Classifying Isometries in Taxicab Geometry

Lillian MacArthur
Mentor: Honglin Zhu
PRIMES Circle

What is an Isometry?

What is an Isometry?

An isometry is a transformation of the plane (or space) such that distance is preserved.

What is an Isometry?

An isometry is a transformation of the plane (or space) such that distance is preserved.

But what is "distance"?

Metric Spaces

Metric Spaces

Definition

A metric space is a set equipped with a distance function such that for any three points X, Y, and Z in the set:

Metric Spaces

Definition

A metric space is a set equipped with a distance function such that for any three points X, Y, and Z in the set:

- $d(X, X) = 0$;
- $d(X, Y) \geq 0$;
- $d(X, Y) = d(Y, X)$;

Metric Spaces

Definition

A metric space is a set equipped with a distance function such that for any three points X, Y, and Z in the set:

- $d(X, X) = 0$;
- $d(X, Y) \geq 0$;
- $d(X, Y) = d(Y, X)$;
- $d(X, Y) + d(Y, Z) \geq d(X, Z)$.

Taxicab Geometry

Taxicab Geometry

Taxicab geometry is the plane equipped with an unusual distance function:

$$d_T = |x_2 - x_1| + |y_2 - y_1|.$$

Taxicab Geometry

Taxicab geometry is the plane equipped with an unusual distance function:

$$d_T = |x_2 - x_1| + |y_2 - y_1|.$$

Compare this to our usual Euclidean distance:

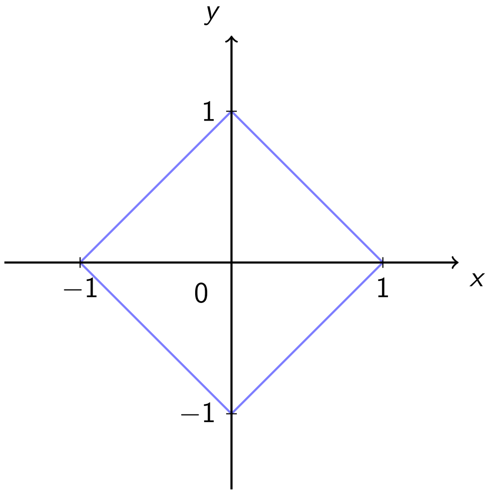
$$d_E = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Features of Taxicab Geometry

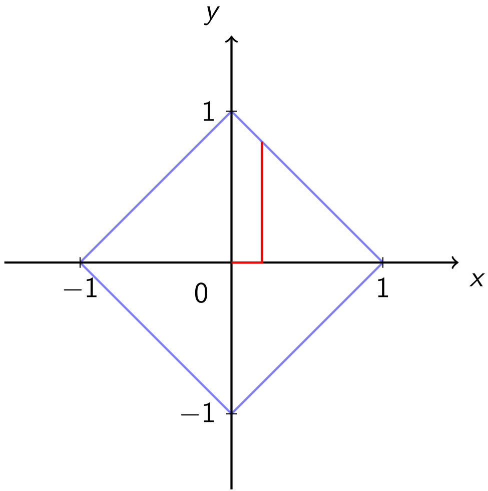
In Taxicab Geometry, circles are shaped like diamonds:

Features of Taxicab Geometry

In Taxicab Geometry, circles are shaped like diamonds:



In Taxicab Geometry, circles are shaped like diamonds:

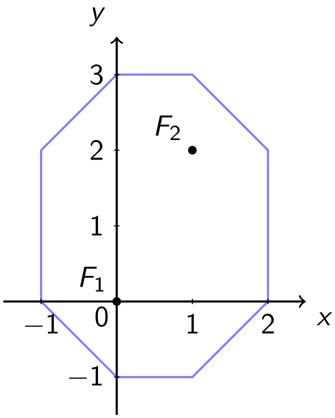


Features of Taxicab Geometry

Other shapes are similarly weirdly shaped:

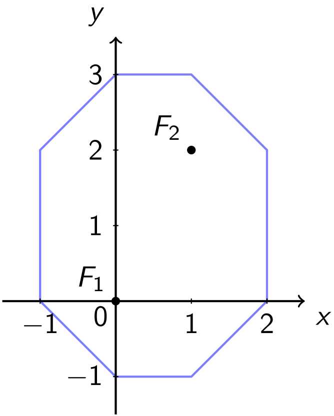
Features of Taxicab Geometry

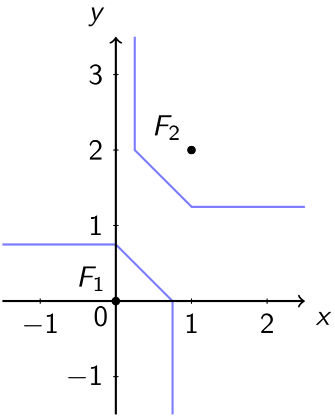
Other shapes are similarly weirdly shaped:



Features of Taxicab Geometry

Other shapes are similarly weirdly shaped:





Isometries in Taxicab Geometry

Isometries in Taxicab Geometry

Our Problem

What isometries exist in taxicab geometry, and how can we completely classify them?

Isometries in Taxicab Geometry

Our Problem

What isometries exist in taxicab geometry, and how can we completely classify them?

Example

A translation is always an isometry in taxicab geometry.

Isometries in Taxicab Geometry

Our Problem

What isometries exist in taxicab geometry, and how can we completely classify them?

Example

A translation is always an isometry in taxicab geometry.

Translations help us by letting us simplify our problem to to classifying the isometries fixing the origin.

Translations

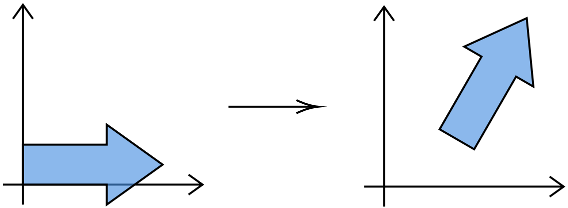
Lemma

Any taxicab isometry is the composition of a taxicab isometry fixing the origin and a translation.

Translations

Lemma

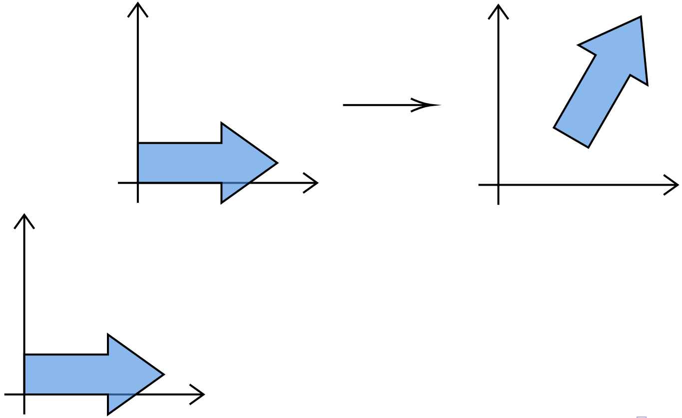
Any taxicab isometry is the composition of a taxicab isometry fixing the origin and a translation.



Translations

Lemma

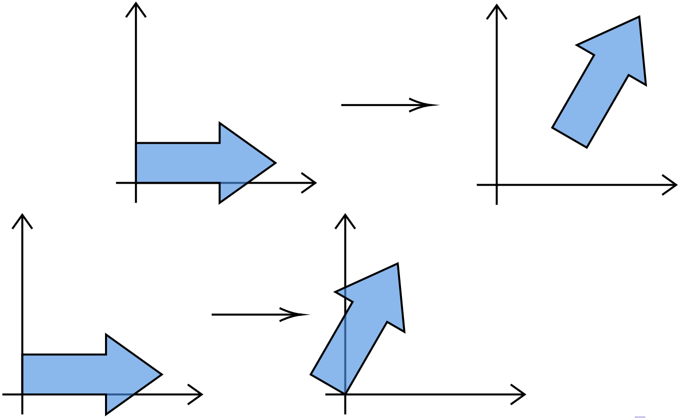
Any taxicab isometry is the composition of a taxicab isometry fixing the origin and a translation.



