The Indecomposable Summands of the Tensor Products of Monomial Modules Over Finite 2-Groups

George Cao
Mentor: Kent Vashaw

Montgomery High School

October 15-16, 2022
MIT PRIMES Conference

A complex sculpture

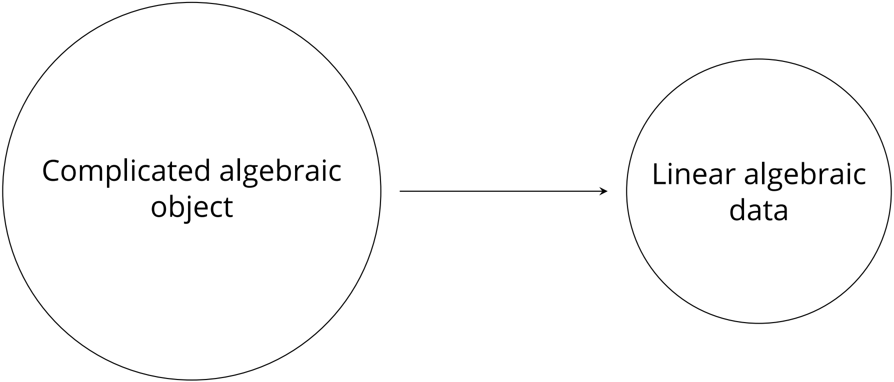
- What is representation theory?
- What are the goals in representation theory?

The sculpture “Threshold” by James Hopkins[1]

---

[1] Source: `https://www.jameshopkinsworks.com/commissions.html`

Representation theory, broadly



Complicated algebraic object

Linear algebraic data

Group representation

Definition

Let G be a finite group. A **representation** of G is a vector space V (over field k) and a group homomorphism $\rho : G \to GL(V)$, where $GL(V)$ is the set of bijective linear transformations $V \to V$.

We write $\rho(g)v \in V$ as gv, where $g \in G$ and $v \in V$.

Example

Let $V = \mathbb{R}^3$. Then V is a representation of $G = C_3 = \langle g \rangle$, where

$$\begin{aligned} \rho(g) : e_1 &\mapsto e_2 \\ e_2 &\mapsto e_3 \\ e_3 &\mapsto e_1 \end{aligned}$$

Direct sums of representations

Let G be a group.

Definition

Let V_1, V_2 be representations of G. The **direct sum** of representations V_1 and V_2 is the vector space $V_1 \oplus V_2$ and the action of G given by $g(v_1 \oplus v_2) = gv_1 \oplus gv_2$.

Definition

Let V be a representation of G. Then V is **indecomposable** if it cannot be written as the direct sum of two nonzero representations, and V is called **irreducible** if it has no nontrivial proper subrepresentations.

Maschke's Theorem

Theorem (Maschke)

Let G be a finite group. Then the characteristic of a field k does not divide $|G|$ if and only if any finite dimensional representation of G can be written as a direct sum of irreducible representations.

Modular representation theory: when the characteristic of k divides $|G|$.

Example

Let $G = C_2 = \langle g \rangle$. Over $\mathbb{C}$, the irreducible representations are $\mathbb{C}_+$ and $\mathbb{C}_-$, given by $\rho(g) = (1)$ and $\rho(g) = (-1)$, respectively. Over $\overline{\mathbb{F}}_2$, the only irreducible representation is $\rho(g) = (1)$. The representation given by $\rho(g) = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$ decomposes into $\mathbb{C}_+ \oplus \mathbb{C}_-$ over $\mathbb{C}$ but is indecomposable over $\overline{\mathbb{F}}_2$.

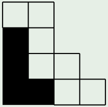
Monomial representations

Let k be an algebraically closed field of characteristic 2.
Let $G = \mathbb{Z}_{2^r} \times \mathbb{Z}_{2^s}$ (a 2-group), with generators x and y.

Choose a partition and remove a sub-partition:

Example

The partition $(4, 4, 2, 1)/(3, 1)$:



- Place a basis vector of V in each cell. The action of $x - 1$ takes a basis vector to the one in the box adjacent to the right. The action of $y - 1$ takes it one cell up.
- Monomial representation is indecomposable if and only if diagram is connected.

Conjecture (Benson and Symonds)

There is a way of “multiplying” representations V and W, denoted $V \otimes W$. The dimension of this is $\dim V \cdot \dim W$.

A consequence of a previously published conjecture is that there is a unique odd-dimensional indecomposable summand of $V^{\otimes n}$. Let this summand be denoted as V_n.

Conjecture

Let $P_V(x)$ be a function such that $P_V(n)$ is the dimension of V_n. Then $P_V(x)$ is a polynomial, or a quasi-polynomial in some cases.

We examine this conjecture for monomial representations.

