Positivity properties of the q-hit numbers

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Abstract

We consider the problem of counting matrices over a finite field with fixed rank and support contained in a fixed set. The count of such matrices gives a q-analogue of the classical rook number, but it is known not to be polynomial in q in general. We use inclusion-exclusion on the support of the matrices and the orbit counting method of Lewis et al. to show that the residues of these functions in low degrees are polynomial. We define a generalization of the rook and hit numbers over certain classes of graphs. This provides us a formula for residues of the q-rook and q-hit numbers in low degrees. We analyze the residues of the q-hit number and show that the coefficient of $q-1$ in the q-hit number is always non-negative.

1 Introduction

The rook and hit numbers are defined by the number of placements of non-attacking rooks on subsets of the squares of a chess board. They were first studied by I. Kaplansky and J. Riordan in [KR46]. The rook and hit numbers were later given a q-analogue by M. Garsia and J.B. Remmel for Ferrers boards in [GR86], called the q-rook and q-hit numbers. These are polynomials in q with non-negative coefficients. A second q-rook number based on counting the number of matrices of a given rank and with zeroes in prescribed positions (outside the "board") was given in [Lew+11]. This q-rook number was shown to not always be a polynomial in q by J.R. Stembridge [Ste98]. These definitions of q-rook number coincide for Ferrers boards by work of Haglund in [Hag97]. These Lewis et. al q-rook numbers have applications in coding theory, explored in [Rav15]. J.B. Lewis and A.H. Morales in [LM20] gave a q-analogue of the hit numbers that, when the board is a Ferrers board, coincides with Garsia and Remmel's hit numbers. Again, these are not always polynomial. The q-hit numbers are defined via inclusion-exclusion in terms of the Lewis et. al q-rook numbers, and are conjectured to have many strong non-negativity properties, including being non-negative for fixed q and any board.

We first define the classical rook and hit numbers. Consider an m by n matrix B whose entries are 0 or 1. We can think of B as a board consisting of a subset of the cells of an $m \times n$ grid (with $m \leq n$) that we denote by $[m] \times [n]$. Define the rook number $r_i(B)$ as the number of ways to place i rooks on the cells of B such that no two attack each other. Also define the hit number $h_i(B)$ as the number of ways to place m non-attacking rooks in the $[m] \times [n]$ grid with exactly i rooks in B [KR46]. They are related by the following equation:

$$\sum_{i=0}^{n} h_i(B)t^i = \sum_{i=0}^{n} r_i(B)\frac{(n-i)!}{(n-m)!}(t-1)^i. \tag{1}$$

Define the support of a matrix A to be the set $\mathrm{supp}(A)$ with $(i,j) \in \mathrm{supp}(A)$ if $A_{i,j}$ is nonzero. We sometimes think of the support as a zero-one matrix for convenience in formulae. For board $B \subseteq [m]\times[n]$, define $\mathfrak{m}_i(B,q)$ as the number of m by n matrices in $\mathbb{F}_q$ with support contained in B and rank i. Lewis et al.[Lew+11, Prop. 5.1] give the following:

$$\mathfrak{m}_i(B,q) \equiv r_i(B)(q-1)^i \mod (q-1)^{i+1}.$$

This means that as a number for fixed q, $\mathfrak{m}_i(B,q)$ is always divisible by $(q-1)^i$. We can define $M_i(B,q) = \mathfrak{m}_i(B,q)/(q-1)^i$. Thus $M_i(B,q)$ is an integer for fixed q. Moreover, if $\mathfrak{m}_i(B,q)$ is a polynomial in q, then $M_i(B,q)$ must also be a polynomial in q. Furthermore, since $M_i(B,q) \equiv r_i(B) \pmod{q-1}$, this means $M_i(B,q)$ is a q-analogue of the rook numbers. We know that $\mathfrak{m}_i(B,q)$ is not necessarily a polynomial in q. However, for certain classes of boards, this count has been proven to be polynomial.

Lewis and Morales defined a q-analogue of the hit numbers in [LM20, Eq. (1.3)] for a board $B \subseteq [m]\times[n]$ and q fixed:

$$\sum_{i=0}^{n} H_i(B,q)t^i = q^{\binom{m}{2}} \sum_{i=0}^{n} M_i(B,q) \cdot \frac{[n-i]!_q}{[n-m]!_q}(t-1)(tq^{-1}-1)\dots(tq^{-(i-1)}-1).$$

The motivation of this paper were the following conjectures given by Lewis and Morales:

Conjecture. *[LM20, Conjecture 6.7] Is it true for every permutation ω and every rank r that the polynomial $H_r(\overline{I_\omega}, x+1)$ has positive coefficients in the variable x?*

and

Conjecture. *[LM20, Conjecture 6.3] Given any board $B \subseteq [m]\times[n]$, rank r, and prime power q, the q-hit number $H_r(B,q)$ is non-negative.*

These led us to our generalized conjecture:

Conjecture (Conjecture 4.1)**.** *Let B be a board, and $P \in \mathbb{Z}[x]$ be a polynomial of degree $k-1$ such that $P(x) = H_i(B,x+1) \pmod{x^k}$ for all x in an unbounded subset of $\mathbb{Z}$. Then, P has non-negative coefficients.*

The q-rook and q-hit numbers are in fact polynomial modulo $(q-1)^6$ (Theorem 3.10). If P is a polynomial satisfying the hypotheses of the conjecture, its constant term is $h_i(B)$ and thus is non-negative, so the $k=1$ case of this conjecture is known. We show the first-degree coefficient is positive by finding a new formula for it, proving our conjecture for the case $k=2$. We also make progress toward the higher-k cases, although we do not expect our methods to extend past $k=6$. As $q=x+1$, these results do not imply analogous results for the coefficients of q, which would be interesting to study.

We define a generalized rook number for bipartite graphs with respect to any arbitrary board B in the following way: for a graph G, let $r_{G,k}(B)$ be the number of subboards $C \subseteq B$ such that the bipartite graph with incidence matrix C has a color-preserving isomorphism to the disjoint union of G with k disjoint edges. Thus, when G is the empty graph, $r_{G,k}(B)$ is the normal $r_k(B)$ rook number. We also define a generalized hit number $h_{G_k}(B)$ for these graphs (see Definition 3.6).

We give a formula for $M_i(B,q) \pmod{(q-1)^2}$ in terms of these new rook numbers in the following theorem.

Theorem (Theorem 3.8)**.** *For board $B \subseteq [m]\times[n]$ and integer d,*

$$M_d(B,q) \equiv (q-1)\big(r_{\mathrm{ZG},d-2}(B) - r_{\mathrm{SG},d-2}(B) + r_{\mathrm{WRG},d-1}(B) + r_{\mathrm{WCG},d-1}(B)\big) + r_d(B) \pmod{(q-1)^2}.$$

This means that if $M_i(B,q)$ is written as a polynomial in $q-1$, then the coefficient of $q-1$ is non-negative and not dependent on q. We also show that $M_i(B,q) \pmod{(q-1)^6}$ is always a polynomial, and similarly get that $H_i(B,q) \pmod{(q-1)^6}$ is always polynomial (see theorem 3.10).

Next, in Theorem 4.6, we give a formula for $H_i(B,q) \pmod{(q-1)^2}$ in terms of our generalized hit numbers. By finding inequalities of these generalized hit numbers, we show the following theorem:

Theorem (Theorem 4.12)**.** *For a board $B \subseteq [n]\times[n]$, if $H_i(B,q) \equiv C_i(B)(q-1) + h_i(B) \pmod{(q-1)^2}$, then $C_i(B) \geq 0$.*

This is partial evidence for Lewis and Morales's conjecture [LM20, Conjecture 6.7], although our result applies to all boards.

