Knot Theory

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MIT PRIMES Reading Group

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Definition (Knot)

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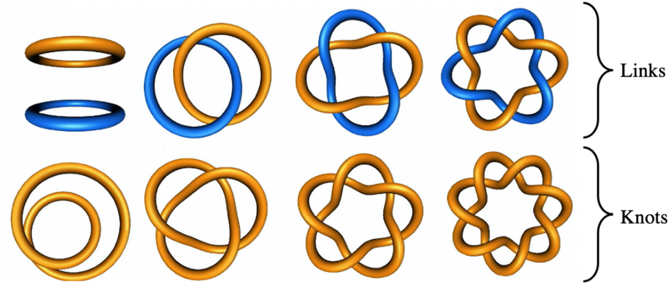
Definition (Link)

Informally, a **link** is a set of knots that are knotted together. The number of knots is called the number of **components**.



Essential Questions in Knot Theory

In this presesntation, we address two important questions in Knot Theory:

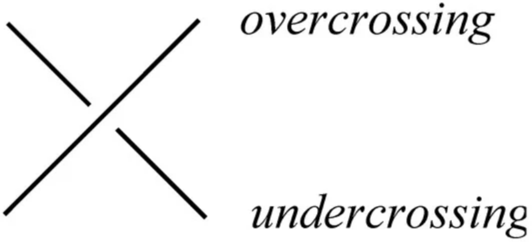
- How can we distinguish knots?
- How can we tell if two knots are the same?

Representing Knots

We use a knot's projection in the 2D plane to represent it. We can use arrows to denote a knot's direction that one is traveling in while they trace the knot.

Definition (Crossing)

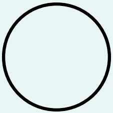
A **crossing** is a point in a knot's projection where it crosses itself.



Important Knots

Example (Unknot)

The **unknot** or **trivial knot** is a knot that has a projection with 0 crossings.



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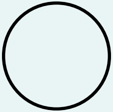

Example (Trefoil Knot)

The **trefoil** knot is represented as below. It is a nontrivial knot with the least number of crossings.

Knot Composition

Definition (Composition)

We define **knot composition** as a process to combine two knots into one according to the following steps:

1. Remove an arc from the outside of each knot projection
2. Connect the four endpoints with two new arcs. If this introduces a new crossing, select a different arc to remove in step 1.

For two knots J and K, we denote their composition as $J\#K$.

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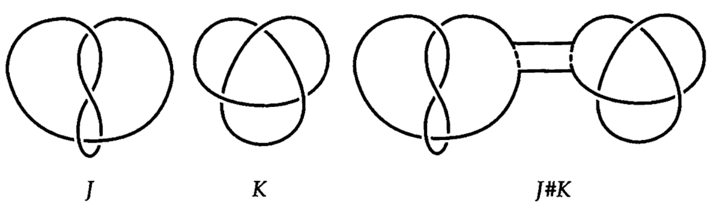
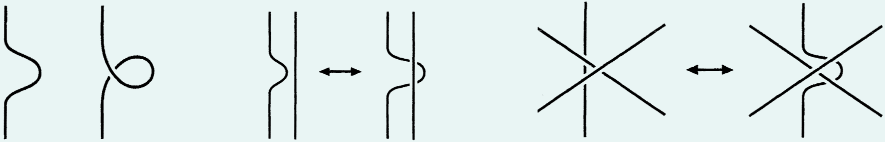


Reidemeister Moves

Definition (Reidemeister Moves)

A **Reidemeister move** is one of three moves that change the projection of a knot while preserving the knot.
Kurt Reidemeister showed that a series of Reidemeister moves and planar isotopies can relate any two projections of the same knot.

Example



Type I Type II Type III

Invariants

Definition (Knot Invariant)

A **knot invariant** is a quantity or a property of a knot that remains unchanged by ambient isotopy, i.e. by Reidemeister moves or planar isotopies.

Tricolorability

Definition (Strand)

For any projection of a link, we define a **strand** to be a segment of the link projection that connects two undercrossings with only overcrossings in between.

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Definition (Tricolorability)

A link is **tricolorable** if we can color each of its strands one of 3 different colors according to the following two rules:

1. At every crossing, either all the strands are the same color or they are all different colors
2. At least 2 of the 3 colors are used

Tricolorability Examples

Example (The Unknot)

The unknot only has one strand, so at most we can use one color to color it. This fails the condition that we use at least 2 out of the 3 colors, so the unknot is not tricolorable.

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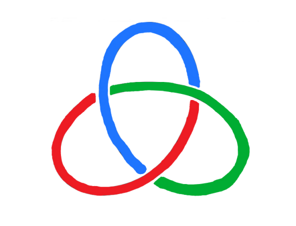
Example (Trefoil Knot)

The trefoil knot is tricolorable, as shown below.



Tricolorability As An Invariant

Theorem

Reidemeister moves preserve tricolorability.

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Proof.

