The Indecomposable Summands of the Tensor Products of Monomial Modules Over Finite 2-Groups

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A complex sculpture

- What is representation theory?
- What are the goals in representation theory?

The sculpture “Threshold” by James Hopkins

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1Source: https://www.jameshopkinsworks.com/commissions.html
Representation theory, broadly

Complicated algebraic object \rightarrow \text{Linear algebraic data}
Definition

Let $G$ be a finite group. A **representation** of $G$ is a vector space $V$ (over field $k$) and a group homomorphism $\rho : G \to GL(V)$, where $GL(V)$ is the set of bijective linear transformations $V \to V$.

We write $\rho(g)v \in V$ as $gv$, where $g \in G$ and $v \in V$.

Example

Let $V = \mathbb{R}^3$. Then $V$ is a representation of $G = C_3 = \langle g \rangle$, where

$$
\rho(g) : e_1 \mapsto e_2 \\
e_2 \mapsto e_3 \\
e_3 \mapsto e_1
$$
Let $G$ be a group.

**Definition**
Let $V_1, V_2$ be representations of $G$. The **direct sum** of representations $V_1$ and $V_2$ is the vector space $V_1 \oplus V_2$ and the action of $G$ given by $g(v_1 \oplus v_2) = gv_1 \oplus gv_2$.

**Definition**
Let $V$ be a representation of $G$. Then $V$ is **indecomposable** if it cannot be written as the direct sum of two nonzero representations, and $V$ is called **irreducible** if it has no nontrivial proper subrepresentations.
Maschke’s Theorem

**Theorem (Maschke)**

Let $G$ be a finite group. Then the characteristic of a field $k$ does not divide $|G|$ if and only if any finite dimensional representation of $G$ can be written as a direct sum of irreducible representations.

Modular representation theory: when the characteristic of $k$ divides $|G|$.

**Example**

Let $G = C_2 = \langle g \rangle$. Over $\mathbb{C}$, the irreducible representations are $\mathbb{C}_+$ and $\mathbb{C}_-$, given by $\rho(g) = (1)$ and $\rho(g) = (-1)$, respectively. Over $\mathbb{F}_2$, the only irreducible representation is $\rho(g) = (1)$. The representation given by $\rho(g) = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$ decomposes into $\mathbb{C}_+ \oplus \mathbb{C}_-$ over $\mathbb{C}$ but is indecomposable over $\mathbb{F}_2$. 
Monomial representations

Let $k$ be an algebraically closed field of characteristic $2$. Let $G = \mathbb{Z}_{2^r} \times \mathbb{Z}_{2^s}$ (a 2-group), with generators $x$ and $y$.

Choose a partition and remove a sub-partition:

**Example**

The partition $(4, 4, 2, 1)/(3, 1)$:

- Place a basis vector of $V$ in each cell. The action of $x - 1$ takes a basis vector to the one in the box adjacent to the right. The action of $y - 1$ takes it one cell up.
- Monomial representation is indecomposable if and only if diagram is connected.
Conjecture (Benson and Symonds)

There is a way of “multiplying” representations $V$ and $W$, denoted $V \otimes W$. The dimension of this is $\dim V \cdot \dim W$.

A consequence of a previously published conjecture is that there is a unique odd-dimensional indecomposable summand of $V \otimes^n$. Let this summand be denoted as $V_n$.

**Conjecture**

Let $P_V(x)$ be a function such that $P_V(n)$ is the dimension of $V_n$. Then $P_V(x)$ is a polynomial, or a quasi-polynomial in some cases.

We examine this conjecture for monomial representations.
Simplest monomial representations to check the conjecture:

**Proposition**

If $V$ is a monomial representation with a monomial diagram that is symmetric by rotation of $180^\circ$, then $V_{\text{odd}} \cong V$ and $V_{\text{even}} \cong k$. Particularly,

$$P_V(n) = \begin{cases} \dim V & \text{if } n \text{ odd} \\ 1 & \text{if } n \text{ even.} \end{cases}$$
Let $V$ be the monomial representation corresponding to the partition $(4, 1)$. 

\[ V = \]

**Proposition**

We have the following decomposition into indecomposable summands:

\[
V_{2k} \otimes V = V_{2k+1} \oplus F \oplus \cdots \oplus F, \\
\text{4k copies}
\]

\[
V_{2k-1} \otimes V = V_{2k} \oplus W \oplus W \oplus F \oplus \cdots \oplus F, \\
\text{4k–3 copies}
\]

where $F$ is a free module of dimension 8 and $W$ is dimension 4. Particularly, $P_V(n) = 4n + 1$. 

George Cao

Summands of Tensor Products

October 2022 10 / 14
(4, 1) monomial representation

(a) $V$

(b) $V_1 \otimes V$

(c) $V_2 \otimes V$

(d) $V_3 \otimes V$

(e) $V_4 \otimes V$

Key

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