Outline. In Section 2, we give definitions of the q-rook and q-hit numbers and recall important results from the literature. In Section 3, we give a formula for $\mathfrak{m}_d(B,q) \pmod{(q-1)^{d+2}}$ in terms of generalized rook numbers of B. We also study the refinement of the set of matrices with rank d and support in B to show that $\mathfrak{m}_d(B,q) \pmod{(q-1)^{d+6}}$ must be a polynomial for any board B. In Section 4, we use our formula for $M_i(B,q) \pmod{(q-1)^2}$ to find an equation for $H_i(B,q) \pmod{(q-1)^2}$ through manipulating the formula relating the q-rook and q-hit numbers. We also show that $H_i(B,q) \pmod{(q-1)^2}$ has non-negative coefficients in $q-1$.

2 Background information

In this section, we give the definitions and background information about the q-rook and q-hit numbers, and then we review important past results.

First, consider the q-analogues of the natural numbers

$$[i]_q = (q^i - 1)/(q-1) = 1 + q + \cdots + q^{i-1}.$$

We define

$$[n]!_q = [n]_q[n-1]_q \dots [1]_q$$

and

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{[n]!_q}{[m]!_q[n-m]!_q}.$$

Define the q-Pochhammer symbol as

$$(t;q)_k = \prod_{i=0}^{k-1}(1-tq^i) = (1-t)(1-tq)\dots(1-tq^{k-1}),$$

and define the q-hit numbers for a board $B \subseteq [m]\times[n]$, where $m \leq n$, with the equation [LM20]:

$$\sum_{i=0}^{n} H_i(B,q)t^i = q^{\binom{m}{2}} \sum_{i=0}^{m} M_i(B,q)\frac{[n-i]!_q}{[n-m]!_q}(-1)^i(t;q^{-1})_i. \tag{2}$$

The q-hit numbers are a valid q-analogue, as shown by Lewis and Morales in [LM20, Prop. 3.3], which states that $H_i(B,q) \equiv h_i(B) \pmod{q-1}$. When q is not 1, there is no known combinatorial interpretation of $H_i(B,q)$ for general boards B. However, the numbers are conjectured to be positive by Lewis and Morales in [LM20, Conjecture 6.3].

Lewis and Morales also showed the following.

Proposition 2.1 ([LM20, Prop. 3.5])**.** *For any board B, we can compute the q-hit and q-rook numbers in terms of each other in the following way:*

$$H_k(B,q) = q^{\binom{k+1}{2}+\binom{m}{2}} \sum_{i=k}^{m} M_i(B,q)\cdot\frac{[n-i]!_q}{[n-m]!_q}\begin{bmatrix} i \\ k \end{bmatrix}_q (-1)^{i+k}q^{-ik}$$

and

$$M_k(B,q) = q^{\binom{k}{2}-\binom{m}{2}}\frac{[n-m]!_q}{[n-k]!_q}\sum_{i=k}^{m} H_i(B,q)\begin{bmatrix} i \\ k \end{bmatrix}_q.$$

By this proposition, if all $M_i(B,q) \in \mathbb{Z}[q]$ for $k \leq i \leq m$, then $H_i(B,q) \in \mathbb{Z}[q]$, and vice versa. In fact, this implies that all $M_i(B,q) \in \mathbb{Z}[q]$ if and only if all $H_i(B,q) \in \mathbb{Z}[q]$. We also have $H_m(B,q) = M_m(B,q)$.

Example 2.2. It is important to note that $H_i(B,q)$ is not a polynomial in q for all choices of B. A counterexample to this is the Fano board $F \subseteq [7] \times [7]$:

$$\begin{bmatrix} * & * & 0 & 0 & 0 & * & 0 \\ 0 & * & * & 0 & 0 & 0 & * \\ * & 0 & * & * & 0 & 0 & 0 \\ 0 & * & 0 & * & * & 0 & 0 \\ 0 & 0 & * & 0 & * & * & 0 \\ 0 & 0 & 0 & * & 0 & * & * \\ * & 0 & 0 & 0 & * & 0 & * \end{bmatrix}.$$

In [Ste98], Stembridge found that

$$H_7(F,q) = M_7(F,q) = (x+1)^3(x^{11} + 17x^{10} + 135x^9 + 650x^8 + 2043x^7 + 4236x^6$$

$$+5845x^5 + 5386x^4 + 3260x^3 + 1236x^2 + 264x + 24 - Z_2x^6),$$

where $x = q-1$ and $Z_2 = 1$ for odd q and 0 for even q.

This shows that the q-rook number $M_r(B,q)$ is not always a polynomial in q. See [Sta] for further discussion on non-polynomiality of this and related counting problems over $\mathbb{F}_q$.

Remark 2.3. The Fano Board is the minimal board (in terms of the dimensions of the board) where $M_r(F,q)$ is not polynomial for some r. In fact, it is not polynomial for all $r \geq 4$ [Ste98, Theorem 8.2].

3 Orbits of matrices

In this section, we study a group action on matrices with a fixed rank and support. Following [Lew+11], we find the size of the orbit of a given matrix with rank d. Many orbits have size divisible by $(q-1)^{d+2}$, and we are able to enumerate the orbits which do not, which gives a formula for $M_d(B,q) \pmod{(q-1)^2}$. We use the same technique to show polynomiality of $\mathfrak{m}_d(B,q)$ modulo $(q-1)^{d+6}$.

3.1 Counting matrices by support

We show a relation between the maximal rook placement of a board and the maximal rank of a matrix with support in that board.

Theorem 3.1. *Consider a board B, and let k be the maximal number of non-attacking rooks that can be placed in B. For any matrix M with support in B, the rank of M is at most k.*

Proof. Let the rank of an m by n matrix M with support in B be k. We show that we can place at least k non-attacking rooks on B. Let $v_1, v_2, \ldots, v_k$ be k rows of M that are linearly independent, which exist because the rank is k. Now, let G be the bipartite graph formed with $a_1, a_2, \ldots, a_k$ nodes on the left and $b_1, b_2, \ldots, b_n$ nodes on the right, with an edge between a_i and b_j if the jth element in v_i is nonzero.

A matching in G with k edges corresponds to k cells in B, no two of which are in the same row nor column. By Hall's marriage theorem [Hal35], if, for every set S of a_is, there are at least $\#S$ nodes incident to some vertex in S, then a maximal matching exists.

For the sake of contradiction, assume there are fewer than $\#S$ nodes incident to at least one vertex in S. Without loss of generality, let these nodes be $a_1, a_2, \ldots, a_{\#S}$ in S, and the nodes incident to these be

$b_1, b_2, \ldots, b_i$ for $i < \#S$. This corresponds to the first $\#S$ rows only having entries in the first i columns for $i < \#S$. Therefore the rank of $v_1, v_2, \ldots, v_{\#S}$ is at most i, so they are not linearly independent, a contradiction.

We conclude that M has rank at most k, where k is the maximal number of non-attacking rooks. ☐

Now, we define the following:

Definition 3.2. Define $S_q(B, d)$ as the set of m by n matrices A such that the rank of A is d and the support of A is *exactly* B.

Since $S_q(B, d)$ is invariant under permutations of the rows and columns of B, we can define $S_q(G, d)$ for a graph G so that $S_q(B, d) = S_q(G(B), d)$.

