Consecutive Patterns in Coxeter Groups

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Dihedral Groups

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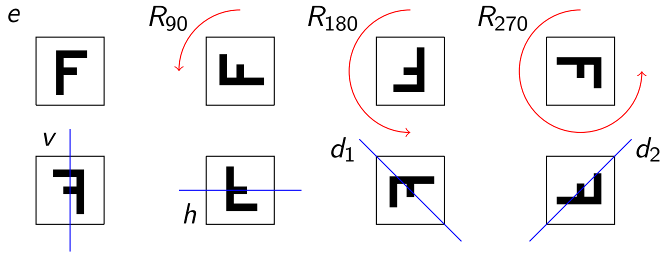

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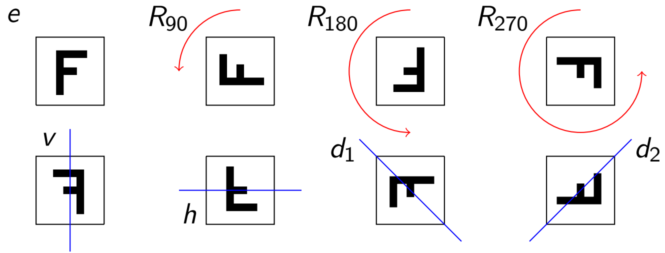

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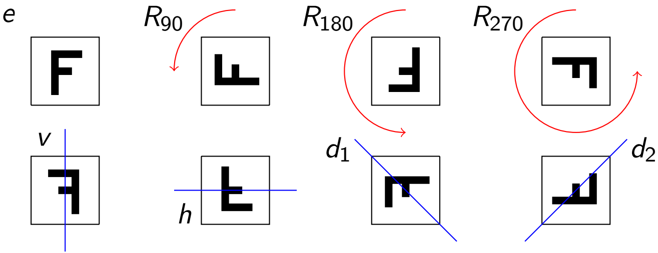

How can we present the group above? One option is, $s_1 = d_1$ and $s_2 = v$, then $D_8 = \langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^4 = e \rangle$.

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$$\mathfrak{S}_4 = \langle s_1, s_2, s_3 \mid s_1^2 = s_2^2 = s_3^2 = (s_1 s_2)^3 = (s_2 s_3)^3 = (s_1 s_3)^2 = e \rangle.$$

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We have

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Accordingly, a **Coxeter group** is a group with presentation,

$$\langle s_1, s_2, \ldots, s_n \mid s_i^2 = e \text{ for } 1 \leq i \leq n, \quad (s_i s_j)^{m_{i,j}} = e \text{ for } 1 \leq i < j \leq n \rangle,$$

where $m_{i,j} \geq 2$.

Coxeter Diagrams and Examples

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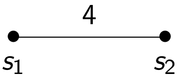

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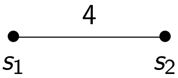

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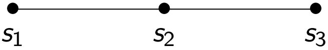

Finite Irreducible Coxeter Groups

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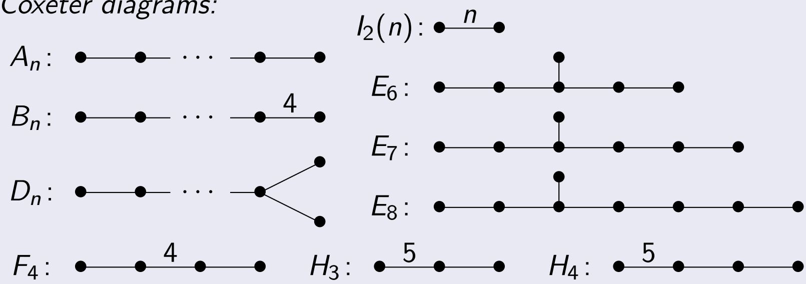
Theorem (Coxeter 1935, [2])

All finite irreducible Coxeter groups are described by the following Coxeter diagrams:



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Permutations

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Goal: Generalize consecutive pattern containment to Coxeter groups.

Reduced Words

Given an element w of a Coxeter group W, we can write it as a product of generators, called a **word**. A word of minimal length is called **reduced**.

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Example

In $\mathfrak{S}_4$, with generators $s_i = (i\ i+1)$ for $i = 1, 2, 3$, we have the following possible words for $w = 4132$:

$$4132 = s_2 s_3 s_2 s_3 s_1 s_3 = s_3 s_2 s_1 s_3 = s_2 s_3 s_2 s_1.$$

Parabolic Decomposition

Given a connected subset (on the Coxeter diagram) J of the set of generators S, we let w_J be the **longest suffix** of any reduced word for w that contains only generators from J

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The longest suffix of a reduced word containing only generators from $J = \{s_1, s_2\}$ is from the reduced word $s_2 s_3 \cdot \mathbf{s_2 s_1}$. Note that $\mathbf{s_2 s_1} = 312$.

