Symmetry and Simplicity in Finite Group Theory

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Roadmap

What is Group Theory?

- Branch of abstract algebra that studies algebraic structures called groups
- Foundation for other interests in mathematics such as representation theory
- Model patterns in nature, manipulations, and puzzles
- Forces, public key cryptography, Rubik's cube

Groups

Definition

A *group* is a finite or infinite set G together with a binary group operation $\circ : G \times G \to G$ that fulfill the group axioms:

i *Closure:* For all $g, h \in G$, the element $g \circ h \in G$.

ii *Associativity:* For $f, g, h \in G$, we have $(f \circ g) \circ h = f \circ (g \circ h)$.

iii *Identity:* There exists an *identity* element $e \in G$, such that $e \circ g = g = g \circ e \quad \forall g \in G$.

iv *Inverse:* For each $g \in G$, there exists an *inverse* element $g^{-1} \in G$ such that $g \circ g^{-1} = e = g^{-1} \circ g$.

Subgroups

Definition

Let G be a group. The subset H of G is a *subgroup* of G if it satisfies the group axioms under the binary operation of G. This relation is denoted as $H \leq G$.

- Subgroups help to "shrink" and simplify groups
- Smaller structures give insight to the whole

Injection, Surjection, and Bijection

- The function $f : X \to Y$ is *injective* if for all $x, x^{'} \in X$, $f(x) = f(x^{'}) \Rightarrow x = x^{'}$.

Injection, Surjection, and Bijection

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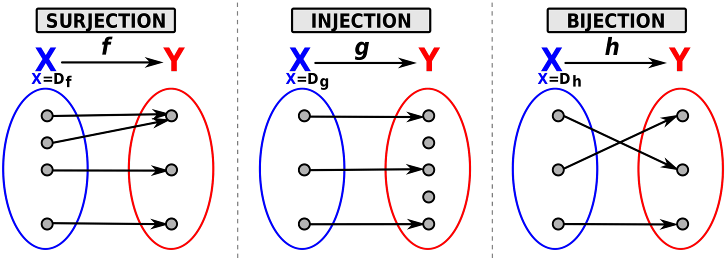
- The function $f : X \to Y$ is *surjective* if for all $y \in Y$, there is $x \in X$ such that $f(x) = y$.

Injection, Surjection, and Bijection

- The function $f : X \to Y$ is *bijective* if for all $y \in Y$, there is a unique $x \in X$ such that $f(x) = y$.



Homomorphisms and Isomorphisms

- The Greek roots "homo" and "morph" mean "same shape."
- A *homomorphism* is a special correspondence between elements of two groups.
- A *isomorphism* is a function that captures a one-to-one relationship between two groups.

Homomorphisms and Isomorphisms

Definition

A *homomorphism* is a map $\phi : G \to H$ between two groups satisfying $\phi(ab) = \phi(a)\phi(b)$, for all $a, b \in G$.

Definition

A group *isomorphism* is a group homomorphism which is a bijection.

Roadmap

Symmetries in Life

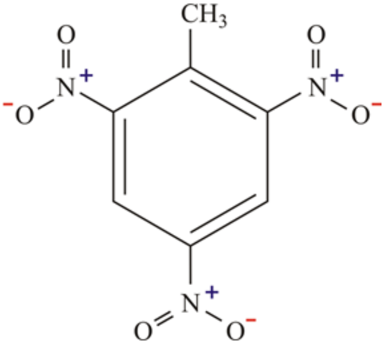


Figure 1: Symmetry in chemistry - Trinitrotoluene (TNT)

Figure 2: Symmetry in architecture - Taj Mahal in Agra, India built in marble from 1634 to 1656

Symmetric Group

Definition (Symmetric Group)

- The elements of the group are permutations on the given set (i.e., bijective maps from the set to itself).
- The product of two elements is their composite as permutations, i.e., function composition.
- The identity element of the group is the identity function from the set to itself.
- The inverse of an element in the group is its inverse as a function.

A group is said to be a *symmetric group* if it is isomorphic to the symmetric group on some set.

Example of Symmetric Groups: log and exp

Example

- Let $\mathbb{R}^x$ be the multiplicative group of positive real numbers, and let $\mathbb{R}$ be the additive group of real numbers.
- The logarithm function $\log : \mathbb{R}^x \to$ satisfies that $\log(xy) = \log(x) + \log(y)$ for all $x, y \in \mathbb{R}^x$, so log is a group homomorphism.

Example of Symmetric Groups: log and exp

Example

- The exponential function exp : $\mathbb{R} \to \mathbb{R}^x$ satisfies $exp(x + y) = exp(x) + exp(y)$ for all x, $y \in \mathbb{R}$ so exponential function is also a homomorphism.
- Logarithm and exponential function are *inverses* of each other. Since log is a homomorphism that has an inverse exp that is also a homomorphism
- So, both log and (exp) are isomorphisms between $\mathbb{R}^x$ and $\mathbb{R}$.

