From Loops to Differential Forms: A Sampler of Algebraic Invariants of Topological Spaces

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Topological Spaces

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What is a topological space more generally?

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In $\mathbb{R}^3$, we have an intuitive idea of what a “space”is.

What is a topological space more generally?
A TOPOLOGICAL SPACE is a collection of points with some sort of notion of “closeness” of points, but no numeric measurement of distance.

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Two topological spaces X and Y are HOMOTOPY EQUIVALENT there exists a continuously deformation of X onto Y, and vice versa.

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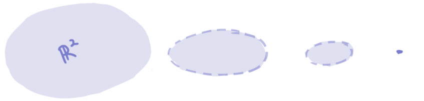

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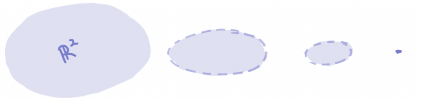
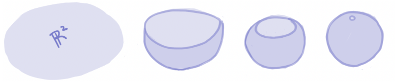


$\mathbb{R}^2$ is homotopy equivalent to a 2-sphere without a point.



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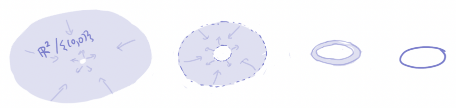
Two topological spaces X and Y are HOMOTOPY EQUIVALENT there exists a continuously deformation of X onto Y, and vice versa.

More Examples:
$\mathbb{R}^n$ with the origin removed is homotopy equivalent to the $(n-1)$-sphere.



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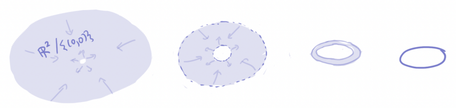
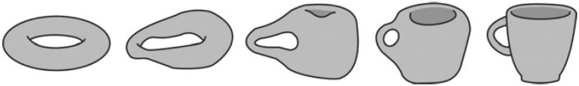


A doughnut is homotopy equivalent to a coffee cup.



Algebraic Invariants

In algebraic topology, we study topological spaces up to homotopy equivalence. We do so by assigning algebraic structures to topological spaces that are invariant up to homotopy equivalence.

In this presentation, we will introduce three different invariants:

- The Fundamental Group
- Singular Homology Groups
- De Rham Cohomology Groups

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- De Rham Cohomology Groups

As an example, we will use these invariants to show that spheres of different dimensions are not homotopy equivalent.

Paths and Loops

Definition

A **path** in a toplogical space X is a continuous map $\gamma : [0,1] \to X$. A path is a **loop** if $\gamma(0) = \gamma(1)$; we call $\gamma(0)$ the **basepoint**.

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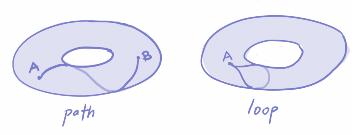
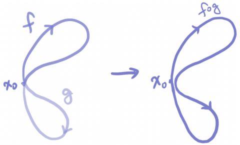


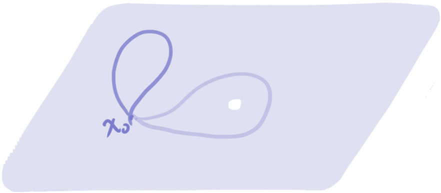
We can CONCATENATE two loops of the same basepoint, f and g, by attaching one to the end of another: $f \circ g(s) = \begin{cases} f(2s) & 0 \leq s \leq \frac{1}{2} \\ g(2s-1) & \frac{1}{2} \leq s \leq 1 \end{cases}$.



The Fundamental Group

Definition (Fundamental Group)

The **fundamental group** of X at basepoint x_0 is $\pi_1(X, x_0)$, where $\pi_1(X, x_0)$ is the set of homotopy equivalence classes of loops with basepoint x_0. The binary operation for the group is concatenation of loops: $[f][g] = [f \circ g]$.



Example: Fundamental Group of the Circle

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What is the fundamental group of the circle, $\pi_1(S^1)$?

