A weaker notion of the finite factorization property

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Outline

1 Preliminaries and Motivation

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1. Preliminaries and Motivation

2. Positive Monoids That Are Length-Finite Factoriation Monoids
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2 Positive Monoids That Are Length-Finite Factoriation Monoids

3 Semirings and an Exponentiation Construction
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1. Preliminaries and Motivation
2. Positive Monoids That Are Length-Finite Factoriation Monoids
3. Semirings and an Exponentiation Construction
4. The LFFM Property and the Exponentiation Construction
General Notation

General notation we will use throughout this talk:

\[ N := \{1, 2, 3, \ldots\} \]
\[ N_0 := \{0\} \cup N = \{0, 1, 2, \ldots\} \]

For \( S \subseteq \mathbb{R} \) and \( r \in \mathbb{R} \), we set
\[ S \geq r := \{s \in S | s \geq r\} \]
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- For $S \subseteq \mathbb{R}$ and $r \in \mathbb{R}$, we set $S \geq r = \{s \in S \mid s \geq r\}$,
- $3 = p_1 < p_2 < \cdots$ is the sequence of odd primes.
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Examples:
The set of nonnegative integers, $\mathbb{N}_0$, is a positive monoid since it is closed under addition and contains 0.
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For convenience, we may call positive monoids as simply monoids in this presentation.
Generating Sets

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**Examples:**

- The positive monoid $\mathbb{N}_0$ can be expressed as $\langle 1 \rangle$.
- The monoid $M = \left\langle \frac{1}{2^n p_n} \mid n \in \mathbb{N} \right\rangle$ is called the Grams’ monoid.
Atomicity

**Definitions:** Let $M$ be a positive monoid.
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Definition

A monoid is called atomic if every element of $M$ is atomic.
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Example 1:

Consider the monoid $M = \{0\} \cup \mathbb{Q}_{\geq 1}$. The set of atoms $A(M) = \mathbb{Q} \cap [1, 2)$ since if $a + b \in \mathbb{Q} \cap [1, 2)$ for some $a, b \in M$ implies either $a < 1$ or $b < 1$ so one of them must be zero. Every $q \in M$ where $q \geq 2$ can be expressed as $q = 1 + (q - 1)$ so it is atomic. Hence, $M = \{0\} \cup \mathbb{Q}_{\geq 1}$ is atomic.

Example 2:

Consider the Grams’ monoid $M = D_1^2_n p_n | n \in \mathbb{N}$. We see that $A(M)$ is precisely the generating set $\{1^2_n p_n | n \in \mathbb{N}\}$ since each element of the set has a unique prime number in the denominator. All elements of $M$ can be written as a sum of elements from the generating set, so $M$ is atomic.
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Finite Factorization Property

Let $M$ be a positive monoid.
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- All formal sums of atoms in $M$ are called factorizations in $M$. 

Definition (Anderson-Anderson-Zafrullah, 1990)

The monoid $M$ is called an FFM if $M$ is atomic and the number of factorizations of $b$ is finite for every $b \in M$.

Examples:

The monoid $\mathbb{N}_0$ is an FFM since every element has only one factorization (the trivial one).

The monoid $M = D^n + 1_{p^n} | n \in \mathbb{N}$ is an FFM since for each $b \in M$, there are a finite number of atoms less than $b$.

Remark:

The finite factorization property is considered one of the most important properties in factorization theory.
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- The monoid $\mathbb{N}_0$ is an FFM since every element has only one factorization (the trivial one).
- The monoid $M = \left\langle n + \frac{1}{p^n} \mid n \in \mathbb{N} \right\rangle$ is an FFM since for each $b \in M$, there are a finite number of atoms less than $b$. 

Remark: The finite factorization property is considered one of the most important properties in factorization theory.
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Positive Monoids That Are Length-Finite Factorization Monoids

Bounded Factorization Property

- If $z := a_1 + \cdots + a_\ell$ is a factorization in $M$ for some $a_1, \ldots, a_\ell \in A(M)$, then we call $\ell$ the length of $z$, and we often denote $\ell$ by $|z|$. 

