Introduction
Jaeyi

Hello, my name is Jaeyi Song and I am a freshman. My interests include science research, music, and playing with my dog.
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Groups
Introduction to Groups

Definition

A group \((G, \ast)\) is a set \(G\) with a binary operation \(\ast\) that has three requirements satisfied:

1. **Associativity:** \(a \ast (b \ast c) = (a \ast b) \ast c\) for all elements \(a, b, c \in G\).

2. **Identity:** there is an element \(e \in G\) in which \(a \ast e = e \ast a = a\) for all elements of \(G\). The identity for groups under multiplication is 1, under addition it is 0.

3. **Inverse:** For an element \(a \in G\), there is the inverse of \(a\) (let's say \(b\)) that satisfies \(a \ast b = b \ast a = e\). 
Introduction to Groups

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Example 1

The group \((\mathbb{Z}/n\mathbb{Z}, +)\), which is the set \(\{0, 1, 2, \ldots, n-1\}\) under addition taken modulo \(n\), is a group.

1. This set satisfies associativity because addition is associative. Addition fulfills \((a + b) + c = a + (b + c)\).

2. Identity is 0 because for addition, 0 will always be identity. Identity is any number that produces \(a\) for \(a + e = e + a\).

3. Inverse of \(x\) will be \(n - x\). This is an example of a cyclic group, which is a special type of group in which every element can be written as iterated copies of a single element \(a\) called a generator of \(G\).
Example 1

**Example**

The group \((\mathbb{Z}/n\mathbb{Z}, +)\), which is the set \(\{0, 1, 2, \ldots, n - 1\}\) under addition taken modulo \(n\), is a group.
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This is an example of a cyclic group, which is a special type of group in which every element can be written as iterated copies of a single element \( a \) called a generator of \( G \).
Example 2

The set $\text{GL}_2(\mathbb{R})$ of invertible $2 \times 2$ real matrices is a group under matrix multiplication.

1. Matrix multiplication is associative, so the binary operation here is associative.
2. The identity matrix is
   \[
   \begin{pmatrix}
   1 & 0 \\
   0 & 1
   \end{pmatrix}
   \]
3. The inverse of the $2 \times 2$ matrix
   \[
   \begin{pmatrix}
   a & b \\
   c & d
   \end{pmatrix}
   \]
   is
   \[
   \begin{pmatrix}
   d & -b \\
   -c & a
   \end{pmatrix}
   \]
   which is in $\text{GL}_2(\mathbb{R})$. 
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3. The inverse of the $2 \times 2$ matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is

$$\frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

which is in $GL(2, \mathbb{R})$.  

Example 3

The free group on two elements $\langle a, b \rangle$ consists of all words formed by $a, b, a^{-1}, b^{-1}$.

1. It is associative because it is essentially concatenation of words.
2. The identity is the empty word, usually denoted $e$.
3. The inverse of every word can be formed by reversing the order and then taking the inverse of each letter.

*Note that this group is not commutative.*
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Nonexample 1

The set $\text{Mat}_2(\mathbb{R})$ is not a group under multiplication because not every matrix has an inverse. For example, \[
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\] does not have a multiplicative inverse because the determinant is 0.
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Integers under multiplication \((\mathbb{Z}, \times)\) are not a group. This set is not a group because the inverse does not exist. For instance, there is no inverse of 2 since \(\frac{1}{2}\) is not an integer.
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Generators and Relations
Free Group

Definition

The free group on elements $\langle x_1, x_2, \ldots, x_n \rangle$ consists of all finite-length words formed by $x_1, x_2, \ldots, x_n, x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}$.

The free group on one element is $\mathbb{Z}$.

The free group on two elements was discussed in Example 3.
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The free group on one element is \( \mathbb{Z} \).