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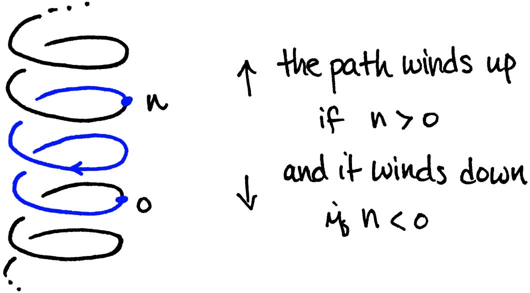
$\pi_1(S^1) \cong \mathbb{Z}$.



Fundamental Group of n-spheres?

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Hence S^1 is not homotopy equivalent to S^n for $n > 1$.

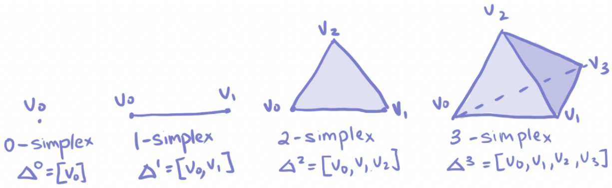
How do we show that different higher dimensional spheres are not homotopic equivalent? We use another algebraic invariant: SINGULAR HOMOLOGY!

Standard n-simplex

Definition (Standard n-simplex)

A **standard n-simplex** is an n-dimensional equilateral triangle.
More formally,

$$\Delta^n = \left\{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} \,\middle|\, \sum_{i=0}^{n} t_i = 1,\ t_i \geq 0\ \forall i \right\}.$$



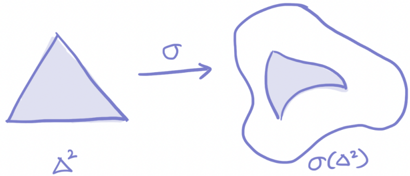
We write: $\Delta^n = [v_0, \dots, v_n]$, where $v_0, \dots, v_n$ are the vertices of the n-simplex.

Singular n-simplex

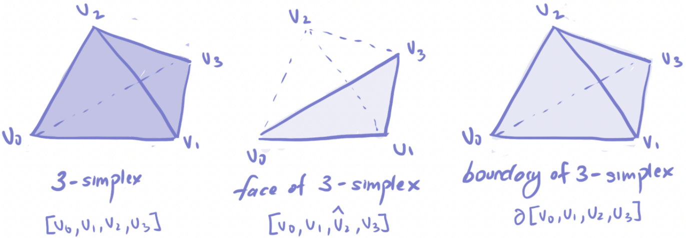
Definition (Singular n-simplex)

A **singular n-simplex** is a standard n-simplex, Δ^n, mapped onto a topological space, X:

$$\sigma : \Delta^n \to X.$$



Faces and Boundaries



Deleting a vertex of a n-simplex gives $(n-1)$-simplex, which we call a FACE. The union of the $n+1$ faces form the BOUNDARY of the n-simplex, notated $\partial\Delta^n$.

We denote the $(n-1)$-simplex with vertex v_i excluded as $[v_0, \ldots, \hat{v}_i, \ldots, v_n]$.

Boundary Map

Let $C_n(X)$ be the free module over $\mathbb{R}$ generated by all singular n-simplicies $\sigma : \Delta^n \to X$.

Definition (Boundary Map)

The **boundary map** $\partial_n : C_n(X) \to C_{n-1}(X)$ is given by

$$\partial_n(\sigma) = \sum_{0 \leq i \leq n} (-1)^i \sigma \big| [v_0, \ldots, \hat{v}_i, \ldots, v_n],$$

where σ is a singular n-simplex, and the right hand side is the alternating sum of the restriction of σ to the faces of the n-simplex.

Chain Complexes

The boundary maps ∂_n satisfy $\partial_n \circ \partial_{n+1} = 0$ for all n. Hence the groups $C_*(X)$ are an example CHAIN COMPLEX.