Definition (Anderson-Anderson-Zafrullah, 1990)

We say that $M$ is a BFM if $M$ is atomic and the set of lengths of $b \in M$ is finite for every $b \in M$. 

Examples: 

The monoid $M = \{0\} \cup \mathbb{Q}_{\geq 1}$ is a BFM since all atoms are at least 1, so for all factorizations $z$ of $b \in M$, the length $|z| \leq b$. 

The Grams' monoid $M = D_{12n^p}$ is not a BFM since the length of factorizations of the element 1 is not bounded.
Bounded Factorization Property

- If \( z := a_1 + \cdots + a_\ell \) is a factorization in \( M \) for some \( a_1, \ldots, a_\ell \in A(M) \), then we call \( \ell \) the length of \( z \), and we often denote \( \ell \) by \( |z| \).

- For \( b \in M \), the set of the lengths of all factorizations of \( b \) is called the set of lengths of \( b \).
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We say that \( M \) is a **BFM** if \( M \) is atomic and the set of lengths of \( b \) is finite for every \( b \in M \).
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- The Grams’ monoid \( M = \left\langle \frac{1}{2^n p_n} \mid n \in \mathbb{N} \right\rangle \) is not a BFM since the length of factorizations of the element 1 is not bounded.
We say that $M$ is a length-finite factorization monoid (LFFM) if $M$ is atomic and for all $b \in M$ and $\ell \in \mathbb{N}$, the number of factorizations of $b$ with length $\ell$ is finite.
Positive Monoids That Are Length-Finite Factorization Monoids

Length-Finite Factorization Property (LFFM)

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**Remarks:**

- From the definitions, we obtain that every FFM is an LFFM.
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- From the definitions, we obtain that every FFM is an LFFM.
- A monoid satisfies the FF property if and only if it satisfies both the LFF property and the BF property.
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- Investigating the LFF property should lead to a better understanding of the finite factorization property.
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Co-well-ordered Monoids

We say that a sequence of real numbers is co-well-ordered if it contains no strictly increasing subsequence.
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**Definition (Harold Polo, 2022)**

We say that a positive monoid $M$ is co-well-ordered if it can be generated by a co-well-ordered sequence.
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Examples:

- The Grams’ monoid $M = \langle \frac{1}{2^n p_n} \mid n \in \mathbb{N} \rangle$ is co-well-ordered since it is generated by a decreasing sequence, and therefore a co-well-ordered sequence.
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- The monoid \( M = \{0\} \cup \mathbb{Q}_{\geq 1} \) is not a co-well-ordered monoid since its set of atoms contain increasing subsequences such as \( a_n = 2 - \frac{1}{n} \) for \( n \in \mathbb{N} \).
A Class of LFFMs

Theorem (LFFM Theorem, Jiang-Kanungo-Kim, 2023)

Every atomic co-well-ordered positive monoid is an LFFM.
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*Every atomic co-well-ordered positive monoid is an LFFM.*

**Remark:** This theorem allows us to find a large class of LFFMs.
A Class of LFFMs

**Theorem (LFFM Theorem, Jiang-Kanungo-Kim, 2023)**

Every atomic co-well-ordered positive monoid is an LFFM.

**Remark:** This theorem allows us to find a large class of LFFMs.

**Examples:**

- The monoid $M = D_{p_n^{|n \in \mathbb{N}}}$ is co-well-ordered since it is generated by a decreasing sequence, and thus an LFFM.
A Class of LFFMs

Theorem (LFFM Theorem, Jiang-Kanungo-Kim, 2023)

Every atomic co-well-ordered positive monoid is an LFFM.