Let $T_q(m, n, B, d)$ be the set of m by n matrices with support contained in B and rank d. We have the following relations:

$$T_q(m, n, B, d) = \bigcup_{C \subseteq B} S_q(C, d)$$

and

$$\mathfrak{m}_d(B, q) = \sum_{C \subseteq B} \#S_q(C, d).$$

Inverting this relationship using Möbius inversion, we get

$$\#S_q(B, d) = \sum_{C \subseteq B} (-1)^{|B|-|C|} \mathfrak{m}_d(C, q).$$

Define $\operatorname{maxhit}(B)$ as the maximum number of non-attacking rooks that can be placed on B. If $\operatorname{maxhit}(C) < d$, then $\mathfrak{m}_d(C, q) = 0$ by Theorem 3.1. Thus our two equations become:

$$T_q(B, d) = \bigcup_{C \subseteq B, \operatorname{maxhit}(C) \geq d} S_q(C, d)$$

and

$$\#S_q(B, d) = \sum_{C \subseteq B, \operatorname{maxhit}(C) \geq d} (-1)^{|B|-|C|} \mathfrak{m}_d(C, q).$$

Remark 3.3. By Remark 2.3 and this relation, the Fano board is also the minimal board such that $\#S_q$ is non-polynomial.

Proposition 3.4. *Let B be a board, and define the bipartite graph $G(B)$ with nodes $v_1, v_2, \ldots, v_m$ and $w_1, w_2, \ldots, w_n$ with edges (v_i, w_j) if $(i, j) \in B$. We show that, for fixed q, the number $\#S_q(B, d)$ is divisible by $(q-1)^{m+n-C(G(B))}$, where $C(G(B))$ is the number of connected components of $G(B)$.*

Proof. We mimic the proof of [Lew+11, Prop. 5.1]. Let $A \in S_q(B, d)$ be a matrix. Let $(\mathbb{F}_q^\times)^l$ be the set of diagonal $l \times l$ matrices with each diagonal entry nonzero. Now, consider the group action $(\mathbb{F}_q^\times)^m \times (\mathbb{F}_q^\times)^n$ on S_q defined by $(X, Y) \cdot A = XAY^{-1}$. The support of XAY^{-1} is still exactly B because X and Y are diagonal matrices. Define $x_1, x_2, \ldots, x_m$ and $y_1, y_2, \ldots, y_n$ as the diagonal entries of X and Y, in that order. We show that (X, Y) stabilizes A if, for each $a_{i,j}$, we have $x_i = y_j$. This is because, if $XAY^{-1} = A$, then $XA = AY$. Then $(XA)_{i,j}$ (the element on the ith row and jth column of XA) is $x_i a_{i,j}$, and similarly $(AY)_{i,j}$ is $a_{i,j} y_j$. Thus if $a_{i,j} \neq 0$, then $x_i = y_j$.

This means there are $(q-1)^{C(G(A))}$ choices for X and Y, because for each connected component in $G(A)$, we have $q-1$ ways to choose those elements in X, Y over $\mathbb{F}_q$. By the orbit-stabilizer theorem, since there are $(q-1)^{m+n}$ ways to choose X and Y, the orbit of A is $(q-1)^{m+n-C(G(A))}$. Finally, this implies that $\#S_q(B, r)$ is divisible by $(q-1)^{m+n-C(G(B))}$, since we can partition $S_q(B, r)$ into orbits of size $(q-1)^{m+n-C(G(B))}$. ☐

3.2 Classifying orbits of size $(q-1)^{d+1}$

In this subsection, we find the number of matrices A such that the orbit of A is either of size $(q-1)^d$ or $(q-1)^{d+1}$. If the size of the orbit is greater than $(q-1)^{d+1}$, then it must be divisible by $(q-1)^{d+2}$.

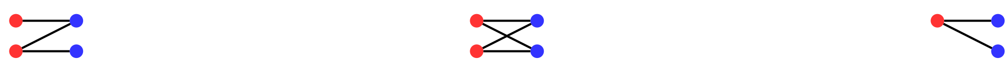
Lewis and Morales [Lew+11, Prop. 5.1] showed that the number of orbit of size $(q-1)^d$ is exactly $\#T_1(B,d)$, where $\#T_1(B,d)$ is defined as the number of ways to place d non-attacking rooks on B, so $\#T_1(B,d) = r_d(B)$. Now, we want to count the number of matrices A of rank d with orbit size exactly $(q-1)^{d+1}$. This means $C(G(\text{supp}(A))) = m+n-d-1$.

In this case, $G(\text{supp}(A))$ is heavily constrained. For each possible graph G, the number of matrices A is the product of $\#S_q(G,d)$ with the number of boards $B' \subseteq B$ such that $G(B')$ is isomorphic to G.

We can define a generalized rook number as follows.

Definition 3.5. Let e be the bicolored graph with two vertices and one edge, and let e^i be the disjoint union $\underbrace{e \sqcup \cdots \sqcup e}_{i \text{ times}}$. For a board $B \subseteq [m] \times [n]$ and a bi-colored graph F with x row and y column vertices, we define $r_{F,i}(B)$ as the number of boards $\sigma \subseteq [m] \times [n]$ with $G(\sigma) \cong F \bigsqcup e^i$ and $\sigma \subseteq B$.

First, we figure out the possible isomorphism classes of G. We know that G is bipartite, so we refer to the vertices of one part as "rows" and the vertices of the other part as "columns," by analogy to the incidence matrix.



(a) Z graph (b) Shoelace Graph (c) 1-wedge graph

Figure 1: Three graphs for ZG, SG, WRG, and WCG

Definition 3.6. We define four classes of graphs as follows, illustrated by Figure 1:

1. Define ZG as the bipartite graph with 2 rows and 2 columns, which forms a path of length 3 (Figure 1a).
2. Define SG as the bipartite graph with 2 rows and 2 columns, which forms a $K_{2,2}$ (Figure 1b).
3. Define WCG as the bipartite graph with 1 row and 2 columns, which forms a row connected to two columns (Figure 1c).
4. Define WRG as the bipartite graph with 2 rows and 1 column, which forms a column connected to two rows (Figure 1c with red and blue flipped).

Lemma 3.7. *If $G = G(A)$ for some A in an orbit of size $(q-1)^{d+1}$, then G is isomorphic to one of* $\text{ZG} \bigsqcup e^{d-2}, \text{SG} \bigsqcup e^{d-2}, \text{WCG} \bigsqcup e^{d-1}$, *or* $\text{WRG} \bigsqcup e^{d-1}$.

Proof. See Appendix A.1 for proof. □

We give a formula for the q-rook number modulo $(q-1)^2$:

Theorem 3.8. *For a board B and non-negative integer d,*

$$\mathfrak{m}_d(B,q) \equiv r_d(B)(q-1)^d + (q-1)^{d+1}(r_{\text{ZG},d-2}(B) + r_{\text{WRG},d-1}(B) \\ + r_{\text{WCG},d-1}(B) + (q-2) r_{\text{SG},d-2}(B)) \pmod{(q-1)^{d+2}}.$$

Proof. We have

$$\mathfrak{m}_d(B,q) \equiv r_d(B)(q-1)^d + \sum_G \sum_{\substack{B' \\ G(B')=G}} \#S_q(B',d) \pmod{(q-1)^{d+2}},$$

but $\#S_q(B',d) = \#S_q(G,d)$ for all these B', so

$$\mathfrak{m}_d(B,q) \equiv r_d(B)(q-1)^d + \sum_G r_G(B)\#S_q(G,d) \pmod{(q-1)^{d+2}}.$$

Case 1: First, we calculate $\#S_q(\mathrm{WCG}\bigsqcup e^{d-1},d)$ and $\#S_q(\mathrm{WRG}\bigsqcup e^{d-1},d)$. Let the two edges connecting the last row to the two columns be e_1, e_2. If each entry is nonzero, then the rank of $\mathrm{WCG}\bigsqcup e^{d-2}$ is exactly d (since each row is linearly independent). Thus $\#S_q(\mathrm{WCG}\bigsqcup e^{d-1},d) = \#S_q(\mathrm{WRG}\bigsqcup e^{d-1},d) = (q-1)^{d+1}$.

