On the Uniqueness of Certain Types of Circle Packings on Translation Surfaces

Nilay Mishra

Mentored by Prof. Sergiy Merenkov

MIT PRIMES-USA

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Overview

What is a translation surface?

- Folding a square.

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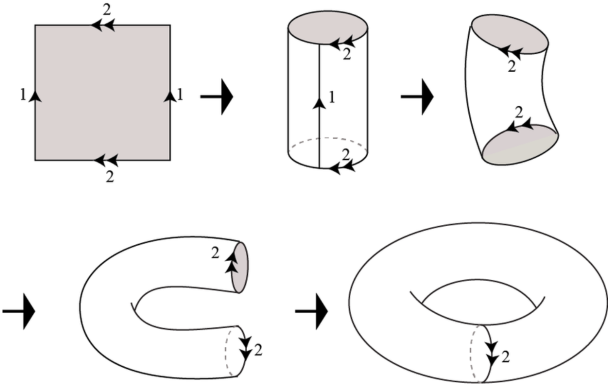
- Folding a square.



What is a translation surface?

- Folding a hexagon.

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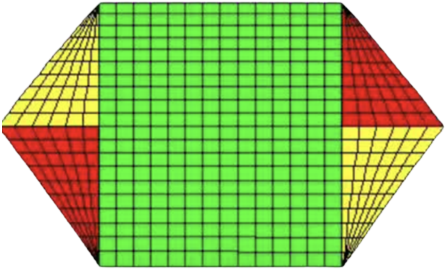

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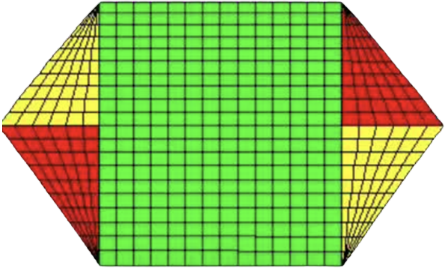

- Animation Link

What is a translation surface?

- Folding an octagon.

What is a translation surface?

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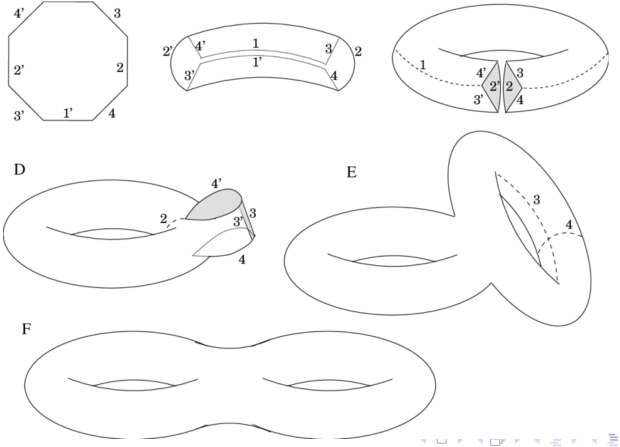

What is a translation surface?

- Start with a polygon that has an even number of sides.
- Opposite sides are parallel and of equal length.
- Identify opposite sides together and fold along them successively.

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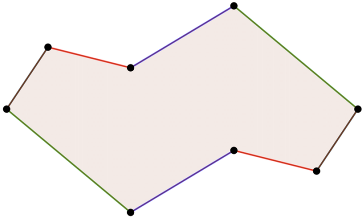

Cone Points

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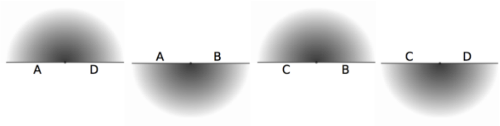
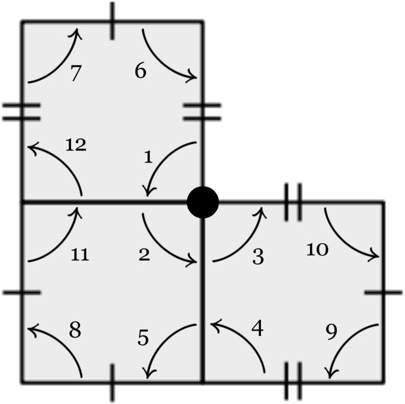
- Angle at cone point of the form $2 \quad (k+1)$ for some $k > 0$.
- Neighborhood around a cone point is isometric to neighborhood around the origin in the following diagram:



Example of a Cone Point



Degrees and Strata

- Suppose that the n cone points have degrees $d_1, d_2, \ldots, d_n$. Then:

$$\sum_{i=1}^{n} d_i = 2g - 2$$

where $g > 1$ is the genus of the translation surface.