Consecutive pattern containment

Definition (W. 2022+)

Suppose π and σ are group elements of Coxeter groups W, W' with set of generators S, S', respectively. Then we say that σ **consecutively contains** π if there exists a connected subset $J \subseteq S'$ such that π "equals" σ_J. Formally, this involves an isomorphism.

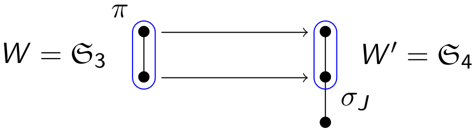

Figure: Consecutive containment in Coxeter groups

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cc-Wilf-Equivalence

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Given two permutations π, τ, we say that they are **c-Wilf-equivalence** if for every n, the number of permutations on n elements consecutively containing π is the same as the number consecutively containing τ.

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Definition (W. 2022+)

We say that two **Coxeter group elements** π and τ of an irreducible Coxeter group are **cc-Wilf-equivalence** if for every finite irreducible Coxeter group W, the number of $\sigma \in W$ consecutively containing π is the same as the number consecutively containing τ.

Automorphisms Induce cc-Wilf-Equivalences

Recall that $4132 = s_2 s_3 \cdot \mathbf{s_2 s_1}$ consecutively contains $312 = \mathbf{s_2 s_1}$.
But it also consecutively contains $231 = \mathbf{s_1 s_2}$.

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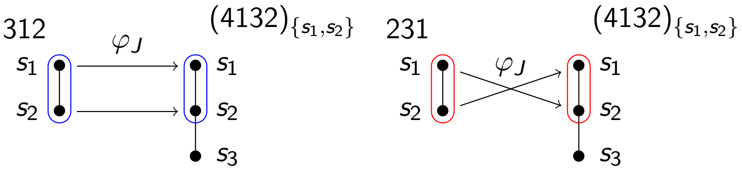

Figure: Isomorphisms for consecutive containment for the Symmetric group

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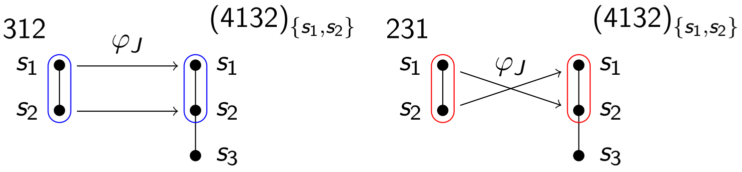

Figure: Isomorphisms for consecutive containment for the Symmetric group

Proposition (W. 2022)

If π is an element of a Coxeter group W, and ϕ is a **diagram automorphism** *of W, then π is cc-Wilf-equivalent to $\phi(\pi)$.*

Maximal Element Induces cc-Wilf-Equivalences

If $\pi = \pi_1\pi_2\cdots\pi_n$ is a permutation on n elements, then the **complement** of π, $\pi^C := (n+1-\pi_1)(n+1-\pi_2)\cdots(n+1-\pi_n)$ is c-Wilf-equivalent to π since σ consecutively contains π if and only if σ^C consecutively contains π^C.

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We can generalize this by writing $\pi^C = n(n-1)\cdots 21 \circ \pi$. Now,

Proposition (Well Known, [1])

Every finite Coxeter group W has a unique element of maximal length. We will denote this element $w_0(W)$.

The permutation $n(n-1)\cdots 21$ is precisely this element in $\mathfrak{S}_n$.

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Using this,

Proposition (W. 2022)

Let π be an element of a Coxeter group W. Then π is cc-Wilf-equivalent to $w_0(W)\pi$.

Nontrivial Families of cc-Wilf-Equivalence classes

Theorem (Duane—Remmel 2011 [4], Dotsenko—Khoroshkin 2013 [3])

We say that a permutation π is **non-overlapping** *if two of its occurrences share in any other permutation σ can share at most one position. Then the first and last entries of a non-overlapping permutation determines its c-Wilf-equivalence class.*

The idea is that π and τ are essentially interchangeable wherever they occur.

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The idea is that π and τ are essentially interchangeable wherever they occur. Skipping over a lot of details, we prove the following:

Theorem (W. 2022)

If π and τ are both strongly difference-disjoint and automorphic-equivalent, then they are cc-Wilf-equivalent.

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My parents for their continued support.

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