Case 2: Next, we calculate $\#S_q(\mathrm{ZG}\bigsqcup e^{d-2},d)$. If every entry is nonzero, then each row would still be linearly independent, so the rank is exactly d. Thus $\#S_q(\mathrm{ZG}\bigsqcup e^{d-2},d) = (q-1)^{d+1}$.

Case 3: Lastly, we calculate $\#S_q(\mathrm{SG}\bigsqcup e^{d-2},d)$. If every entry is nonzero, then the SG submatrix of the $\mathrm{SG}\bigsqcup e^{d-2}$ matrix must have rank exactly 2. This is a 2 by 2 with rank exactly 2. There are $(q-1)^3(q-2)$ such matrices: choose 3 cells, then the remaining cell can be all but one value. Thus $\#S_q(\mathrm{SG}\bigsqcup e^{d-2},d) = (q-1)^{d+1}(q-2)$.

Now, we can write $\mathfrak{m}_d(B,q)$ modulo $(q-1)^{d+2}$ as the following:

$$\begin{aligned}\mathfrak{m}_d(B,q) &\equiv r_d(B)(q-1)^d + (q-1)^{d+1}(r_{\mathrm{ZG},d-2}(B) + r_{\mathrm{WRG},d-1}(B) \\ &\quad + r_{\mathrm{WCG},d-1}(B) + (q-2)r_{\mathrm{SG},d-2}(B)) \pmod{(q-1)^{d+2}}.\end{aligned} \tag{3}$$

□

Example 3.9. For example, in the $B = [2]\times[2]$ board with $r=2$,

$$\#T_q(B) = (q^2-1)(q^2-q) \equiv 2(q-1)^2 + 3(q-1)^3 \pmod{(q-1)^4}.$$

3.3 Polynomiality of $\mathfrak{m}_d(B,q)$ modulo $(q-1)^{d+6}$

In this subsection, we show that the q-rook and q-hit numbers are polynomial modulo low powers of $q-1$.

Theorem 3.10. *For any board B, we have that $\mathfrak{m}_d(B,q) \pmod{(q-1)^{d+6}}$ is a polynomial with integer coefficients.*

Proof. Consider $\mathfrak{m}_d(B,q) \pmod{(q-1)^{d+6}}$. By Proposition 3.4, we know each matrix A is part of an orbit of size $(q-1)^c$, for some integer c. Then, if a board A is part of an orbit of size greater than $(q-1)^{d+6}$, the size of the orbit would be divisible by $(q-1)^{d+6}$, so we do not need to consider it.

Thus consider a board A that is part of an orbit of $d+t$ for some integer $0 \le t \le 5$. This means that $G(A)$, the bipartite graph with incidence matrix A, has $m+n-d-t$ connected components. Let there be $d+a$ nonzero rows and $d+b$ nonzero columns. Since the rank is d, we must have $a,b \ge 0$. Let T be the bipartite subgraph of G formed by these $d+a$ rows and $d+b$ columns. Each of the other rows and columns form their own connected component, so there are $m+n-2d-a-b$ such connected components. This means that T has $d+a+b-t$ connected components.

Let there be $d+c$ rows in T that are part of a connected component of size 2; this forms a mapping from the rows to columns of size $d+c$, so there are $d+c$ such columns as well. This corresponds to a

$d+c$ by $d+c$ submatrix, where each column and each row have exactly one cell. The whole matrix can be written like

$$\left(\begin{array}{c|cccc} R & & 0 & & \\ \hline & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ & \vdots & & \ddots & \vdots \\ & 0 & 0 & \cdots & 1 \end{array}\right),$$

for some matrix R up to permutations of rows and columns. Each of these $d+c$ rooks has $(q-1)^{d+c}$ ways of choosing the entries, which is polynomial in q.

Next, we consider $G(R)$. It is a bipartite graph with $a-c$ vertices on the left and $b-c$ on the right. Furthermore, there must be

$$(d+a+b-t)-(d+c)=a+b-c-t$$

connected components in $G(R)$.

Denote the graphs of these connected components as $g_1, g_2, \dots, g_{a+b-c-t}$. We know the rank of R to be $-c$. Since each connected component is independent of the other, if the matrix formed by g_i has rank d_i, then the total rank of R is $d_1+\cdots+d_{a+b-c-t}$, so

$$-c=\sum_{i=1}^{a+b-c-t} d_i.$$

Therefore we must have

$$\#S_q(G(R),-c)=\sum_{\substack{d_1,\dots,d_{a+b-c-t}\geq 0\\ \sum d_i=-c}}\prod_{i=1}^{a+b-c-t}\#S_q(g_i,d_i).$$

We show that the maximal number of rows in some g_i is $t+1$. This is because, each g_i has at least 1 row, and there are $a-c$ rows to choose from, so the maximal number of rows in some g_i is

$$a-c-(a+b-c-t-1)=t+1-b.$$

Since b is non-negative, this means there are at most $t+1$ rows in g_i. Similarly, there are at most $t+1$ columns in g_i. However, since $t\leq 5$, there are at most 6 rows and columns in g_i. By the minimality of the Fano plane, we must have $\#S_q(g_i,d_i)\in\mathbb{Z}[q]$ for all i.

This means $\#S_q(G(R),-c)\in\mathbb{Z}[q]$, so

$$\#S_q(G(T),r)=(q-1)^{d+c}\#S_q(G(R),-c)\in\mathbb{Z}[q].$$

Summing over all boards T, of which there are a constant number of them dependent on n; this means that for all boards T with at least $m+n-d-5$ connected components, or of orbit with size at most $(q-1)^{d+5}$, we get $\#S_q(G(T),d)\in\mathbb{Z}[q]$. Therefore $\mathfrak{m}_d(B,q) \pmod{(q-1)^{d+6}}$ is always a polynomial in q. □

Corollary 3.11. *For a board B, we know that $M_d(B,q)$ and $H_d(B,q) \pmod{(q-1)^6}$ are both polynomials in q.*

Remark 3.12. In fact, $d+6$ is the maximal number c such that $\mathfrak{m}_d(B,q) \pmod{(q-1)^c}$ is always a polynomial. Modulo $(q-1)^{d+7}$, taking the Fano board F and $d=7$, the coefficient of $(q-1)^{13}$ depends on the residue class of $q \pmod 2$. Therefore $\mathfrak{m}_7(F,q) \pmod{(q-1)^{14}}$ can be non-polynomial.

4 Hit numbers modulo $(q-1)^2$

In this section, we consider the q-hit number $H_i(B,q)$. Motivated by [LM20, Conjecture 6.7], we conjecture the following.

