The Probabilistic Method and the Lovász Local Lemma

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Presentation Overview

Problem

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Basic Idea

Let S be a random set and A be the property we want to find.
$\Pr[S \text{ has } A] > 0 \implies$ there exists some set with the property A.

Ramsey Numbers

Definition

The Ramsey number $R(k, l)$ is the smallest $n \in \mathbb{N}$ such that any edge two-coloring of K_n contains either a red K_k or a blue K_l.

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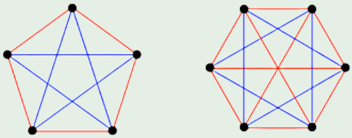
Example

$R(3, 3) = 6$. First note that $R(3, 3) > 5$:



Consider any vertex v in K_6. WLOG, it has 3 red edges to u_1, u_2, u_3. To not form a red triangle, all edges between these three must be blue, which would form a blue triangle. So $R(3, 3) \leq 6$.

Ramsey Numbers

Theorem

For all $k \geq 3$,

$$R(k, k) > 2^{k/2}.$$

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Proof.

Randomly color the edges of K_n.

For any set S of k vertices, let A_S be the event that S is monochromatic.

$\Pr[A_S] = 2^{1-\binom{k}{2}}$.

We want $\Pr\left[\bigcap \overline{A_S}\right] \geq 1 - (\sum \Pr[A_S]) = 1 - \binom{n}{k} 2^{1-\binom{k}{2}} > 0$.

$\binom{n}{k} 2^{1-\binom{k}{2}} < \frac{n^k}{k!} \cdot \frac{2^{1+k/2}}{2^{k^2/2}} < \frac{n^k}{2^{k^2/2}}$ for $k \geq 3$. □

Dependence

We start with independence...

Let $A_1, A_2, ..., A_n$ be mutually independent events defined on an arbitrary probability space with $\Pr[A_i] = x_i$, then we have:

$$\Pr\left[\bigcap_{i=1}^{n} \overline{A_i}\right] = \prod_{i=1}^{n}(1 - x_i).$$

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Problem

What would happen if $A_1, A_2, ..., A_n$ are not mutually independent?

The Symmetric Lovász Local Lemma

Lemma

Let $A_1, A_2, ..., A_n$ *be events such that for each* $1 \leq i \leq n$, A_i *is mutually independent with all but at most* d *other events* A_j, *and* $Pr[A_i] \leq p$. *If*

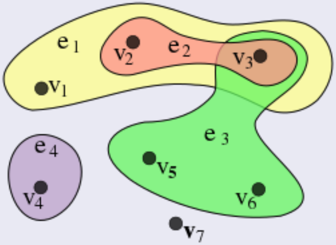
$$ep(d+1) \leq 1$$

then we have $\Pr\left[\bigcap_{i=1}^{n} \overline{A_i}\right] > 0$.

2-Colorable Hypergraphs

Definition

A hypergraph $H = (V, E)$ is a generalization of a graph, where V is a set of vertices, and E is a set of non-empty subsets of V.



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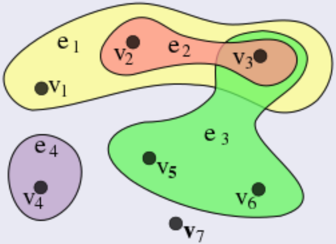

H is vertex 2-colorable if V can be colored with two colors such that no edge is monochromatic.

2-Colorable Hypergraphs

Theorem

Let $H = (V, E)$ be a hypergraph where every edge has at least k elements, and each edge intersects with at most d other edges. If

$$e(d+1) \leq 2^{k-1},$$

then H is vertex 2-colorable.

Proof.

Randomly color the vertices of H.
For any edge $f \in E$, let A_f be the event that f is monochromatic.
$\Pr[A_f] = 2^{1-|f|} \leq 2^{1-k}$.
A_f is independent with all but at most d other events $A_{f'}$.
By the Symmetric Local Lemma, if $e(d+1)2^{1-k} \leq 1$, then
$\Pr\left[\bigcap \overline{A_f}\right] > 0$. □

Ramsey Numbers (continued)

Theorem

If $e\left(\binom{k}{2}\binom{n-2}{k-2}+1\right)2^{1-\binom{k}{2}} \leq 1$, *then* $R(k,k) > n$. *So,*

$$R(k,k) > \frac{\sqrt{2}}{e}(1+o(1))k2^{k/2}.$$

Ramsey Numbers (continued)

Theorem

If $e\left(\binom{k}{2}\binom{n-2}{k-2}+1\right)2^{1-\binom{k}{2}} \leq 1$, *then* $R(k,k) > n$. *So,*

$$R(k,k) > \frac{\sqrt{2}}{e}(1+o(1))k2^{k/2}.$$

Proof.

Randomly color the edges of K_n.
For any set S of k vertices, let A_S be the event that S is monochromatic.
$\Pr[A_S] = 2^{1-\binom{k}{2}}$.
A_S is dependent on A_T only if they share an edge: $|S \cap T| \geq 2$.
Fixing S, the number of dependent T is $d \leq \binom{k}{2}\binom{n-2}{k-2}$.
If $e\left(\binom{k}{2}\binom{n-2}{k-2}+1\right)2^{1-\binom{k}{2}} \leq 1$, then $\Pr\left[\bigcap \overline{A_S}\right] > 0$.

□

Dependence

Definition

The dependency graph of a set of events $A_1, ..., A_n$ is a graph $D = (V, E)$, which satisfies $V = \{1, 2, ..., n\}$, and for every $1 \leq i \leq n$, the event A_i is mutually independent with all A_j for $(i, j) \notin E$.

The Asymmetric Lovász Local Lemma

Lemma

Let $A_1, A_2, ..., A_n$ *be events and* $D = (V, E)$ *be their dependency digraph. If there exist real numbers* $x_1, x_2, ..., x_n$ *such that* $0 \leq x_i < 1$ *and* $\Pr[A_i] \leq x_i \prod_{(i,j)\in E}(1 - x_j)$ *for all* $1 \leq i \leq n$*, then*

$$\Pr\left[\bigcap_{i=1}^{n} \overline{A_i}\right] \geq \prod_{i=1}^{n}(1 - x_i).$$

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References

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