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A **chain complex** (D_*, ∂_*) is sequence of homomorphisms of abelian groups

$$\cdots \to D_{n+1} \xrightarrow{\partial_{n+1}} D_n \xrightarrow{\partial_n} D_{n-1} \to \cdots \to D_1 \xrightarrow{\partial_1} D_0 \to \cdots$$

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where $\partial_n \circ \partial_{n+1} = 0$ for all n.

Definition (Homology Group)

The n**th homology group** of the chain complex (D_*, ∂_*) is the quotient $\text{Ker } \partial_n / \text{Im } \partial_{n+1}$.

Singular Homology

Definition (Singular Homology Group)

The n**th singular homology group** (with coefficients in $\mathbb{R}$) $H_n(X;\mathbb{R})$ is the nth homology group of the chain complex $(C_*(X), \partial_*)$:

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If two topological spaces are homotopy equivalent, then they have the same singular homology groups.

Example

The singular homology groups of any contractible topological space X (such as $\mathbb{R}^n$) are isomorphic to those of a point, i.e., $H_0(X;\mathbb{R}) = \mathbb{R}$ and $H_k(X;\mathbb{R}) = 0$ for $k > 0$.

Singular Homology of Sphere

Proposition (Homology of S^n for $n > 0$)

For $n > 0$,

$$H_0(S^n; \mathbb{R}) \cong H_n(S^n; \mathbb{R}) \cong \mathbb{R}$$

and $H_k(S^n; \mathbb{R}) \cong 0$ for $k \neq 0, n$.

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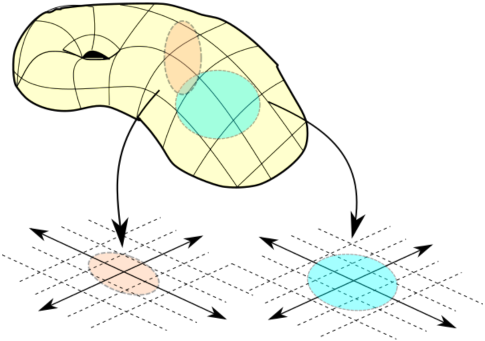
Therefore, two spheres S^m and S^n are homotopy equivalent if and only if $m = n$!

Manifolds



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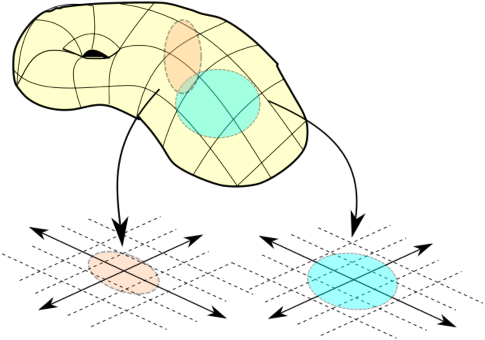

Definition (Manifold)

Intuitively, an n-dimensional **manifold** M is a topological space such that each point in M has a neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$.

Differential Forms

Let $(x_1, ..., x_n)$ be the standard coordinates in $\mathbb{R}^n$, and $dx_1, \ldots, dx_n$ the standard basis of the cotangent space at the origin.
Then Ω^* is the graded exterior algebra on $\{dx_1, \ldots, dx_n\}$. As vector spaces,

$$\begin{aligned} \Omega^0 &= \mathbb{R} \\ \Omega^1 &= \mathbb{R}\{dx_1, \ldots, dx_n\} \\ \Omega^2 &= \mathbb{R}\{dx_i dx_j, \forall i < j\} \\ &\vdots \\ \Omega^n &= \mathbb{R}\{dx_1 \ldots dx_n\}. \end{aligned}$$

Definition (Differential Form)

We define the **differential forms** on $\mathbb{R}^n$ to be elements of

$$\Omega^*(\mathbb{R}^n) = \{\text{smooth functions on } \mathbb{R}^n\} \otimes_{\mathbb{R}} \Omega^*.$$

If $\omega \in \Omega^q(\mathbb{R}^n)$, we say that ω is a q-**form** over $\mathbb{R}^n$.

