Vanishing Polynomials and Polynomial Functions

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A **ring** is a set $A$ with operation $+$ and $\times$ such that:

- $A$ is closed under $+$ and $\times$.
- $+$ is commutative and has inverses (so $-$ exists).
- There is an additive identity (denoted 0).
- Both operations are associative.
- The distributive law holds ($(a + b) \times c = a \times c + b \times c$).

We will be working with commutative rings (so $\times$ is commutative).
**Definition (Polynomial)**

A polynomial $F(x)$ in a polynomial ring $R[x]$ is a formal sum

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

for some nonnegative integer $n$, where each $a_i \in R$ and $x$ is an indeterminate.

**Definition (Vanishing polynomial)**

A vanishing polynomial $F(x) \in R[x]$ is a polynomial such that $F(a) = 0$ for all $a \in R$. By definition, 0 itself is a vanishing polynomial.
Example
Consider the polynomial $F(x) = x^2 + x$ over $\mathbb{Z}_2$. Notice that $F(0) = 0$ and $F(1) = 2 = 0$.

Example
Consider the ring $R = \prod_{n=1}^\infty \mathbb{Z}_2$. Notice that $x^2 + x$ is vanishing in this ring as well.

Example
Over the ring $\mathbb{Z}_6$, the polynomial $x(x - 1)(x - 2)(x - 3)(x - 4)(x - 5)$ clearly vanishes; however, the lower degree $x(x - 1)(x - 2)$ and $3(x - 1)(x - 2)$ also vanish.
Definition (Polynomial function)

A **polynomial function** $f : \mathbb{R} \rightarrow \mathbb{R}$ is a function on $\mathbb{R}$ for which there exists a polynomial $F(x) \in \mathbb{R}[x]$ such that $f(r) = F(r)$ for all $r \in \mathbb{R}$.

Polynomials are denoted with uppercase letters while polynomial functions are denoted with lowercase letters.

Thus, $F(x)$ is a polynomial but $f(x)$ is a polynomial function.
Example

Over $\mathbb{Z}_6$, $F(x) = x^2 + 1$ is a polynomial while the mapping induced, namely $f$ which maps $0 \rightarrow 1, 1 \rightarrow 2, 2 \rightarrow 5, 3 \rightarrow 4, 4 \rightarrow 5, 5 \rightarrow 2$, is a polynomial function.
Ideal of Vanishing Polynomials

Definition (Ideal)

A subring $S \subseteq R$ is an ideal if $rs \in S$ for all $r \in R$ and $s \in S$.

Lemma (Well-known)

Vanishing polynomials form an ideal.
Vanishing Polynomials Over $\mathbb{Z}_n$

**Theorem (Singmaster, 1974)**

Any element of the ideal of vanishing polynomials over $\mathbb{Z}_n$ is of the form

$$G(x) = F(x)B_s(x) + \sum_{k=0}^{s-1} a_k \cdot \frac{n}{\text{gcd}(k!, n)} \cdot B_k(x)$$

where $B_k(x) = (x + 1)(x + 2) \ldots (x + k)$ with $B_0(x) = 1$, and $s$ is the smallest integer such that $n \mid s!$. $F(x)$ is a polynomial which is uniquely defined based on $G(x)$, and $a_k$'s are integers also uniquely defined in the range $0 \leq a_k < \text{gcd}(k!, n)$. 
Definition

A polynomial $P(x)$ is **integer valued** if for all integers $n$, $P(n)$ is an integer.

- Any vanishing polynomial $F(x)$ corresponds to an integer valued polynomial $G(x) = F(x)/n$.
- Conversely, in order for an integer-valued polynomial $G(x)$ to correspond to a polynomial $F(x) = nG(x)$, all resulting coefficients in $F(x)$ must be integers.
\( \binom{x}{k} = \frac{x(x-1)\cdots(x-k+1)}{k!} \) denotes the "choose" function.

**Lemma (Well-known)**

Any integer-valued polynomial can be uniquely expressed as a linear sum with integer coefficients of functions of the form \( \binom{x}{k} \).

By the lemma, every such \( G(x) \) can be uniquely represented as a sum \( G(x) = \sum_{k=1}^{m} c_k \binom{x}{k} \). Thus, every vanishing polynomial \( F(x) \) over \( \mathbb{Z}_n \) can be uniquely represented as

\[
F(x) = \sum_{k=1}^{m} nc_k \binom{x}{k},
\]

where \( c_k, m \in \mathbb{Z} \).
Vanishing Polynomials Over $\mathbb{Z}_n$

Theorem (Borodin, Liu, Zhang, 2022)

If any term in the summation $\sum_{k=1}^{m} nck\binom{x}{k}$ has a non-integer coefficient then the resulting polynomial cannot have integer coefficients.