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- The Grams’ monoid \( M = \langle \frac{1}{2^n p_n} \mid n \in \mathbb{N} \rangle \) is co-well-ordered, and hence an LFFM.
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Comparing LFFM to BFM

Example of a BFM that is not an LFFM:

We have already seen that the positive monoid
\[ M := \{0\} \cup \mathbb{Q}_{\geq 1} \]
is a BFM. The equality
\[ 3 = 3^2 - 1^n + 3^2 + 1^n \]
yields a length-2 factorization of 3 for each
\( n \in \mathbb{N} \) with \( n \geq 3 \). Thus
\( M \) is not an LFFM.

Example of an LFFM that is not a BFM:
Consider the Grams' monoid:
\[ M = D_{1^2 n} E. \]
We have already seen that the Grams' monoid is an LFFM but not a BFM.
Thus, we can conclude that BFM and LFFM properties are
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- Consider the Grams’ monoid: $M = \left\langle \frac{1}{2^n p_n} \mid n \in \mathbb{N} \right\rangle$.
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- We have already seen that the positive monoid $M := \{0\} \cup \mathbb{Q}_{\geq 1}$ is a BFM.
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Example of an LFFM that is not a BFM:

- Consider the Grams’ monoid: $M = \left\langle \frac{1}{2^n p_n} \middle| n \in \mathbb{N} \right\rangle$.
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Thus, we can conclude that BFM and LFFM properties are not comparable.
Positive Semirings

Let $S$ be a subset of the nonnegative reals.
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**Definition**

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Note that every positive semiring forms a positive monoid under addition.
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- $S$ contains 0 and 1, and
- $S$ is closed under addition and multiplication.

Note that every positive semiring forms a positive monoid under addition. If the monoid formed by the nonzero elements of $S$ under multiplication is a BFM, we say $S$ is a BFS. Similarly define FFS and LFFS properties.
The $E(M)$ Construction

Let $M$ be a monoid whose elements are algebraic numbers.
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The $E(M)$ Construction

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Let $E(M)$ be the positive monoid generated by $\{e^m \mid m \in M\}$.

- $E(M)$ contains 1 and is closed under multiplication, so $E(M)$ is a positive semiring.
The $E(M)$ Construction

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Let $E(M)$ be the positive monoid generated by $\{e^m \mid m \in M\}$.

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A natural question to ask is:

If $M$ satisfies the BFM (resp. FFM, LFFM) property, is $E(M)$ a BFS (resp. FFS, LFFS)?

In our research, we have completely answered this question.
Let $M$ be a monoid whose elements are algebraic numbers.

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In our research, we have completely answered this question.
For the BFS and FFS properties, the answer to the question posed in the previous slide is “yes”. We have the following two propositions:

Proposition (Jiang-Kanungo-Kim, 2023)
If $M$ is a positive monoid satisfying the BFM property, then $E(M)$ also satisfies the BFS property.

Proposition (Jiang-Kanungo-Kim, 2023)
If $M$ is a positive monoid satisfying the FFM property, then $E(M)$ also satisfies the FFS property.

Unfortunately we do not have time to prove these propositions, but you can find a proof in our paper, “A Weaker Notion of the Finite Factorization Property” (https://arxiv.org/abs/2307.09645).
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LFFMs and the $E(M)$ Construction

The FFM and BFM properties ascend to the $E(M)$ construction,
The LFFM Property and the Exponentiation Construction

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The FFM and BFM properties ascend to the $E(M)$ construction, but the LFFM property does not.

Proposition (Jiang-Kanungo-Kim, 2023)

There exists a monoid $M$ of positive rationals such that $M$ is an LFFM, but $E(M)$ is not an LFFS.

Some notable facts:

Particularly, the monoid we found is $M = \{\prod_{i \in N, i \geq 3} p_i \cup \prod_{i \in N, i \geq 3} p_i - 1 p_i^2 \}$.

(recall $3 \leq p_1 < p_2 < \cdots$ are the odd primes).

Notice that $M$ is generated by the union of two decreasing sequences, so it is not hard to show $M$ itself is decreasing.

Then, we can conveniently directly see $M$ is an LFFM by the LFFM theorem mentioned before.

