Unitary Conditions of Heun and Lamé Differential Equations

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Complex Analysis Crash Course

Don’t worry, there are emergency supplies if you do crash.
What is Complex Analysis?

- TLDR: Calculus with complex numbers
- Intuitively similar to multivariable calculus (imagine two dimensions, one for the real component and one for the imaginary)
- Many real functions \((e^x, \sin(x), \text{etc.})\) can be nicely extended to the complex plane.
We define a complex derivative as

$$f'(z) := \lim_{h \to 0} \frac{f(z + h) - f(z)}{h}$$

Unlike multivariable partial derivatives, with the complex derivative, it does not depend on the direction from which $h$ approaches 0.

Therefore, complex differentiable functions locally look like just a rotation and scaling of the complex plane.

Complex differentiable functions are called **analytic**.
 Principle of Analytic Continuation

- Start with an **analytic** (i.e. differentiable, expressible as a power series) function $f$ at a point
- There is a little area around $P$ where $f$ is analytic
- We can extend this along a curve to point $Q$ to get a value for $f(Q)$
- Doesn’t matter which path is taken (usually)
What about Loops?

- This value is not uniquely-defined if $f$ is not analytic inside the loop.
- If $f$ is not analytic inside a loop, we get a matrix mapping $f \rightarrow Mf$ by going around the loop once.
- By going around the opposite direction, we invert this map.
Differential Equations

Making a difference

\[ C_y = C_{yi}(\alpha - \alpha_i) \]

\[ P = m \cdot V \cdot T = 2 \pi \sqrt{\frac{L}{g}} \]
Differential Equations

- Equations relating complex functions and their derivatives.
- Define a function on small area of the complex plane.

\[ \frac{d^2 w}{dz^2} + P(z) \frac{dw}{dz} + Q(z)w = 0 \]
The Heun Equation

\[ \frac{d^2w}{dz^2} + \left( \frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\varepsilon}{z-a} \right) \frac{dw}{dz} + \frac{\alpha_1 \beta z - B}{z(z-1)(z-a)} w = 0. \]

- It has two independent solutions: called \( f \) and \( g \)
- Lamé Equation when \( \gamma, \delta, \varepsilon \in \mathbb{Z} + \frac{1}{2} \)
- Singular at 0, 1, and \( a \)
- How do the solutions change as we continue around each singularity?
Take the solution $f$

As we go around $0$, $f$ morphs into a different solution, say $f + g$.

Similarly $g$ might morph into $2f - g$.

This gives us the map

\[
\begin{pmatrix}
1 & 1 \\
2 & -1
\end{pmatrix}
\]
Monodromies

- From looping, we get three matrices $M_0, M_1,$ and $M_a$
- We refer to these maps as monodromies (monodromy, singular).

\[ M_0, M_1, M_a : f \rightarrow f^* \]
Our Project

To make the world a better place.
Define the **monodromy group** $G$ of an equation as the group generated by its monodromy matrices (under multiplication).

A **Hermitian matrix** $H$ is preserved complex-conjugate transposition:

$$\overline{H^T} = H$$

We call the group of monodromies **unitary** if, for all $g \in G$, there is an $H$ that satisfies:

$$\overline{g^T H g} = H$$
Research Question

\[
\frac{d^2 w}{dz^2} + \left( \frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\varepsilon}{z-a} \right) \frac{dw}{dz} + \frac{\alpha \beta z - B}{z(z-1)(z-a)} w = 0.
\]

- We call this \( B \) the **accessory parameter**.
- We seek to answer: For what values \( B \) does this equation possess a unitary monodromy group?
- Our question is equivalent to: For what values \( B \) does this above equation admit real-analytic solutions?
- **Significance:** This question is important for the analytic Langlands correspondence, a theory attempting to link complex curves with algebraic data.
Analysis

Made a difference?
We extend a method from (Beukers, ‘07) to compute desired values of $B$ using computational methods. **Works when all monodromy matrices are reflections (eigenvalues 1, -1).**

**Computational algorithm:**
- Guess an initial value of $B$
- Repeatedly gets refined using a gradient-descent style method and traces of matrix-pair products
Our Heun Unitarity Theorem

**Theorem.** Let $G$ be the monodromy group generated by the matrices $P, Q, R \in \text{GL}(2, \mathbb{C})$, and assume that $PQR$ is parabolic. Then, for the statements

1. $G$ is unitary,

2. For $\lambda_P = e^{-\pi i \gamma}$, $\lambda_Q = e^{-\pi i \delta}$, $\lambda_R = e^{-\pi i \epsilon}$, we have

   $$\frac{\text{tr}(PQ)}{\lambda_P \lambda_Q}, \frac{\text{tr}(QR)}{\lambda_Q \lambda_R}, \frac{\text{tr}(PR)}{\lambda_P \lambda_R} \in \mathbb{R},$$

(1) $\implies$ (2) in general and (2) $\implies$ (1) when two of $P, Q, R$ are reflections.
So what?

- Previously, \textbf{ALL of} $P, Q, R$ \textbf{had to be reflections} (i.e eigenvalues 1, -1). In other words, it only works for the Lamé Equation.

- Our theorem allows us to rigorously extend the algorithm to the case where only \textbf{TWO of} $P, Q, R$ \textbf{need to be reflections} (the third matrix is free to vary). This extends our results to an infinite class of Heun Equations.

- Furthermore, our theorem shows that in all cases of the Heun Equation, our algorithm greatly restricts the possible values of $B$ for which the monodromy group is unitary (i.e the set of values we find MUST contain the unitary values of $B$).
Numerical Results

Ooh, pretty!
Results

- The following colorings of the complex plane represent the outputs of our algorithm.
- The position represents the initial guess we used for $B$
- The coloring represents the final complex value of $B$ (using an HSV Color Transform)
- Beautiful Spiral Patterns
More results

Heun Equation with $\epsilon=0.125$

Heun Equation with $\epsilon=0.625$
Lattice of Gaussian Integers

- By Beukers’s predictions the unitary values should be the squares of distorted Gaussian Integers \((a + b\hat{i}, \text{ where } a, b \text{ are integers})\)
- Indeed, taking the square roots of our numerical results, we get the following lattices:
Thank you for hanging in there!

We hope you enjoyed!
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(Beukers (2007) has worked on this problem for a specific version of our equation (the Lamé Equation))