Conjecture 4.1. *Let B be a board, and $P \in \mathbb{Z}[x]$ be a polynomial of degree $k-1$ such that $P(x) = H_i(B, x+1) \pmod{x^k}$ for all x in an unbounded subset of $\mathbb{Z}$. Then, P has non-negative coefficients.*

In this section we verify our conjecture for the two lowest-degree coefficients of such a polynomial $P(x)$. By [LM20, Prop. 3.3], we know that $H_i(B, x+1) \equiv h_i(B) \pmod{x}$, which is manifestly non-negative. We wish to find an expression for the residue of $H_i(B, x+1)$ modulo x^2, in the form of

$$H_i(B, x+1) \equiv C_i(B)x + h_i(B) \pmod{x^2}.$$

A priori, $C_i(B)$ might depend on x, as $H_i(B, x+1)$ may not be a polynomial. However, because $H_i(B,q)$ modulo $(q-1)^2$ is an integer polynomial (see Corollary 3.11), we know that $C_i(B, x+1)$ does not depend on x and is an integer. We seek a formula for $C_i(B)$, using Equation (2) and reducing both sides modulo $(q-1)^2$, and then extracting the coefficient of t^i.

We then show that this coefficient $C_i(B)$ is non-negative for any board B.

4.1 Finding $H_i(B,q)$ modulo $(q-1)^2$

In this subsection, we compute $H_i(B,q) \pmod{(q-1)^2}$ from our formula 3.

Lemma 4.2. *We have the following identities:*

$$\begin{aligned} q^n &\equiv n(q-1)+1 \pmod{(q-1)^2}, \\ (n)_q &\equiv \frac{n(n-1)}{2}(q-1)+n \pmod{(q-1)^2}, \\ [n]!_q &\equiv \frac{n!\cdot n\cdot(n-1)}{4}(q-1)+n! \pmod{(q-1)^2}. \end{aligned}$$

Lemma 4.3. *We have the following:*

$$\begin{bmatrix} n \\ i \end{bmatrix}_q \equiv (q-1)\binom{n}{i}\frac{i(n-i)}{2} + \binom{n}{i} \pmod{(q-1)^2}.$$

Proposition 4.4. *For integers a, q, n, we have that*

$$(a;q)_n \equiv (1-a)^n + (q-1)\left(-\binom{n}{2}(1-a)^{n-1}a\right) \pmod{(q-1)^2}.$$

Proof. The proofs of these lemmas are computational and are omitted. ☐

Using $M_i(B,q) \pmod{(q-1)^2}$ and our previous formulas, we find $H_i(B,q) \pmod{(q-1)^2}$. We use the following definitions.

Definition 4.5. For a board $B \subseteq [m]\times[n]$, and a bi-colored graph F with x row and y column vertices, we define $h_{F,i}(B)$ as the number of boards $\sigma \subseteq [m]\times[n]$ with $G(\sigma) \cong F \bigsqcup e^{\min(m-x,n-y)}$, such that $G(\sigma \cap B) \cong F \bigsqcup e^i$.

Now we can state our formula for $C_i(B)$.

Theorem 4.6. *The q-hit number satisfies*

$$H_i(B,q) \equiv C_i(B)(q-1) + h_i(B) \pmod{(q-1)^2},$$

where $C_i(B)$ is given by one of the following formulas.

For rectangular boards $B \subseteq [m]\times[n]$ with $m<n$,

$$\begin{aligned}C_i(B) = h_{\mathrm{ZG},i-2}(B) - h_{\mathrm{SG},i-2}(B) + (n-m+1)h_{\mathrm{WRG},i-1}(B) + \frac{n-i}{n-m}h_{\mathrm{WCG},i-1}(B) - 2h_{\mathrm{ZG},i-1} + 2h_{\mathrm{SG},i-1}\\ - (n-m+1)h_{\mathrm{WRG},i} + \frac{2i-n-1}{n-m}h_{\mathrm{WCG},i} + h_{\mathrm{ZG},i}(B) - h_{\mathrm{SG},i}(B) + \frac{i-1}{n-m}h_{\mathrm{WCG},i+1}(B)\\ + \frac{1}{4}\Big(h_i(B)(-i^2-2in+3i+m^2+2mn-3m) + h_{i+1}(B)(2i+2)(n-1) + h_{i+2}(B)(i+2)(i+1)\Big).\end{aligned}$$

For square boards $B \subseteq [n]\times[n]$,

$$\begin{aligned}C_i(B) = h_{\mathrm{ZG},i-2}(B) - h_{\mathrm{SG},i-2}(B) + h_{\mathrm{WRG},i-1}(B) + h_{\mathrm{WCG},i-1}(B) - 2h_{\mathrm{ZG},i-1}(B) + 2h_{\mathrm{SG},i-1}(B)\\ - h_{\mathrm{WRG},i}(B) - h_{\mathrm{WCG},i}(B) + h_{\mathrm{ZG},i}(B) - h_{\mathrm{SG},i}(B)\\ + \frac{1}{4}\big(h_i(B)(n-i)(3n+i-3) + h_{i+1}(B)(2i+2)(n-1) + h_{i+2}(B)(i+2)(i+1)\big).\end{aligned}$$

Remark 4.7. A priori, $C_i(B)$ is just an integer, but in Theorem 4.12 we show it is non-negative.

We prove this theorem after giving a few preliminary lemmas.

Example 4.8. For example, in $B=[2]\times[2]$ and $i=2$, we have

$$h_i(B) = q^2+q \equiv 3q-1 \pmod{(q-1)^2},$$

so $C_i(B) = 3 = h_{\mathrm{ZG},0}(B) - h_{\mathrm{SG},0}(B)$.

Consider Equation (2):

$$\sum_{i=0}^{m} H_i(B,q)t^i = q^{\binom{m}{2}}\sum_{i=0}^{m} M_i(B,q)\frac{[n-i]!_q}{[n-m]!_q}(-1)^i(t;q^{-1})_i.$$

We wish to evaluate the right hand side modulo $(q-1)^2$ then extract the coefficients for t^i modulo $(q-1)^2$. Let

$$\begin{aligned}M_i(B,q) &\equiv a_i(q-1)+b_i \pmod{(q-1)^2},\\ \frac{[n-i]!_q}{[n-m]!_q} \equiv \binom{n-i}{n-m}_q [m-i]!_q &\equiv c_i(q-1)+d_i \pmod{(q-1)^2},\end{aligned}$$

and

$$(-1)^i(t;q^{-1})_i \equiv e_i(q-1)+f_i \pmod{(q-1)^2}.$$

Then, we have

$$\begin{aligned}\sum_{i=0}^{m} H_i(B,q)t^i &= \sum_{i=0}^{m} M_i(B,q)\frac{[n-i]!_q}{[n-m]!_q}(-1)^i(t;q^{-1})_i\\ &\equiv \sum_{i=0}^{m}(a_i(q-1)+b_i)(c_i(q-1)+d_i)(e_i(q-1)+f_i) \pmod{(q-1)^2}\\ &\equiv (q-1)\sum_{i=0}^{m} a_id_if_i + (q-1)\sum_{i=0}^{m} b_ic_if_i + (q-1)\sum_{i=0}^{m} b_id_ie_i + \sum_{i=0}^{m} b_id_if_i \pmod{(q-1)^2}.\end{aligned}$$

By Equation (3), we know that

$$a_i = r_{\mathrm{ZG},i-2}(B) - r_{\mathrm{SG},i-2}(B) + r_{\mathrm{WRG},i-1}(B) + r_{\mathrm{WCG},i-1}(B), b_i = r_i(B).$$