Proving that $E(M)$ is not an LFFS is technical and not insightful, so we will skip it.
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\[ M = \left\langle \left\{ \frac{1}{p_i} \mid i \in \mathbb{N}, i \geq 3 \right\} \cup \left\{ \frac{7}{6p_i} - \frac{1}{p_i^2} \mid i \in \mathbb{N}, i \geq 3 \right\} \right\rangle. \]

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When LFFM Ascends

However, we can impose some conditions on the monoid $M$ so that $E(M)$ is indeed an LFFS when $M$ is an LFFM.

Theorem (LFFM Ascent Theorem, Jiang-Kanungo-Kim, 2023)

Let $M$ be an atomic positive LFFM of rational numbers. Assume that there exists an integer $N$ such that, for all $n > N$, there is at most one atom of $M$ with denominator divisible by $n$ in simplest form. Then $E(M)$ is an LFFS.

We will give some examples of monoids satisfying the condition:

$M = D_1 p_i | i \geq 1$ satisfies this condition.

$M = D_1 p_i p_{i+1} | i \geq 1$ satisfies this condition.

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- $M = \left\langle \frac{1}{p_i p_{i+1}} \mid i \geq 1 \right\rangle$ satisfies this condition.
- $M = \left\langle \frac{3^i}{2p_i} \mid i \geq 2 \right\rangle$ satisfies the condition.
We can also give examples of monoids not satisfying the condition:

\[ M = \{ 0 \} \cup \mathbb{Q}_{\geq 1} \] does not satisfy the condition; the atoms are all rationals in \([1, 2)\) so every positive integer divides infinitely many denominators of atoms in simplest form.

Grams' monoid \[ M := \mathbb{N} \] does not satisfy the condition, since all powers of two divide infinitely many denominators of atoms in simplest form.

However, Grams' monoid is a useful example, so we will also prove:

**Proposition (Jiang-Kanungo-Kim, 2023)**

If \( M \) is Grams' monoid, defined as \[ M := \mathbb{N} \], then \( \mathcal{E}(M) \) is an LFFS.

We will skip the proofs, as they are technical.
When LFFM Ascends, Cont.

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- Grams’ monoid \( M := \{1\} \cup \mathbb{Z} \forall n \in \mathbb{N} \) does not satisfy the condition, since all powers of two divide infinitely many denominators of atoms in simplest form.

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\[ M := \left\langle \frac{1}{2^n p_n} \mid n \in \mathbb{N} \right\rangle \]

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When LFFM Ascends, Cont.

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$$M := \left\langle \frac{1}{2^n p_n} \mid n \in \mathbb{N} \right\rangle,$$

then $E(M)$ is an LFFS.*

We will skip the proofs, as they are technical.
Corollaries of Propositions

Because of the above results, we are able to extend the examples given for monoids to show that LFFS is independent from BFS and for semirings, as well.
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Example of an LFFS that is not a BFS:
Corollaries of Propositions

Because of the above results, we are able to extend the examples given for monoids to show that LFFS is independent from BFS and for semirings, as well.

Example of an LFFS that is not a BFS:

- Consider the Grams’ monoid: \( M := \langle \frac{1}{2^n p_n} \mid n \in \mathbb{N} \rangle \).
Corollaries of Propositions

Because of the above results, we are able to extend the examples given for monoids to show that LFFS is independent from BFS and for semirings, as well.

Example of an LFFS that is not a BFS:

- Consider the Grams’ monoid: \( \langle \frac{1}{2^n p_n} \mid n \in \mathbb{N} \rangle \).
- By our previous proposition, \( E(M) \) is an LFFS.
Corollaries of Propositions

Because of the above results, we are able to extend the examples given for monoids to show that LFFS is independent from BFS and for semirings, as well.

Example of an LFFS that is not a BFS:

- Consider the Grams’ monoid: \( M := \left\langle \frac{1}{2^n p_n} \mid n \in \mathbb{N} \right\rangle. \)

- By our previous proposition, \( E(M) \) is an LFFS.