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Generators and Relations

Definition
Consider the free group on $n$ elements, $x_1, x_2, \ldots, x_n$. Let $r_1, r_2, \ldots, r_m$ be elements in this group (these are just words).
The group $\langle x_1, x_2, \ldots, x_n | r_1, r_2, \ldots, r_m \rangle$ is the quotient we get by setting each $r_i$ equal to identity.
Generators and Relations

Definition

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\[
\langle x_1, x_2, \ldots, x_n \mid r_1, r_2, \ldots, r_m \rangle
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Presentation of a Group

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A presentation of a group, $G$, is an expression of $G$ in terms of generators and relations (shown in previous slide).
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Example 1

The group \( \mathbb{Z}/3\mathbb{Z} \) has a presentation \( \langle x | x^3 = e \rangle \).

The \( x \) represents the element 1, so \( x^3 = e \) just means that \( 1 + 1 + 1 = 0 \) (mod 3).
Example 1

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The $x$ represents the element 1, so $x^3 = e$ just means that $1 + 1 + 1 = 0 \pmod{3}$. 
Example 2

The group $\mathbb{Z}_2$ has a presentation $\langle x, y \mid xy = yx \rangle$.

The $x$ and $y$ represent elements $(1, 0)$ and $(0, 1)$, and the relation $xy = yx$ just means that $x$ and $y$ commute.
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The group $\mathbb{Z}^2$ has a presentation $\langle x, y \mid xy = yx \rangle$. 
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Example 3

The symmetric group \( S_4 \) has presentation

\[
\langle x_1, x_2, x_3 | x_2^2 = x_2 x_1 x_2 = x_2 x_3 x_2 = e \rangle.
\]

The \( x_i \) represents the transpositions \((i, i+1)\).
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The symmetric group $S_4$ has presentation

$$\langle x_1, x_2, x_3 \mid x_1^2 = x_2^2 = x_3^2 = (x_1x_2)^3 = (x_2x_3)^3 = e \rangle.$$
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The symmetric group $S_4$ has presentation

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The $x_i$ represents the transpositions $(i, i + 1)$. 
Propositions

Every group has a presentation. Every group has presentation
\[ \langle \{ x_g \} \mid x_g x_g' = x_g g g' \forall g, g' \in G \rangle. \]
(Note that the number of generators and relations may be infinite, which is ok).
This is really large to work with by hand, so our examples have much nicer presentations!
Propositions

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Fun problems
Problem 1: Unlooping a rope

Consider two poles that extend infinitely to the sky (and imagine we are living on the \( \mathbb{R}^2 \) plane). You want to loop a rope around the two poles and connect the ends such that the rope cannot be removed from the poles. One simple way to do this is: However, you want to be able to remove the rope by removing either one of the poles. In the picture above, removing a pole does not untangle the rope from the remaining pole. How can you do it?
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![Diagram of two poles connected by a loop.]

However, you want to be able to remove the rope by removing either one of the poles. In the picture above, removing a pole does not untangle the rope from the remaining pole.
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![Diagram of two poles with a rope looped around them.]

However, you want to be able to remove the rope by removing either one of the poles. In the picture above, removing a pole does not untangle the rope from the remaining pole. How can you do it?
Problem 1: Loops as group elements

Fix a base point away from the poles. A loop (beginning and ending at this base point) going counterclockwise around the left pole is denoted as \(a\) and a loop going counterclockwise around the right pole is denoted as \(b\). Loops (beginning and ending at the base point, up to homotopy) form a group by concatenation, with inverse being the reverse direction of the loop. This is called the fundamental group: in this case, the group is \(\langle a, b \rangle\).
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Loops (beginning and ending at the base point, up to homotopy) form a group by concatenation, with inverse being the reverse direction of the loop. This is called the fundamental group: in this case, the group is $\langle a, b \rangle$. 
Problem 1: Reformulation in group theory

A loop is an element of this group \langle a, b \rangle. A loop that is entangled around the poles and cannot be removed is an element that is not the identity. Removing the left pole is the same as setting \(a\) to be the identity element. Similarly, removing the right pole is the same as setting \(b\) to be the identity element.