Translations

Lemma

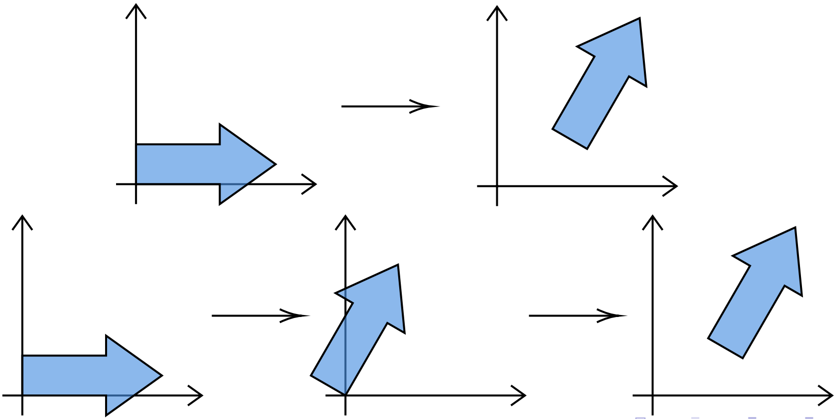
Any taxicab isometry is the composition of a taxicab isometry fixing the origin and a translation.



Translations

Lemma

Any taxicab isometry is the composition of a taxicab isometry fixing the origin and a translation.



Isometries in Taxicab Geometry

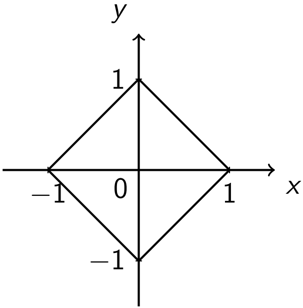
Our lemma implies that it suffices to classify isometries fixing the origin.

Isometries in Taxicab Geometry

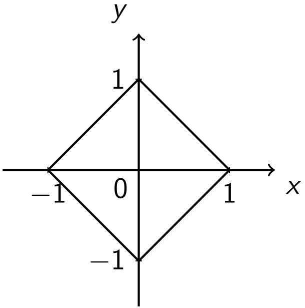
Our lemma implies that it suffices to classify isometries fixing the origin. Recall the shape of the unit circle:

Isometries in Taxicab Geometry

Our lemma implies that it suffices to classify isometries fixing the origin. Recall the shape of the unit circle:



Our lemma implies that it suffices to classify isometries fixing the origin. Recall the shape of the unit circle:

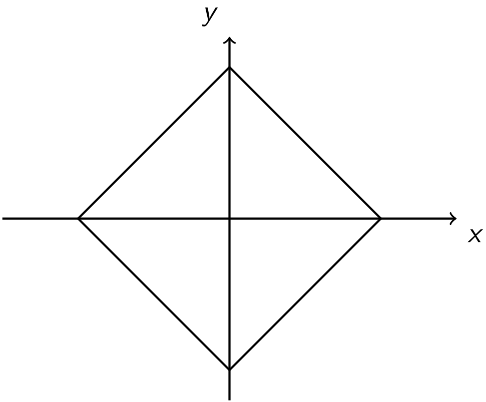


If the origin is fixed, the unit circle must be mapped back to itself.

Isometries in Taxicab Geometry

Theorem

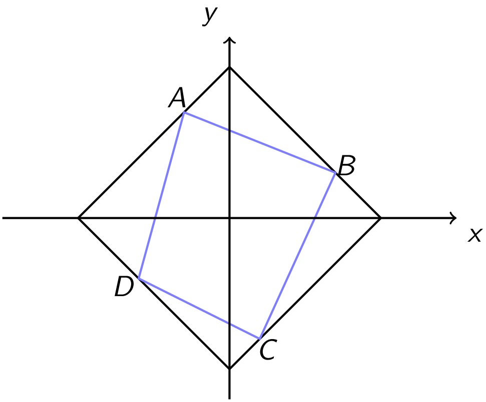
A taxicab isometry fixing the origin must permute the four corners of the unit circle: $(1,0),(0,1),(-1,0),(0,-1)$.



Isometries in Taxicab Geometry

Theorem

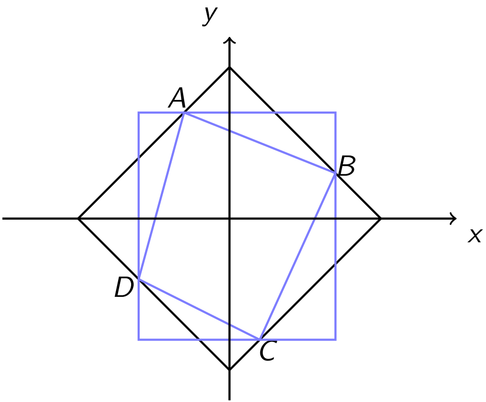
A taxicab isometry fixing the origin must permute the four corners of the unit circle: $(1,0),(0,1),(-1,0),(0,-1)$.