De Rham Complex

Definition (de Rham Complex)

The **de Rham complex** of $\mathbb{R}^n$ is a chain complex $\Omega^*(\mathbb{R}^n)$ equipped with a differential $d : \Omega^q(\mathbb{R}^n) \to \Omega^{q+1}(\mathbb{R}^n)$:

$$0 \to \Omega^0(\mathbb{R}^n) \to \ldots \to \Omega^{n-1}(\mathbb{R}^n) \to \Omega^n(\mathbb{R}^n) \to 0.$$

Example

Let $f \in \Omega^0(\mathbb{R}^n)$, so f is a smooth function. Then, we have

$$df = \sum_{1 \leq i \leq n} \frac{\partial f}{\partial x_i} \otimes dx_i \in \Omega^1(\mathbb{R}^n).$$

De Rham Cohomology

Because a manifold is locally Euclidean and differential forms can be defined by gluing local forms together, we can extend the definition of $(\Omega^*(\mathbb{R}^n), d)$ to a chain complex $(\Omega^*(M), d)$ for an arbitrary smooth manifold M, called the DE RHAM COMPLEX of M.

Definition (de Rham Cohomology)

The q**th de Rham cohomology group** of M is the q-th homology group of the chain complex $(\Omega^*(M), d)$, i.e.,

$$H^q_{DR}(M) = \frac{\ker\{d : \Omega^q(M) \to \Omega^{q+1}(M)\}}{\operatorname{im}\{d : \Omega^{q-1}(M) \to \Omega^q(M)\}}.$$

Theorem (Homotopy Invariance)

If two smooth manifolds are homotopy equivalent, then they have the same de Rham cohomology groups.

Poincaré Duality

Definition (Compact)

A manifold is **compact** if it is closed and bounded. For example, S^n is compact, but $\mathbb{R}^n$ is not.

Definition (Orientability)

A manifold is **orientable** if and only if it has a global non-vanishing n-form. In other words, M is orientable if and only if there exists a form $\omega \in \Omega^n(M)$ such that $\omega \neq 0$ for all points in M.

Theorem (Poincaré Duality)

If M is a n-dimensional, compact, smooth, and orientable manifold, then there exists an isomorphism

$$H^k_{DR}(M) \cong H^{n-k}_{DR}(M).$$

Relation to Singular Homology with coefficients in $\mathbb{R}$

Theorem (De Rham Theorem)

Given a compact, smooth manifold M, the De Rham cohomology groups of M are isomorphic to the the singular homology groups of M:

$$H^k_{\mathrm{DR}}(M) \cong H_k(M; \mathbb{R}).$$

Relation to Singular Homology with coefficients in $\mathbb{R}$

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Theorem (Poincaré Duality for Singular Homology)

If M is a n-dimensional, compact, smooth, and orientable manifold, then there exists an isomorphism

$$H_k(M; \mathbb{R}) \cong H_{n-k}(M; \mathbb{R}).$$

Examples Revisited

Example (n-sphere)

Earlier, we saw that the singular homology of a sphere for $n > 0$ are

$$H_0(S^n;\mathbb{R}) \cong H_n(S^n;\mathbb{R}) \cong \mathbb{R}$$

$$\text{and } H_k(S^n;\mathbb{R}) \cong 0 \text{ for } 0 < k \neq n,$$

which matches the Poincaré Duality for singular homology!

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Example ($\mathbb{R}^n$)

The singular homology of $\mathbb{R}^n$ is

$$H_*(\mathbb{R}^n; \mathbb{R}) = \begin{cases} \mathbb{R} & \text{in dimension } 0 \\ 0 & \text{elsewhere.} \end{cases}$$

This does not match the Poincaré Duality for singular homology because $\mathbb{R}^n$ is not compact.

Acknowledgements

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- The MIT PRIMES program
- Our families
- The audience for listening to this presentation :)

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