To calculate c_i and d_i, observe that

$$\frac{[n-i]!_q}{[n-m]!_q} \equiv (q-1)c_i + d_i \equiv \binom{n-i}{n-m}_q [m-i]!_q \pmod{(q-1)^2}.$$

Using Lemma 4.3 and Lemma 4.2, we get

$$\begin{aligned}(q-1)c_i + d_i &\equiv \left((q-1)\binom{n-i}{m-i}\frac{(m-i)(n-m)}{2} + \binom{n-i}{m-i}\right)\left((m-i)!\left(\frac{(m-i)(m-i-1)}{4}(q-1)+1\right)\right) \\ &\equiv (q-1)\binom{n-i}{n-m}(m-i)!\frac{(m-i)(2n-m-i-1)}{4} + \binom{n-i}{m-i}(m-i)! \pmod{(q-1)^2}.\end{aligned}$$

Finally, to calculate e_i and f_i, we get

$$e_i(q-1) + f_i \equiv (-1)^i (t;q^{-1})_i \equiv t^i \left(\frac{1}{t};q\right)_i q^{-\binom{i}{2}}.$$

By Lemma 4.2 and Proposition 4.4, we get

$$\begin{aligned}e_i(q-1) + f_i &\equiv t^i\left(1-\binom{i}{2}(q-1)\right)\left(\left(1-\frac{1}{t}\right)^i - (q-1)\left(\binom{i}{2}\left(1-\frac{1}{t}\right)^{i-1}\frac{1}{t}\right)\right) \\ &\equiv (t-1)^i - (q-1)\left(\binom{i}{2}(t-1)^i + \binom{i}{2}(t-1)^{i-1}\right) \\ &\equiv (t-1)^i - (q-1)(t-1)^{i-1}\binom{i}{2}t \pmod{(q-1)^2}.\end{aligned}$$

Denote $adf = \sum_i a_i d_i f_i, bcf = \sum_i b_i c_i f_i$, and $bde = \sum_i b_i d_i e_i$. First we calculate adf. We have

$$adf = \sum_{i=0}^{m} (r_{\mathrm{ZG},i-2}(B) - r_{\mathrm{SG},i-2}(B) + r_{\mathrm{WRG},i-1}(B) + r_{\mathrm{WCG},i-1}(B))\frac{(n-i)!}{(n-m)!}(t-1)^i.$$

Lemma 4.9. *For boards $B \subseteq [m]\times[n]$ with $m<n$,*

$$\begin{aligned}adf = \sum_{i=0}^{m} t^{i-2}(t-1)\Big[(t-1)\big(h_{\mathrm{ZG},i-2} - h_{\mathrm{SG},i-2} - \frac{i-1}{n-m}h_{\mathrm{WCG},i-1}(B)\big) + \\ t\big((n-m+1)h_{\mathrm{WRG},i-1}(B) + \frac{n-1}{n-m}h_{\mathrm{WCG},i-1}(B)\big)\Big].\end{aligned}$$

For boards $B \subseteq [m]\times[n]$ with $m=n$,

$$adf = \sum_{i=0}^{m} t^{i-2}(t-1)\Big[(t-1)(h_{\mathrm{ZG},i-2} - h_{\mathrm{SG},i-2}) + t(h_{\mathrm{WRG},i-1}(B) + h_{\mathrm{WCG},i-1}(B))\Big].$$

Proof. First we handle the case of rectangular boards inside $[m]\times[n]$. Consider the set S of pairs (ω,σ) of boards $\omega,\sigma \subseteq [m]\times[n]$, with $\omega \subseteq B$, $\omega \subseteq \sigma \subseteq [m]\times[n]$, $G(\omega) \cong \mathrm{ZG}\bigsqcup e^{i-2}$ for some i, and $G(\sigma) \cong \mathrm{ZG}\bigsqcup e^{m-2}$. Assign each pair a weight of $(t-1)^i$, and sum over the weights of all possible pairings; i.e., consider the sum

$$T = \sum_{(\omega,\sigma)\in S} (t-1)^{i(\omega)},$$

where $i(\omega)$ is the unique number such that $G(\omega) \cong \mathrm{ZG} \bigsqcup e^{i-2}$. We can count this by first counting the number of subboards $\omega \subseteq B$ such that $G(\omega) \cong \mathrm{ZG} \bigsqcup e^{i-2}$, then counting the number of boards $\sigma \subseteq [m] \times [n]$ with $G(\sigma) \cong \mathrm{ZG} \bigsqcup e^{m-2}$ such that $\omega \subseteq \sigma$. To count the number of boards $\sigma \subseteq [m] \times [n]$ with $G(\sigma) \cong \mathrm{ZG} \bigsqcup e^{m-2}$ such that $\omega \subseteq \sigma$, consider adding cells to ω in the $m-i$ and $n-i$ empty rows and columns. We need to choose $m-i$ cells on different rows and columns in $[m-i] \times [n-i]$, which can be done in $\frac{(n-i)!}{(n-m)!}$ ways. This gives the formula

$$T = \sum_{i=0}^{n} r_{\mathrm{ZG},i-2} \frac{(n-i)!}{(n-m)!}(t-1)^i.$$

We can also obtain a formula for T by first counting the boards $\sigma \subseteq [m] \times [n]$ with $G(\sigma) \cong \mathrm{ZG} \bigsqcup e^{m-2}$ and then counting the subboards $\omega \subseteq B$ with $\omega \subseteq \sigma$ and $G(\omega) \cong \mathrm{ZG} \bigsqcup e^{j-2}$ for some j. Note that $G(\sigma \cap B)$ is guaranteed to be isomorphic to $\mathrm{ZG} \bigsqcup e^{i-2}$ for some i. If $G(\sigma \cap B) \cong \mathrm{ZG} \bigsqcup e^{i-2}$, then we can sum over all subboards $\omega \subseteq \sigma \cap B$ with $G(\omega) \cong \mathrm{ZG} \bigsqcup e^{j-2}$ for some j. For each j, there are $\binom{i-2}{j-2}$ such subboards: Thinking of ω and $\sigma \cap B$ as incidence matrices, ω must contain the three cells of $\sigma \cap B$ which represent a disconnected ZG, and so ω is given by a choice of $j-2$ cells in the $i-2$ cells of $\sigma \cap B$ which represent disconnected edges. All of these ω have a weight of $(t-1)^j$, and so we get the formula

$$T = \sum_{i=0}^{m} \sum_{\substack{\sigma \\ G(\sigma\cap B)\cong \mathrm{ZG}\bigsqcup e^{i-2}}} \sum_{j=0}^{i-2} \binom{i-2}{j-2}(t-1)^j = \sum_{i=0}^{m} h_{\mathrm{ZG},i-2}(B)(t-1)^2 t^{i-2}.$$

Since these are both formulas for T, we have the relation

$$\sum_{i=0}^{m} r_{\mathrm{ZG},i-2}(B) \frac{(n-i)!}{(n-m)!}(t-1)^i = \sum_{i=0}^{m} h_{\mathrm{ZG},i-2}(B)(t-1)^2 t^{i-2}, \tag{4}$$

which resembles the relation 1 for the classical rook and hit numbers.