We must find an element \(x \in \langle a, b \rangle\) that is not identity, but when either \(a\) or \(b\) is set to identity, \(x\) becomes the identity.
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Removing the left pole is the same as setting $a$ to be the identity element. Similarly, removing the right pole is the same as setting $b$ to be the identity element.
We must find an element $x \in \langle a, b \rangle$ that is not identity, but when either $a$ or $b$ is set to identity, $x$ becomes the identity.
Problem 1: Resolution

An element that satisfies these conditions is $aba^{-1}b^{-1}$.

Generalization: If instead of 2 poles, there are $n$ poles, can you find a loop which cannot be disentangled, but once any of the $n$ poles are removed, then the loop can be removed?
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Let \( a_1, a_2, \ldots, a_n \) be the generators of the fundamental group, where \( a_i \) is the counterclockwise loop around the \( i \)th pole.
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Let \( a_1, a_2, \ldots, a_n \) be the generators of the fundamental group, where \( a_i \) is the counterclockwise loop around the \( i \)th pole.
Let \( x_{n-1} \) be the solution representing \( n - 1 \) poles. The element \( x_{n-1}a_nx_{n-1}^{-1}a_n^{-1} \) represents the solution for \( n \) poles.
Problem 1: Resolution

Let $a_1, a_2, \ldots, a_n$ be the generators of the fundamental group, where $a_i$ is the counterclockwise loop around the $i$th pole. Let $x_{n-1}$ be the solution representing $n - 1$ poles. The element $x_{n-1}a_nx_{n-1}^{-1}a_n^{-1}$ represents the solution for $n$ poles. Why? When either one of the poles from 1 to $n - 1$ are removed, $x_{n-1}$ becomes the identity and the element becomes $a_n a_n^{-1}$, which is identity. If the $n$th pole is removed, the element becomes $x_{n-1}x_{n-1}^{-1}$, which is also identity.
The Alphabet group

Let's consider the free group generated by 26 generators, say $a, b, c, d, \ldots, x, y, z$. Now impose the relations of homophones: that is, for every pair of words which are homophones (i.e., read and red), set them equal (i.e., read = red, where the generators are being multiplied). What is this group?

Answer: There are many ways to arrive at the same answer. Here is one plausible solution.
The Alphabet group

Let’s consider the free group generated by 26 generators, say a,b,c,d,...,x,y,z. Now impose the relations of homophones: that is, for every pair of words which are homophones (i.e. read and red), set them equal (i.e., read=red, where the generators are being multiplied). What is this group?
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(1) $by = bye \implies e = 1$
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(5) $whole = hole \quad \Rightarrow \quad w = 1$
(6) $hour = our \quad \Rightarrow \quad h = 1$
(7) $in = inn \quad \Rightarrow \quad n = 1$
(8) $knot = not \quad \Rightarrow \quad k = 1$
(9) $die = dye \quad \Rightarrow \quad y = 1$
(10) $ad = add \quad \Rightarrow \quad d = 1$
(11) $all = awl \quad \Rightarrow \quad l = 1$
(12) $arc = ark \quad \Rightarrow \quad c = 1$
Possible Solutions continued...
Possible Solutions continued...

(13) \( ate = eight \implies g = 1 \)
(14) \( base = bass \implies s = 1 \)
(15) \( berry = bury \implies r = 1 \)
(16) \( boos = booze \implies s = 1 \)
(17) \( bat = batt \implies t = 1 \)
(18) \( check = cheque \implies q = 1 \)
(19) \( idle = idol \implies o = 1 \)
(20) \( lam = lamb \implies b = 1 \)
(21) \( coo = coup \implies p = 1 \)
(22) \( faze = phase \implies f = 1 \)
(23) \( genes = jeans \implies j = 1 \)
(24) \( flex = flecks \implies x = 1 \)
(25) \( gamma = gama \implies m = 1 \)
Explanation of Solution:

All letters except $v$ are identity. Merriam-Webster finds that there are also no relations in $v$, so it turns out the quotient group is just $\langle v \rangle \cong \mathbb{Z}$. 
Explanation of Solution:

All letters except $\nu$ are identity. Merriam-Webster finds that there are also no relations in $\nu$, so it turns out the quotient group is just $\langle \nu \rangle \cong \mathbb{Z}$. 
Acknowledgements

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