The Probabilistic Method and the Lovász Local Lemma

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Problem
We want to prove the existence of a certain combinatorial structure.
The Probabilistic Method

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Basic Idea
Let $S$ be a random set and $A$ be the property we want to find. 
$\Pr[S \text{ has } A] > 0 \iff$ there exists some set with the property $A$. 

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Definition

The **Ramsey number** $R(k, l)$ is the smallest $n \in \mathbb{N}$ such that any edge two-coloring of $K_n$ contains either a red $K_k$ or a blue $K_l$. 

Example

$R(3, 3) = 6$.

First note that $R(3, 3) > 5$:
Consider any vertex $v$ in $K_6$. WLOG, it has 3 red edges to $u_1, u_2, u_3$. To not form a red triangle, all edges between these three must be blue, which would form a blue triangle. So $R(3, 3) \leq 6$. 

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Ramsey Numbers

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**Example**

$R(3, 3) = 6$. First note that $R(3, 3) > 5$:

Consider any vertex $v$ in $K_6$. Without loss of generality (WLOG), it has 3 red edges to $u_1, u_2, u_3$. To not form a red triangle, all edges between these three must be blue, which would form a blue triangle. So $R(3, 3) \leq 6$. 
Ramsey Numbers

**Theorem**

For all \( k \geq 3 \),

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R(k, k) > 2^{k/2}.
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Ramsey Numbers

Theorem

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\[ R(k, k) > 2^{k/2}. \]

Proof.

Randomly color the edges of \( K_n \).
For any set \( S \) of \( k \) vertices, let \( A_S \) be the event that \( S \) is monochromatic.
\[ \Pr[A_S] = 2^{1-\binom{k}{2}}. \]

We want
\[ \Pr \left[ \bigcap \overline{A_S} \right] \geq 1 - \left( \sum \Pr[A_S] \right) = 1 - \left( \binom{n}{k} 2^{1-\binom{k}{2}} \right) > 0. \]

\[ \frac{n^k}{k!} \cdot \frac{2^{1+k/2}}{2^{k^2/2}} < \frac{n^k}{2^{k^2/2}} \text{ for } k \geq 3. \]
We start with independence...

Let $A_1, A_2, ..., A_n$ be mutually independent events defined on an arbitrary probability space with $\Pr[A_i] = x_i$, then we have:

$$\Pr\left[\bigcap_{i=1}^{n} \overline{A_i}\right] = \prod_{i=1}^{n} (1 - x_i).$$
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Problem

What would happen if $A_1, A_2, ..., A_n$ are not mutually independent?
The Symmetric Lovász Local Lemma

Let $A_1, A_2, \ldots, A_n$ be events such that for each $1 \leq i \leq n$, $A_i$ is mutually independent with all but at most $d$ other events $A_j$, and $\Pr[A_i] \leq p$. If

$$e p(d + 1) \leq 1$$

then we have $\Pr \left[ \bigcap_{i=1}^{n} \overline{A_i} \right] > 0$. 
A hypergraph $H = (V, E)$ is a generalization of a graph, where $V$ is a set of vertices, and $E$ is a set of non-empty subsets of $V$. A hypergraph $H = (V, E)$ is vertex 2-colorable if $V$ can be colored with two colors such that no edge is monochromatic.
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$H$ is vertex 2-colorable if $V$ can be colored with two colors such that no edge is monochromatic.
**Theorem**

Let $H = (V, E)$ be a hypergraph where every edge has at least $k$ elements, and each edge intersects with at most $d$ other edges. If
e(d + 1) \leq 2^{k-1},$

then $H$ is vertex 2-colorable.

**Proof.**

Randomly color the vertices of $H$.

For any edge $f \in E$, let $A_f$ be the event that $f$ is monochromatic.

$Pr[A_f] = 2^{1-|f|} \leq 2^{1-k}$.

$A_f$ is independent with all but at most $d$ other events $A_{f'}$. By the Symmetric Local Lemma, if $e(d + 1)2^{1-k} \leq 1$, then

$Pr \left[ \bigcap A_f \right] > 0.$
Theorem

If \( e \left( \binom{k}{2} \binom{n-2}{k-2} + 1 \right) 2^{1-\binom{k}{2}} \leq 1 \), then \( R(k, k) > n \). So,

\[
R(k, k) > \frac{\sqrt{2}}{e} (1 + o(1))k2^{k/2}.
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Proof.

Randomly color the edges of \( K_n \).
For any set \( S \) of \( k \) vertices, let \( A_S \) be the event that \( S \) is monochromatic.
\[
\Pr[A_S] = 2^{1-\binom{k}{2}}.
\]
\( A_S \) is dependent on \( A_T \) only if they share an edge: \( |S \cap T| \geq 2 \).
Fixing \( S \), the number of dependent \( T \) is \( d \leq \binom{k}{2} \binom{n-2}{k-2} \).
If \( e \left( \binom{k}{2} \binom{n-2}{k-2} + 1 \right) 2^{1-\binom{k}{2}} \leq 1 \), then \( \Pr \left[ \bigcap \overline{A_S} \right] > 0 \).
Definition

The **dependency graph** of a set of events \( A_1, ..., A_n \) is a graph \( D = (V, E) \), which satisfies \( V = \{1, 2, ..., n\} \), and for every \( 1 \leq i \leq n \), the event \( A_i \) is mutually independent with all \( A_j \) for \((i, j) \notin E\).
Lemma

Let $A_1, A_2, ..., A_n$ be events and $D = (V, E)$ be their dependency digraph. If there exist real numbers $x_1, x_2, ..., x_n$ such that $0 \leq x_i < 1$ and $\Pr[A_i] \leq x_i \prod_{(i,j) \in E} (1 - x_j)$ for all $1 \leq i \leq n$, then

$$\Pr \left[ \bigcap_{i=1}^{n} \overline{A_i} \right] \geq \prod_{i=1}^{n} (1 - x_i).$$
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