Using the same counting technique, we can obtain identical relations for the other isomorphism classes of graph. We have

$$\sum_{i=0}^{m} r_{\mathrm{SG},i-2}(B) \frac{(n-i)!}{(n-m)!}(t-1)^i = \sum_{i=0}^{m} h_{\mathrm{SG},i-2}(B)(t-1)^2 t^{i-2} \tag{5}$$

for the graphs SG,

$$\sum_{i=0}^{m} r_{\mathrm{WRG},i-1}(B) \frac{(n-i)!}{(n-m)!}(t-1)^i = (n-m+1) \sum_{i=0}^{m} h_{\mathrm{WRG},i-1}(B)(t-1) t^{i-1} \tag{6}$$

for the graphs WRG, and

$$\sum_{i=0}^{m} r_{\mathrm{WCG},i-1}(B) \frac{(n-i)!}{(n-m)!}(t-1)^i = \frac{1}{n-m} \sum_{i=0}^{m} h_{\mathrm{WCG},i-1}(B)(t^{i-1}(t-1)(n-1) - (i-1)t^{i-2}(t-1)^2) \tag{7}$$

for the graphs WCG.

In the square board case, the formulas are almost the same. The only one which changes is the final relation for the graphs WCG. If $B \subseteq [n] \times [n]$ is a square board, we have

$$\sum_{i=0}^{n} r_{\mathrm{WCG},i-1}(B)(n-i)!(t-1)^i = \sum_{i=0}^{n} h_{\mathrm{WCG},i-1}(B)(t-1)t^{i-1} \tag{8}$$

for the graphs WCG.

Now we can put these relations together to compute *adf*. To compute *adf* for a board B with $m < n$, using Equation (4), Equation (5), Equation (6), and Equation (7), we get

$$adf = \sum_{i=0}^{m} \Big[h_{\mathrm{ZG},i-2}(B)(t-1)^2t^{i-2} - h_{\mathrm{SG},i-2}(B)(t-1)^2t^{i-2} + h_{\mathrm{WRG},i-1}(B)(n-m+1)(t-1)t^{i-1} + \frac{1}{n-m}h_{\mathrm{WCG},i-1}(B)(t^{i-1}(t-1)(n-1)-(i-1)t^{i-2}(t-1)^2)\Big].$$

This means we have

$$adf = \sum_{i=0}^{m} t^{i-2}(t-1)\Big[(t-1)\big(h_{\mathrm{ZG},i-2}(B) - h_{\mathrm{SG},i-2}(B) - \frac{i-1}{n-m}h_{\mathrm{WCG},i-1}(B)\big) + t\big((n-m+1)h_{\mathrm{WRG},i-1}(B) + \frac{n-1}{n-m}h_{\mathrm{WCG},i-1}(B)\big)\Big].$$

For boards with $m = n$, using Equation (4), Equation (5), Equation (6), and Equation (8), we get

$$adf = \sum_{i=0}^{m} t^{i-2}(t-1)\Big[(t-1)(h_{\mathrm{ZG},i-2}(B) - h_{\mathrm{SG},i-2}(B)) + t(h_{\mathrm{WRG},i-1}(B) + h_{\mathrm{WCG},i-1}(B))\Big].$$

□

To calculate *bcf* and *bde*, we use the following lemma.

Lemma 4.10. *For a board B and fixed k, we have*

$$\sum_{i=0}^{m} r_i(B)\frac{(n-i)!}{(n-m)!}i(i-1)\dots(i-k+1)(t-1)^i = \sum_{i=0}^{m} i(i-1)\dots(i-k+1)(t-1)^k t^{i-k} h_i(B).$$

Proof. Take k derivatives of the equation (1) and then multiply by $(t-1)^k$. □

Now, we calculate *bcf* and *bde*.

Lemma 4.11. *For boards $B \subseteq [m] \times [n]$, we have*

$$bcf = \frac{1}{4}\sum_{i=0}^{m} h_i(B)t^{i-2}\left((2n-m-1)mt^2 - (2n-2)i(t-1)t + 2(t-1)^2\binom{i}{2}\right)$$

and

$$bde = -\sum_{i=0}^{m} h_i(B)\binom{i}{2}t^{i-1}(t-1).$$

Proof. We have:

$$bcf = \sum_{i=0}^{m} b_i c_i f_i = \sum_{i=0}^{m} r_i(B)\frac{(n-i)!}{(n-m)!}\frac{(m-i)(2n-m-i-1)}{4}(t-1)^i$$
$$= \frac{1}{4}\sum_{i=0}^{m} r_i(B)\frac{(n-i)!}{(n-m)!}\Big(i(i-1) - i(2n-2) + (2n-m-1)m\Big).$$

We can apply Lemma 4.10 with $k = 0, 1, 2$ to get

$$bcf = \frac{1}{4}\sum_{i=0}^{m} h_i(B)t^{i-2}\left((2n-m-1)mt^2 - (2n-2)i(t-1)t + 2(t-1)^2\binom{i}{2}\right).$$

For bde, we get

$$bde = -\sum_{i=0}^{m} r_i(B)\frac{(n-i)!}{(n-m)!}(t-1)^{i-1}\binom{i}{2}t.$$

By Lemma 4.10, with $k = 2$, we get

$$bde = -\sum_{i=0}^{m} h_i(B)\binom{i}{2}t^{i-1}(t-1).$$

□

We are now ready to complete the proof of Theorem 4.6.

Proof of Theorem 4.6. Putting everything together, we have

$$\sum_{i=0}^{m} H_i(B,q)t^i \equiv q^{\binom{m}{2}}\big((q-1)(adf+bcf+bde)+bdf\big) \equiv (q-1)(adf+bcf+bde+\binom{m}{2}bdf)+bdf \pmod{(q-1)^2}.$$

We now extract the coefficient of t^i from adf, bcf, bde, and bdf. First, consider the case where $m < n$. The coefficient of bdf is $h_i(B)$. By Lemma 4.9, the coefficient of t^i in adf is

$$h_{\mathrm{ZG},i-2}(B) - h_{\mathrm{SG},i-2}(B) + (n-m+1)h_{\mathrm{WRG},i-1}(B) + \frac{n-i}{n-m}h_{\mathrm{WCG},i-1}(B) - 2h_{\mathrm{ZG},i-1} + 2h_{\mathrm{SG},i-1}$$
$$-(n-m+1)h_{\mathrm{WRG},i} + \frac{2i-n-1}{n-m}h_{\mathrm{WCG},i} + h_{\mathrm{ZG},i}(B) - h_{\mathrm{SG},i}(B) + \frac{i-1}{n-m}h_{\mathrm{WCG},i+1}(B).$$

This means that

$$bcf + bde = \frac{1}{4}\sum_{i=0}^{m} h_i(B)t^{i-2}\big(t^2(-i^2-2in+3i-m^2+2mn-m) + t(2i)(n-1) + i(i-1)\big).$$

We can extract the coefficient of t^i from this. Adding it to the coefficient of t^i from adf, and adding $h_i(B)\binom{m}{2}$, we get, for $m < n$,

$$\begin{aligned}C_i(B) = {} & h_{\mathrm{ZG},i-2}(B) - h_{\mathrm{SG},i-2}(B) + (n-m+1)h_{\mathrm{WRG},i-1}(B) + \frac{n-i}{n-m}h_{\mathrm{WCG},i-1}(B) - 2h_{\mathrm{ZG},i-1} + 2h_{\mathrm{SG},i-1}\\ & - (n-m+1)h_{\mathrm{WRG},i} + \frac{2i-n-1}{n-m}h_{\mathrm{WCG},i} + h_{\mathrm{ZG},i}(B) - h_{\mathrm{SG},i}(B) + \frac{i-1}{n-m}h_{\mathrm{WCG},i+1}(B)\\ & + \frac{1}{4}\Big(h_i(B)(-i^2-2in+3i+m^2+2mn-3m) + h_{i+1}(B)(2i+2)(n-1) + h_{i+2}(B)(i+2)(i+1)\Big).\end{aligned}$$