- $nc_k\binom{x}{k}$ has integer coefficients $\implies k! \mid nc_k$.
- The smallest such $c_k$ is $k!/\gcd(n, k!)$ and any greater $c_k$ would be a multiple of this, so any valid $c_k$ can be written as $a_k \cdot (k!/\gcd(n, k!))$ for integer $0 \leq a_k < \gcd(n, k!)$ (any $a_k$ outside this range is redundant in $\mathbb{Z}_n$).
- If we define $s$ to be the smallest integer such that $n \mid s!$, any polynomial $n \cdot a_k \frac{k!}{\gcd(k!, n)}\binom{x}{k}$, where $k \geq s$, is a polynomial multiple of $\binom{x}{s}$.
If we define $s$ to be the smallest integer such that $n \mid s!$, any polynomial $n \cdot a_k \frac{k!}{\gcd(k!, n)} \binom{x}{k}$, where $k \geq s$, is a polynomial multiple of $\binom{x}{s}$.

Therefore we have arrived at Singmaster’s formulation, except with $B_k(x)$ written in the form $\binom{x}{k} \cdot k!$.

**Theorem (Singmaster 1974)**

Any element of the ideal of vanishing polynomials over $\mathbb{Z}_n$ is of the form

$$G(x) = F(x)B_s(x) + \sum_{k=0}^{s-1} a_k \cdot \frac{n}{\gcd(k!, n)} \cdot B_k(x)$$

where $B_k(x) = (x + 1)(x + 2) \ldots (x + k)$ with $B_0(x) = 1$, and $s$ is the smallest integer such that $n \mid s!$. $F(x)$ is a polynomial which is uniquely defined based on $G(x)$, and $a_k$’s are integers also uniquely defined in the range $0 \leq a_k < \gcd(k!, n)$. 
Corollary

The generating set for the ideal of vanishing polynomials over $\mathbb{Z}_n$ is

$$\left\{ \frac{n}{\gcd(k!, n)} \cdot B_k(x) \mid k \in \mathbb{Z}_{\geq 0} \right\}$$

for either definition of $B_k(x)$.

- If $k$ is less than the smallest prime divisor of $n$, the only element in the above set is the zero polynomial.
- We can immediately find the degree of the minimal degree monic vanishing polynomial and minimal degree non-monic vanishing polynomial, which would be $s$ and the smallest prime factor of $n$, respectively.
- The minimal degree non-monic polynomial must be unique up to multiplication by a constant since the generating set only contains a single nonzero polynomial of that degree or lower.
Corollary

The generating set for the ideal of vanishing polynomials over \( \mathbb{Z}_n \) is

\[
\left\{ \frac{n}{\gcd(k!, n)} \cdot B_k(x) \mid k \in \mathbb{Z}_{\geq 0} \right\}
\]

for either definition of \( B_k(x) \).

- Many of the elements in the generating set are redundant.
- In particular, if we have two polynomials \( a \cdot \binom{x}{i} \cdot i! \) and \( a \cdot \binom{x}{j} \cdot j! \) for some integer \( a \), and \( i < j \) then the polynomial containing \( j \) is a polynomial multiple of the other and therefore redundant in a generating set.
- Therefore, in order to minimize our generating set we can remove any polynomials \( \left( \frac{n}{\gcd(k!, n)} \right) \binom{x}{k} \cdot k! \) for which \( k \) is not the minimal integer which gives the same value of \( \gcd(k!, n) \).
A quotient of a ring $R$ by an ideal $I$ is a partitioning of the ring $R$ into cosets of the form $r_1 + I$, $r_2 + I$, $r_3 + I$, ..., which form a ring under $(a + I) + (b + I) = ((a + b) + I)$ and $(a + I) \times (b + I) = ((a \times b) + I)$.
### Example

$5\mathbb{Z}$ is the ideal generated by 5 consisting of all integer multiples of 5. The quotient $\mathbb{Z}/5\mathbb{Z}$ is summarized in the following table:

<table>
<thead>
<tr>
<th>Representative</th>
<th>Coset</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$0 + 5\mathbb{Z} = {\ldots, -5, 0, 5, 10, \ldots}$</td>
</tr>
<tr>
<td>1</td>
<td>$1 + 5\mathbb{Z} = {\ldots, -4, 1, 6, 11, \ldots}$</td>
</tr>
<tr>
<td>2</td>
<td>$2 + 5\mathbb{Z} = {\ldots, -3, 2, 7, 12, \ldots}$</td>
</tr>
<tr>
<td>3</td>
<td>$3 + 5\mathbb{Z} = {\ldots, -2, 3, 8, 13, \ldots}$</td>
</tr>
<tr>
<td>4</td>
<td>$4 + 5\mathbb{Z} = {\ldots, -1, 4, 9, 14, \ldots}$</td>
</tr>
</tbody>
</table>
(y) = \{ay : a \in R\}, the ideal generated by y.

**Definition**

For a \( y \in R \) such that \( R/(y) \) is finite and creates the cosets \( a_1 + (y), a_2 + (y), \ldots, a_k + (y) \) we define

\[
F_y(x) = (x - a_1)(x - a_2) \ldots (x - a_k).
\]

**Lemma (Borodin, Liu, Zhang, 2022)**

*Given nonzero \( y_1y_2 = 0 \), the polynomials*

\[
G(x) = y_2 F_{y_1}(x)
\]

*and*

\[
H(x) = F_{y_1}(x)F_{y_2}(x)
\]

*are vanishing.*
Theorem (Borodin, Liu, Zhang, 2022)

Given nonzero $y_1 y_2 \ldots y_m = 0$ and an indexing set $N$ such that if $i \in N$ then $R/(y_i)$ is finite and $M$ containing all other indices the polynomial

$$H(x) = \prod_{j \in M} y_j \cdot \prod_{i \in N} F_{y_i}(x)$$

is vanishing.
Note that we often get duplicate terms which can be removed.

**Example**

Over \( \mathbb{Z}_{35} \), we get the vanishing polynomial

\[
G(x) = (x)(x - 1)(x - 2)(x - 3)(x - 4) \
\cdot (x)(x - 1)(x - 2)(x - 3)(x - 4)(x - 5)(x - 6)
\]

which can be reduced to

\[
G(x) = (x)(x - 1)(x - 2)(x - 3)(x - 4)(x - 5)(x - 6).
\]
Some duplicate terms cannot be removed.

**Example**

Consider the polynomial \((x)(x - 1) \cdot (x)(x - 1)\) over the ring \(\mathbb{Z}_4\) using the zero divisors \(2 \cdot 2 = 0\). These duplicate terms cannot be removed since \((x)(x - 1)\) is not vanishing over \(\mathbb{Z}_4\).

Precise description of when terms can be removed is more complicated.
This method allows us to find vanishing polynomials not only for finite rings but also for infinite ones.

**Example**

- Consider $R = \prod_{n=1}^{\infty} \mathbb{Z}_2$.
- $(0, 1, 1, 1, 1 \ldots) \cdot (1, 0, 0, 0, 0 \ldots) = 0$.
- $R/((0, 1, 1, 1, 1 \ldots)) \cong \mathbb{Z}_2$ so it is finite.
- $(1, 0, 0, 0, 0 \ldots)(x - (0, 0, 0, 0, 0 \ldots))(x - (1, 0, 0, 0, 0 \ldots))$ is vanishing.
- $(0, 0, 0, 0, 0 \ldots)$ and $(1, 0, 0, 0, 0 \ldots)$ can be replaced by any representatives from the corresponding cosets.
Theorem (Borodin, Liu, Zhang, 2022)

If \( R = \mathbb{Z}/n \), this description is sufficient to give a generating set of all vanishing polynomials if we take \( y_1 \cdot \ldots \cdot y_k = n \) to be the prime factorization of \( n \).

Proof sketch.

Number theoretic proof from before uses the fact that \( x(x - 1)(x - 2) \ldots (x - k) \) is divisible by \( \gcd(n, k!) \). Now we can instead use a product of \( F_{y_i}(x) \)'s where \( y_i \)'s multiply to \( \gcd(n, k!) \) to achieve the same result. Removing duplicates gives the desired degree.
Vanishing Polynomials Over Product Rings

We now have a classification of vanishing polynomials for $\mathbb{Z}_n$.

- How to extend to more general rings?
- Extend to direct products of rings of integers modulo a number.

