \left\langle \frac{1}{p_i} \mid i \in \mathbb{N} \right\rangle$ satisfies this condition.
- $M = \left\langle \frac{3^i}{2p_i} \mid i \geq 2 \right\rangle$ satisfies the condition.

When LFFM Ascends, Cont.

We can also give examples of monoids not satisfying the condition:

- $M = \left\langle \frac{1}{p_i p_{i+1}} \mid i \in \mathbb{N} \right\rangle$ does not satisfy this condition as every prime divides two atoms.
- Grams' monoid

  $$M := \left\langle \frac{1}{2^n p_n} \mid n \in \mathbb{N} \right\rangle$$

  does not satisfy the condition, since all powers of two divide infinitely many denominators of atoms in simplest form.

However, Grams' monoid is a useful example, so we will also prove:

Proposition (Jiang-Kanungo-Kim, 2023)

If M is Grams' monoid, defined as

$$M := \left\langle \frac{1}{2^n p_n} \mid n \in \mathbb{N} \right\rangle,$$

then $E(M)$ is an LFFS.

We will skip the proofs, as they are technical.

Corollaries of Propositions

Because of the above results, we are able to extend the examples given for monoids to show that LFFS is independent from BFS for semirings, as well.

Example of an LFFS that is not a BFS:

- Consider the Grams' monoid: $M := \left\langle \frac{1}{2^n p_n} \;\middle|\; n \in \mathbb{N} \right\rangle$.
- By our previous proposition, $E(M)$ is an LFFS.
- However, $E(M)$ is not a BFS since there are arbitrarily long factorizations of e (since there are arbitrarily long factorizations of 1 in M).

Example of a BFS that is not an LFFS:

- Consider the positive monoid $M := \{0\} \cup \mathbb{Q}_{\geq 1}$.
- We have previously seen that M is a BFM with $\mathcal{A}(M) = \mathbb{Q} \cap [1, 2)$ and also that M is not an LFFM.
- By our previous proposition that the BFM property ascends to $E(M)$, we know that $E(M)$ is a BFS.
- However, we can now factorize the element e^3 of $E(M)$ as $e^{\frac{3}{2}-\frac{1}{n}} \cdot e^{\frac{3}{2}+\frac{1}{n}}$ for any positive integer $n > 2$, so that element has infinitely many length-2 factorizations.

The Monoid Algebra

We define a structure related to the $E(M)$ construction, the monoid algebra.

Definition

For a positive monoid M, we call $\mathbb{Q}[M]$ to be the set of sums of elements of

$$\{qx^m \mid q \in \mathbb{Q}, m \in M\},$$

called a monoid algebra of M over $\mathbb{Q}$.

Note that we use a general variable x instead of e, so M does not necessarily consist of rationals.

Intuitively, this is like the $E(M)$ construction, but the coefficients are rational numbers instead of the nonnegative integers.

We can see that $\mathbb{Q}[M]$ is a semiring (in fact, it is a ring, which just means that the negative of every element in $\mathbb{Q}[M]$ is also in $\mathbb{Q}[M]$).

Problems Related to the Monoid Algebra

We have already proved that

Proposition (Jiang-Kanungo-Kim, 2023)

There is a positive LFFM of rationals M where $\mathbb{Q}[M]$ is not an LFFS.

It turns out we can just use the same monoid for which $E(M)$ is not an LFFS, and the proof is similar.

An open problem we will work on is:

Question

Does the FFM property ascend from M to $\mathbb{Q}[M]$?

The main added complication that some coefficients may be negative.

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Thank you!