Cayley Table

	$\mathbf{e}$	$\mathbf{R_1}$	$\mathbf{R_2}$	$\mathbf{S_1}$	$\mathbf{S_2}$	$\mathbf{S_3}$
$\mathbf{e}$	e	R_1	R_2	S_1	S_2	S_3
$\mathbf{R_1}$	R_1	R_2	e	S_3	S_1	S_2
$\mathbf{R_2}$	R_2	e	R_1	S_2	S_3	S_1
$\mathbf{S_1}$	S_1	S_2	S_3	e	R_1	R_2
$\mathbf{S_2}$	S_2	S_3	S_1	R_2	e	R_1
$\mathbf{S_3}$	S_3	S_1	S_2	R_1	R_2	e

Cayley Table: describes the structure of a finite group by arranging all the possible products of all the group's elements in a square table reminiscent of an addition or multiplication table.

Cayley's Theorem

Theorem (Cayley)

Every group G is isomorphic to a subgroup of a symmetric group. Specifically, G is isomorphic to a subgroup of the symmetric group whose elements are the permutations of the underlying set of G.

Cayley's Theorem

Proof.

To prove Cayley's theorem we need to find a subgroup H of $Sym(G)$ and a bijective homomorphism $f : G \to H$. The roadmap for the proof is:

- Define $\phi_a : G \to G$ for each a $\in G$ and show that ϕ_a is a bijection
- Define H - ϕ_a $|a \in G$ and show that $H \leq Sym(G)$
- Define $f : G \to H$ and show that f is both a bijection and homomorphism

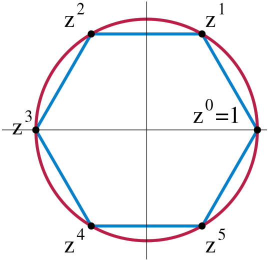
$\square$

Roadmap

Cyclic Groups

Definition

A group G is *cyclic* if G can be generated by a single element, that is, if there is some element $g \in G$ such that $G = \{g^n \mid n \in \mathbb{Z}\}$. We say that G is *generated* by g.



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Theorem

Any two cyclic groups of the same order are isomorphic.

We often express the cyclic group of order n as Z_n. Every infinite cyclic group is isomorphic to the additive group of $\mathbb{Z}$, the integers. Every finite cyclic group of order n is isomorphic to the additive group of $\mathbb{Z}/n\mathbb{Z}$, the integers modulo n.

Cyclic Groups

Theorem (Lagrange)

If G is a finite group and $H \leq G$, then the order of H divides the order of G.

Cyclic Groups

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Corollary

If G is a group of prime order p, then G is cyclic. Then $G \cong Z_p$.

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Corollary

If G is a group of prime order p, then G is cyclic. Then $G \cong Z_p$.

Proof.

Let $g \in G$, $g \neq e_G$. Thus $|\langle g \rangle| > 1$ and $|\langle g \rangle|$ divides $|G|$. Since $|G|$ is prime we must have $|\langle g \rangle| = |G|$. Hence $G = \langle g \rangle$ is cyclic. Since cyclic groups of equal order are isomorphic, we have $G \cong Z_p$. □

Roadmap

Normal Subgroups

Definition

The element gng^{-1} is the *conjugate* of $n \in N$ by g. The set $gNg^{-1} = \{gng^{-1} \mid n \in N\}$ is the *conjugate* of N by g. If $gNg^{-1} = N$, the element g is said to *normalize* N.

Definition

The subgroup N of a group G is *normal* if $gNg^{-1} = N$, or equivalently $gN = Ng$, for all $g \in G$, i.e. if every element of G normalizes N. This relation is denoted as $N \trianglelefteq G$.

Simple Groups

Definition

A nontrivial group G is *simple* if its only normal subgroups are the identity and itself.

Theorem (Feit-Thompson)

If G is a simple group of odd order, then $G \cong Z_p$ for some prime p.

Classification of Finite Simple Groups

Theorem (Classification Theorem, Gorenstein)

Every finite simple group is isomorphic to one of the following:

i *A cyclic group of prime order;*

ii *An alternating group;*

iii *A member of one of sixteen infinite families of groups of Lie type; or*

iv *One of twenty-six sporadic groups not isomorphic to any of the above groups.*

Roadmap

Abelian Simple Groups

Theorem

Abelian simple groups are cyclic groups of prime order.

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Every subgroup of an abelian group is normal.

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Lemma

Every subgroup of an abelian group is normal.

Proof.

Let G be an abelian group and let $H \leq G$. Consider an element $x \in gHg^{-1}$. Since G is abelian and $g, h \in G$, we have

$$x = (gh)g^{-1} = (hg)g^{-1} = h \in H \Rightarrow gHg^{-1} \subseteq H \Rightarrow H \trianglelefteq G$$

Abelian Simple Groups

Theorem

Abelian simple groups are cyclic groups of prime order.

Proof.

($\Rightarrow$) *If G is a simple abelian group, then the order of G is prime.*

- Suppose that G is a simple abelian group and consider $\langle g \rangle \leq G$ where $g \in G$ is a nonidentity element.
- Since G is abelian, every subgroup of G is normal. Since G is simple, we must have $\langle g \rangle = G$ and G be of finite order.
- Let $|g| = |G| = p$. FSOC assume that $p = mn$ is a composite number. Then $\langle g^m \rangle \trianglelefteq G$, but G is simple, so p must be prime.

Abelian Simple Groups

Proof.

($\Leftarrow$) *If the order of G is prime, then G is a simple abelian group.*

- Similar to the forward direction

Abelian simple groups are cyclic groups of prime order.

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