- However, \( E(M) \) is not a BFS since there are arbitrarily long factorizations of 1 (since there are arbitrarily long factorizations of 1 in \( M \)).
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Example of a BFS that is not an LFFS:
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**Example of a BFS that is not an LFFS:**

- Consider the positive monoid \( M := \{0\} \cup \mathbb{Q}_{\geq 1}. \)
Corollaries of Propositions

Because of the above results, we are able to extend the examples given for monoids to show that LFFS is independent from BFS and for semirings, as well.

**Example of an LFFS that is not a BFS:**

- Consider the Grams’ monoid: \( M := \langle \frac{1}{2^n p_n} \mid n \in \mathbb{N} \rangle. \)
- By our previous proposition, \( E(M) \) is an LFFS.
- However, \( E(M) \) is not a BFS since there are arbitrarily long factorizations of \( e \) (since there are arbitrarily long factorizations of 1 in \( M \)).

**Example of a BFS that is not an LFFS:**

- Consider the positive monoid \( M := \{0\} \cup \mathbb{Q}_{\geq 1}. \)
- We have previously seen that \( M \) is a BFM with \( A(M) = \mathbb{Q} \cap [1, 2) \) and also that \( M \) is not an LFFM.
Corollaries of Propositions

Because of the above results, we are able to extend the examples given for monoids to show that LFFS is independent from BFS and for semirings, as well.

Example of an LFFS that is not a BFS:

- Consider the Grams’ monoid: \( M := \langle 1 \cdot 2^n p_n \mid n \in \mathbb{N} \rangle. \)
- By our previous proposition, \( E(M) \) is an LFFS.
- However, \( E(M) \) is not a BFS since there are arbitrarily long factorizations of \( e \) (since there are arbitrarily long factorizations of 1 in \( M \)).

Example of a BFS that is not an LFFS:

- Consider the positive monoid \( M := \{0\} \cup \mathbb{Q}_{\geq 1}. \)
- We have previously seen that \( M \) is a BFM with \( A(M) = \mathbb{Q} \cap [1, 2) \) and also that \( M \) is not an LFFM.
- By our previous proposition that the BFM property ascends to \( E(M) \), we know that \( E(M) \) is a BFS.
Corollaries of Propositions

Because of the above results, we are able to extend the examples given for monoids to show that LFFS is independent from BFS and for semirings, as well.

Example of an LFFS that is not a BFS:

- Consider the Grams’ monoid: \( M := \langle \frac{1}{2^n p_n} \mid n \in \mathbb{N} \rangle. \)
- By our previous proposition, \( E(M) \) is an LFFS.
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- By our previous proposition that the BFM property ascends to \( E(M) \), we know that \( E(M) \) is a BFS.
- However, we can now factorize the element \( e^{\frac{3}{2}} \) of \( E(M) \) as \( e^{\frac{3}{2}} - \frac{1}{p} \cdot e^{\frac{3}{2}} + \frac{1}{p} \) for any odd prime \( p \), so that element has infinitely many length-2 factorizations.
The Monoid Algebra

We define a structure related to the $E(M)$ construction, the monoid algebra.

**Definition**

For a positive monoid $M$, we call $\mathbb{Q}[M]$ to be the set of sums of elements of

$$\{qx^m \mid q \in \mathbb{Q}, m \in M\},$$

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We can see that $\mathbb{Q}[M]$ is a semiring (in fact, it is a ring, which just means that the negative of every element in $\mathbb{Q}[M]$ is also in $\mathbb{Q}[M]$).
We have already proved that

**Proposition (Jiang-Kanungo-Kim, 2023)**

There is a positive Puiseux LFFM $M$ where $\mathbb{Q}[M]$ is not an LFFS.
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\textbf{Question}

\textit{Does the FFM property ascend from }$M$\textit{ to }$\mathbb{Q}[M]$\textit{?}
Problems Related to the Monoid Algebra

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The main added complication that some coefficients may be negative.


Thank you!