**Definition (Direct Product)**

The **direct product** $A \times B$ of rings $A$ and $B$ is the set of elements $(a, b)|a \in A, b \in B$ such that

$$(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$$

where $a_1 + a_2$ is the sum of $a_1$ and $a_2$ in $A$. Similarly,

$$(a_1, b_1) \cdot (a_2, b_2) = (a_1 \cdot a_2, b_1 \cdot b_2).$$
Example

Consider the two elements $a = (0, 1)$ and $b = (1, 0)$ in $\mathbb{Z}_2 \times \mathbb{Z}_2$. We have

$$a + b = (1, 1) \text{ and } ab = (0, 0).$$

Notice that $\mathbb{Z}_4$ is not the same as $\mathbb{Z}_2 \times \mathbb{Z}_2$: in $\mathbb{Z}_4$, the identity added to itself gives $1 + 1 = 2$, while in $\mathbb{Z}_2 \times \mathbb{Z}_2$ it gives $(1, 1) + (1, 1) = (0, 0)$, the zero element.
Besides extending above results, what can direct product be used for?

- Any finite ring can be decomposed into prime power order rings.
- Very small set of distinct prime power order rings for any given prime power.
Lemma (Well-known)

Let $R$ be the direct product of $k$ rings $R_1, \ldots, R_k$. Then, we have

$$R[x] \cong R_1[x] \times \cdots \times R_k[x].$$

Theorem (Borodin, Liu, Zhang, 2022)

Let $R$ be the direct product of $k$ rings $R_1, \ldots, R_k$. Then, the ring of polynomial functions on $R$ has the same ring structure as the direct product of the rings of polynomial functions on $R_1, \ldots, R_k$. 
Example

Consider $R = \mathbb{Z}_2 \times \mathbb{Z}_2$.

We can then express any element of $R[x]$ as an element of $\mathbb{Z}_2[x] \times \mathbb{Z}_2[x]$ and vice versa.

<table>
<thead>
<tr>
<th>$R[x]$</th>
<th>$\mathbb{Z}_2[x] \times \mathbb{Z}_2[x]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1,0)x$</td>
<td>$(x,0)$</td>
</tr>
<tr>
<td>$(1,0)x^4 + (0, 1)x^3 + (1, 1)x$</td>
<td>$(x^4 + x, x^3 + x)$</td>
</tr>
<tr>
<td>$(1,0)x^3 + (1, 1)x^2 + (0, 1)x$</td>
<td>$(x^3 + x^2, x^2 + x)$</td>
</tr>
</tbody>
</table>

Notice that all vanishing polynomials in $R[x]$ correspond to pairs of vanishing polynomials in $\mathbb{Z}_2[x] \times \mathbb{Z}_2[x]$. 
Example

We now apply this theorem to find the set of polynomial functions over $\mathbb{Z}_2 \times \mathbb{Z}_2$.

<table>
<thead>
<tr>
<th>(a,b)</th>
<th>0</th>
<th>1</th>
<th>x</th>
<th>x+1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(0,0)</td>
<td>(1,0)</td>
<td>(1,0)x</td>
<td>(1,0)x + (1,0)</td>
</tr>
<tr>
<td>1</td>
<td>(0,1)</td>
<td>(1,1)</td>
<td>(1,0)x + (0,1)</td>
<td>(1,0)x + (1,1)</td>
</tr>
<tr>
<td>x</td>
<td>(0,1)x</td>
<td>(0,1)x + (1,0)</td>
<td>(1,1)x</td>
<td>(1,1)x + (1,0)</td>
</tr>
<tr>
<td>x+1</td>
<td>(0,1)x + (0,1)</td>
<td>(0,1)x + (1,1)</td>
<td>(1,1)x + (0,1)</td>
<td>(1,1)x + (1,1)</td>
</tr>
</tbody>
</table>

The set of polynomial functions over $\mathbb{Z}_2 \times \mathbb{Z}_2$ is

$$(0, 0), (1, 0), (0, 1), (1, 1),$$
$$(1, 0)x, (1, 0)x + (1, 0), (1, 0)x + (0, 1), (1, 0)x + (1, 1)$$
$$(0, 1)x, (0, 1)x + (1, 0), (0, 1)x + (0, 1), (0, 1)x + (1, 1)$$
$$(1, 1)x, (1, 1)x + (1, 0), (1, 1)x + (0, 1), (1, 1)x + (1, 1).$$
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References