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- Suppose that the n cone points have degrees $d_1; d_2; \quad ; d_n$. Then:

$$\sum_{i=1}^{n} d_i = 2g \quad 2$$

  where $g > 1$ is the genus of the translation surface.
- Let $g > 1$ and consider a partition of $2g \quad 2$. We define a *stratum* $H(\ )$ to be a collection of translation surfaces such that the order of each cone point is given by .

Genus Two Strata

- When $g = 2$, we have two possible cases.
- One cone point of degree 2, denoted $H(2)$ or two cone points of degree 1, denoted $H(1;1)$.

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- One cone point of degree 2, denoted $H(2)$ or two cone points of degree 1, denoted $H(1,1)$.
- Every translation surface M of genus 2 is hyperelliptic (i.e. admits a conformal involution $: M \to M$ with exactly six fixed points).

Doubled Slit Torus

Theorem (McMullen, 2007)

Let M be a translation surface of genus 2. Then M contains a geodesic J such that $J \neq \ (J)$ and splits along $J \cup \ (J)$ into the connected sum of two slit tori.

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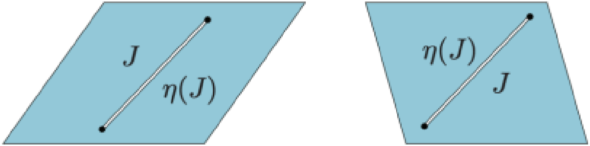

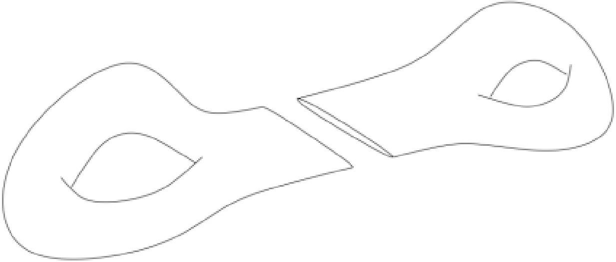

Triangulations

- A triangulation of a surface S is a locally finite decomposition of S into a collection of topologically closed triangles such that any two either:
  - are entirely disjoint
  - intersect at one or two vertices
  - intersect at a single edge

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Triangulations are allowed to be degenerate (loops and bigons).

Contacts Graph

A contacts graph is a graph with n vertices $v_1; v_2; :::; v_n$ corresponding to the generalized circles $c_1; c_2; :::; c_n$ such that v_i and v_j are connected if and only if c_i and c_j are externally tangent

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Circle Packing

A circle packingis a con guration of generalized circles on the surface such that the contacts graph is a triangulation.

Circle Packing Theorem

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Let K be a simple planar graph.

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This circle con guration is univalent and unique (up to the Möbius transformation).

Guiding Questions

For a given triangulation of a translation surface in $\mathcal{H}(1,1)$, are circle packings unique up to the hyperelliptic involution?

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For a given triangulation of a translation surface in $\mathcal{H}(1,1)$, are circle packings unique up to the hyperelliptic involution?

Given an arbitrary triangulation T of a genus 2 translation surface M, can one always find a circle packing of some M' with contacts graph T such that M and M' lie in the same stratum?

Our Work

Theorem

Suppose that there exists a circle packing on the doubled slit torus with an associated triangulation.

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Suppose that the packing contains two externally tangent double circles C_1 and C_2 such that the slit connects the centers of the two circles.

If C_1 and C_2 are fixed in place on the doubled slit torus, the packing can vary in only finitely many ways.

Diagram

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My parents

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