Since Reidemeister moves preserve tricolorability, there is no sequence of Reidemeister moves that will change a projection of a tricolorable knot to a knot that is not tricolorable. It follows that the unknot and trefoil knot are distinct up to Reidemeister moves, as desired. □

Linking Number

Definition (Linking number)

The **linking number** is a measure of how intertwined two knots in a link are.

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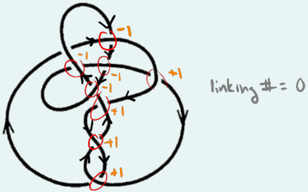
Example (6_3 knot)



Linking Number as an Invariant

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Reidemeister moves do not change linking number, hence linking number is an invariant of any oriented link.

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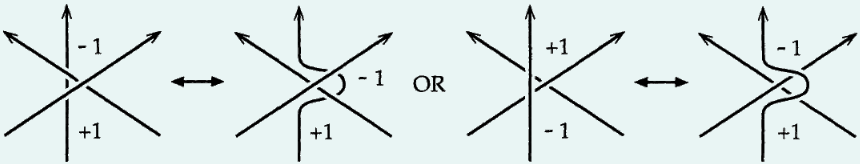
Example



Type III Reidemeister moves don't change linking number

Surfaces

Definition (Surfaces)

Informally, a **surface** or two-manifold is defined as an object such that every point on the object has a neighborhood contained within the object that is "equivalent" to a disk.

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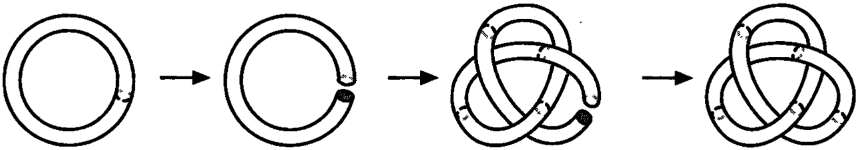
Definition (Homeomorphism)

A **homeomorphism** is a bijective, continuous map between surfaces such that its inverse map is also continuous.



Genus

Definition (Genus of a surface)

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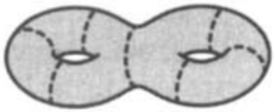

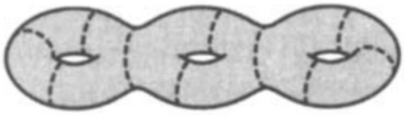

Surfaces With Boundary

Definition (Boundary Component)

A surface with boundary is a surface without boundary where at least one open disk, called a **boundary component**, has been removed.

Orientability

Definition (Orientable surface)

A surface in a three-dimensional space is **orientable** if it has two sides we can paint two different colors so that the two colors only meet along the boundary of a surface. A surface that does not satisfy this property is **nonorientable**.

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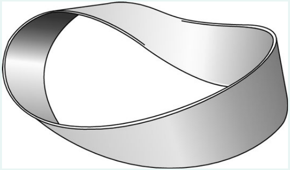
Example (Möbius Band)

A Möbius Band is *nonorientable* because if we were to start painting a side one color, we would find that we've painted the whole surface that same color because it only has one side.



Seifert Surfaces

Definition (Seifert surface)

A **Seifert surface** for a knot K is an orientable surface with a boundary component that is the given knot K.

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Definition (Genus of a knot)

The **genus** of a knot K is the minimal genus of any Seifert surface of K.

Seifert's Algorithm

Algorithm (Seifert's Algorithm)

Seifert's algorithm constructs a Seifert surface S for any given projection of a knot K.

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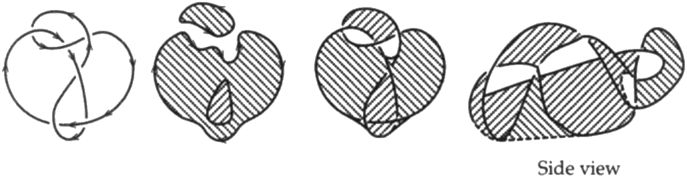

The Trivial Knot is Prime

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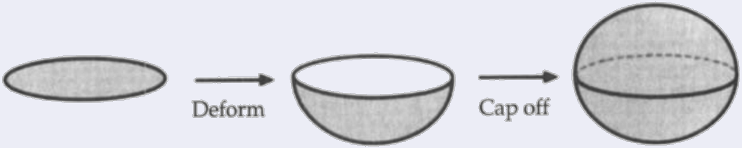
Proposition

The trivial knot is the only knot with genus 0, because it is the only knot that is the boundary of an embedded disk.



Theorem

Given a knot K, let $g(K)$ be the genus of K. Then,

$$g(J\#K) = g(J) + g(K).$$

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Theorem

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Proof.

Because genus is nonnegative, all nontrivial knots have genus ≥ 1. By our earlier theorem, the composition of two nontrivial knots then has genus ≥ 2 and therefore is not the trivial knot, which has genus 0. Hence, the trivial knot must be prime. □

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- Our families
- The audience for listening to this presentation

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Image Sources:

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