Isometries in Taxicab Geometry

Theorem

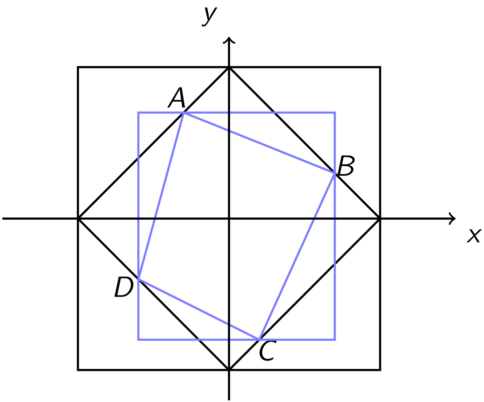
A taxicab isometry fixing the origin must permute the four corners of the unit circle: $(1,0),(0,1),(-1,0),(0,-1)$.



Isometries in Taxicab Geometry

Theorem

A taxicab isometry fixing the origin must permute the four corners of the unit circle: $(1,0),(0,1),(-1,0),(0,-1)$.



Every Isometry in Taxicab Geometry

Every Isometry in Taxicab Geometry

With the knowledge that the corners must be mapped back onto themselves, we have a full list of isometries:

Every Isometry in Taxicab Geometry

With the knowledge that the corners must be mapped back onto themselves, we have a full list of isometries:

- The identity (doing nothing).

Every Isometry in Taxicab Geometry

With the knowledge that the corners must be mapped back onto themselves, we have a full list of isometries:

- The identity (doing nothing).
- A reflection across the x-axis.

Every Isometry in Taxicab Geometry

With the knowledge that the corners must be mapped back onto themselves, we have a full list of isometries:

- The identity (doing nothing).
- A reflection across the x-axis.
- A reflection across the y-axis.

Every Isometry in Taxicab Geometry

With the knowledge that the corners must be mapped back onto themselves, we have a full list of isometries:

- The identity (doing nothing).
- A reflection across the x-axis.
- A reflection across the y-axis.
- A reflection across the line $y = x$.

Every Isometry in Taxicab Geometry

With the knowledge that the corners must be mapped back onto themselves, we have a full list of isometries:

- The identity (doing nothing).
- A reflection across the x-axis.
- A reflection across the y-axis.
- A reflection across the line $y = x$.
- A reflection across the line $y = -x$.

Every Isometry in Taxicab Geometry

With the knowledge that the corners must be mapped back onto themselves, we have a full list of isometries:

- The identity (doing nothing).
- A reflection across the x-axis.
- A reflection across the y-axis.
- A reflection across the line $y = x$.
- A reflection across the line $y = -x$.
- A rotation of $\frac{\pi}{2}$ radians (90 degrees).

Every Isometry in Taxicab Geometry

With the knowledge that the corners must be mapped back onto themselves, we have a full list of isometries:

- The identity (doing nothing).
- A reflection across the x-axis.
- A reflection across the y-axis.
- A reflection across the line $y = x$.
- A reflection across the line $y = -x$.
- A rotation of $\frac{\pi}{2}$ radians (90 degrees).
- A rotation of π radians (180 degrees).

Every Isometry in Taxicab Geometry

With the knowledge that the corners must be mapped back onto themselves, we have a full list of isometries:

- The identity (doing nothing).
- A reflection across the x-axis.
- A reflection across the y-axis.
- A reflection across the line $y = x$.
- A reflection across the line $y = -x$.
- A rotation of $\frac{\pi}{2}$ radians (90 degrees).
- A rotation of π radians (180 degrees).
- A rotation of $\frac{3\pi}{2}$ radians (270 degrees).

Every Isometry in Taxicab Geometry

With the knowledge that the corners must be mapped back onto themselves, we have a full list of isometries:

- The identity (doing nothing).
- A reflection across the x-axis.
- A reflection across the y-axis.
- A reflection across the line $y = x$.
- A reflection across the line $y = -x$.
- A rotation of $\frac{\pi}{2}$ radians (90 degrees).
- A rotation of π radians (180 degrees).
- A rotation of $\frac{3\pi}{2}$ radians (270 degrees).

All of which can be done alone or composed with a translation.

Any questions?