Symmetric monomial representations

Simplest monomial representations to check the conjecture:

Proposition

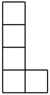
If V is a monomial representation with a monomial diagram that is symmetric by rotation of 180°, then $V_{\text{odd}} \cong V$ and $V_{\text{even}} \cong k$. Particularly,

$$P_V(n) = \begin{cases} \dim V & \text{if } n \text{ odd} \\ 1 & \text{if } n \text{ even.} \end{cases}$$

$(4, 1)$ monomial representation

Let V be the monomial representation corresponding to the partition $(4, 1)$.

$V =$



Proposition

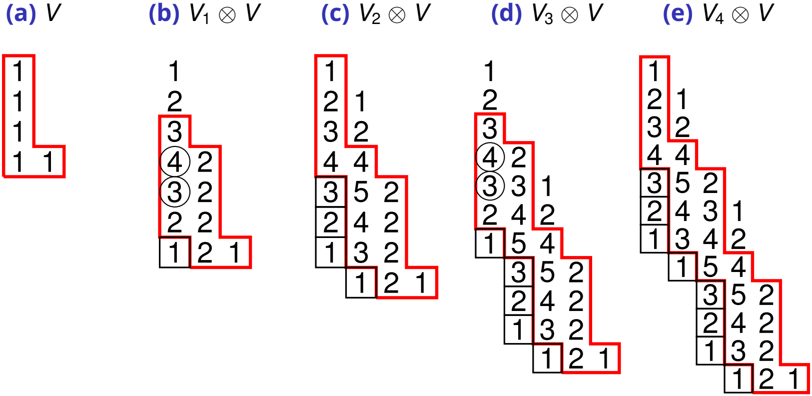
We have the following decomposition into indecomposable summands:

$$V_{2k} \otimes V = V_{2k+1} \oplus \underbrace{F \oplus \cdots \oplus F}_{4k \text{ copies}},$$

$$V_{2k-1} \otimes V = V_{2k} \oplus W \oplus W \oplus \underbrace{F \oplus \cdots \oplus F}_{4k-3 \text{ copies}},$$

where F is a free module of dimension 8 and W is dimension 4. Particularly, $P_V(n) = 4n + 1$.

$(4, 1)$ monomial representation



(a) V **(b)** $V_1 \otimes V$ **(c)** $V_2 \otimes V$ **(d)** $V_3 \otimes V$ **(e)** $V_4 \otimes V$

Key	
1	1 1
1	1 1
1	1 1
①	$\boxed{1}$ 1

Data computed with MAGMA

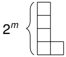
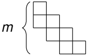
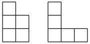
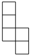
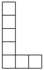
Diagram	Computed QP
	$2^m x + 1$
	$2(m-1)x + 1$
	$[10x-5, 6x+1]$
	$[6x-1, 6x+1]$
	$2x^2 + 4x + 1$
	$[18x-11, 10x+1]$

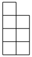
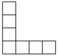
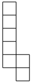
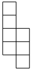
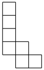
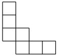
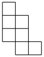
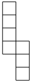
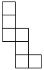
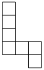
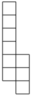
Diagram	Computed QP
	$[4x+3, 4x-1]$
	$[8x-1, 8x+1]$
	$[10x-3, 10x+1]$
	$[12x-5, 12x-7]$
	$6x + 1$
	$[20x-13, 12x+1]$

Diagram	Computed QP
	$[12x-4, 12x+1]$
	$[4x+3, 8x+1]$
	$[8x-1, 12x+1]$
	$[10x-3, 10x+1]$
	$12x^2 - 4x + 1$

Acknowledgements

I would like to thank:

- Dr. Kent Vashaw for mentoring me.
- Prof. Etingof for proposing this project and giving valuable advice.
- Prof. Benson for his useful discussions.
- Dr. Gerovitch, Dr. Khovanova, and the MIT PRIMES program for making this wonderful research opportunity possible.

References

- J. L. Alperin. *Local Representation Theory*: Modular representations as an introduction to the local representation theory of finite groups. Cambridge Studies in Advanced Mathematics, 11. Cambridge University Press, Cambridge, 1986. x+178 pp.
- D. Benson. *Some conjectures and their consequences for tensor products of modules over a finite p-group*. J. Algebra, **558** (2020), 24–42, DOI 10.1016/j.jalgebra.2019.10.012.
- D. Benson and P. Symonds. *The non-projective part of the tensor powers of a module*. J. London Math. Soc., **101** (2019), no. 2, 828–856, DOI 10.1112/jlms.12288.
- P. Etingof, O. Golberg, S. Hensel, T. Liu, A. Schwendner, D. Vaintrob, E. Yudovina. *Introduction to Representation Theory* with historical interludes by Slava Gerovitch. Student Mathematical Library, 59. American Mathematical Society, Providence, RI, 2011. viii+228 pp.