Now for the square case, we know that the coefficients of t^i for $bcf + bde + \binom{m}{2}bdf$ are still the same. However, since $m = n$, the formula simplifies to

$$\frac{1}{4}\big(h_i(B)(n-i)(3i+i-3) + h_{i+1}(B)(2i+2)(n-1) + h_{i+2}(B)(i+2)(i+1)\big).$$

For adf, we have from Lemma 4.9 that the coefficient of t^i is

$$\begin{aligned}& h_{\mathrm{ZG},i-2}(B) - h_{\mathrm{SG},i-2}(B) + h_{\mathrm{WRG},i-1}(B) + h_{\mathrm{WCG},i-1}(B) - 2h_{\mathrm{ZG},i-1}(B) + 2h_{\mathrm{SG},i-1}(B)\\ &\quad - h_{\mathrm{WRG},i}(B) - h_{\mathrm{WCG},i}(B) + h_{\mathrm{ZG},i}(B) - h_{\mathrm{SG},i}(B),\end{aligned}$$

which means

$$\begin{aligned}C_i(B) = {} & h_{\mathrm{ZG},i-2}(B) - h_{\mathrm{SG},i-2}(B) + h_{\mathrm{WRG},i-1}(B) + h_{\mathrm{WCG},i-1}(B) - 2h_{\mathrm{ZG},i-1}(B) + 2h_{\mathrm{SG},i-1}(B) \\ & - h_{\mathrm{WRG},i}(B) - h_{\mathrm{WCG},i}(B) + h_{\mathrm{ZG},i}(B) - h_{\mathrm{SG},i}(B) \\ & + \frac{1}{4}\big(h_i(B)(n-i)(3n+i-3) + h_{i+1}(B)(2i+2)(n-1) + h_{i+2}(B)(i+2)(i+1)\big).\end{aligned}$$

□

4.2 Positivity in shifted q-hit number coefficient

We prove the $k = 2$ case of Conjecture 4.1. As a special case, we confirm part of a conjecture of Lewis and Morales. In [LM20, Conjecture 6.7] they conjectured that the coefficient of $(q-1)^k$ for the q-hit number of complements of diagram boards $H_r(\overline{I}_\omega, q)$ was non-negative for all k. Our result shows the coefficient of $q-1$ is non-negative.

Theorem 4.12. *For a board $B \subseteq [n] \times [n]$, if $H_i(B, q) \equiv C_i(B)(q-1) + h_i(B) \pmod{(q-1)^2}$, then $C_i(B) \geq 0$.*

Corollary 4.13. *The coefficient of $q-1$ in $H_r(\overline{I}_\omega, q)$ is non-negative for all ω.*

We prove that $C_i(B)$ is positive using a series of inequalities relating our generalized hit numbers to the usual ones.

Lemma 4.14. *For boards $B \subseteq [n] \times [n]$, we have*

$$h_{\mathrm{WRG},i-1}(B) + h_{\mathrm{WCG},i-1}(B) - 2h_{\mathrm{ZG},i-1}(B) + 2h_{\mathrm{SG},i-1}(B) + \frac{1}{4}(2i+2)ih_{i+1}(B) \geq 0 \tag{9}$$

and

$$\begin{aligned} & h_{\mathrm{ZG},i}(B) - h_{\mathrm{SG},i}(B) - h_{\mathrm{WRG},i}(B) - h_{\mathrm{WCG},i}(B) \\ & + \frac{1}{4}(2i+2)(n-i-1)h_{i+1}(B) + \frac{1}{4}(i+1)(i+2)h_{i+2}(B) \geq 0. \end{aligned} \tag{10}$$

Proof. Define a square-chain as a set of cells $t \subset [n] \times [n]$ whose associated graph $G(t)$ consists of the union of $n-2$ disjoint edges and a $K_{2,2}$. For a square-chain t, let $T(t)$ be the set of cells in t corresponding to the $K_{2,2}$. For a board $B \subseteq [n] \times [n]$, define $S_i(B)$ as the set of square chains t such that $|(t \backslash T(t)) \cap B| = i-2$.

For each board $\omega \subseteq [n] \times [n]$ with $G(\omega) \cong \mathrm{ZG} \bigsqcup e^{n-2}$ and $G(\omega \cap B) \cong \mathrm{ZG} \bigsqcup e^{i-2}$, there is exactly one $t \in S_i(B)$ such that $\omega \subseteq t$, and there are no $t \in S_k(B)$ for $i \neq k$ such that $\omega \subseteq t$. The same is true mutatis mutandis for the isomorphism classes SG, WRG, and WCG.

For board $\omega \subseteq [n] \times [n]$ with $G(\omega)$ consisting of n disjoint edges and $G(\omega \cap B)$ consisting of i disjoint edges, there are exactly $\binom{i}{2}$ different $t \in S_i(B)$ such that $\omega \subseteq t$. This is because we choose two cells $c_1, c_2 \in \omega \cap B$ that are part of $T(t)$ in $\binom{i}{2}$ ways, and such a choice fixes the other two cells in $T(t)$. Similarly, there are exactly $i(n-i)$ such $t \in S_{i+1}(B)$ such that $\omega \subseteq t$.

For a square-chain t, let $C_{t,\mathrm{ZG},i}(B)$ be defined as the number of boards $\omega \subseteq [n] \times [n]$ with $G(\omega) \cong \mathrm{ZG} \bigsqcup e^{n-2}$ such that $G(\omega \cap B) \cong \mathrm{ZG} \bigsqcup e^i$ and $\omega \subseteq t$. Define $C_{t,\mathrm{SG},i}(B), C_{t,\mathrm{WRG},i}(B), C_{t,\mathrm{WCG},i}(B)$ similarly. Also, define $C_{t,i}(B)$ as the number of boards $\omega \subseteq [n] \times [n]$, where $G(\omega)$ consists of n disjoint edges such that $\omega \subseteq t$ and $|\omega \cap B| = i$.

We clearly have

$$\sum_{t \in S_i(B)} C_{t,\mathrm{ZG},i-2}(B) = h_{\mathrm{ZG},i-2}(B),$$

because a choice of $\sigma \subset [n] \times [n]$ such that $G(\sigma) \cong \mathrm{ZG} \bigsqcup e^{i-2}$ fixes a unique choice of t in $S_i(B)$. We get similar results for SG, WRG, and WCG.

Similarly, for the usual hit numbers, we have the relations

$$\sum_{t \in S_i(B)} C_{t,i}(B) = \binom{i}{2} h_i(B)$$

and

$$\sum_{t \in S_{i+1}(B)} C_{t,i}(B) = i(n-i)h_i(B).$$

Observe that if $t \in S_i(B)$, then the values $C_{t,\mathrm{ZG},i-2}(B), C_{t,\mathrm{SG},i-2}(B), C_{t,\mathrm{WRG},i-2}(B)$, $C_{t,\mathrm{WCG},i-2}(B)$, $C_{t,i}(B)$, $C_{t,i-1}(B)$, $C_{t,i-2}(B)$ can be determined by the intersection $T(t) \cap B$.

First, for $t \in S_{i+1}(B)$, we have the inequality

$$C_{t,\mathrm{WRG},i-1}(B) + C_{t,\mathrm{WCG},i-1}(B) - 2C_{t,\mathrm{ZG},i-1}(B) + 2C_{t,\mathrm{SG},i-1}(B) + C_{t,i+1}(B) \geq 0.$$

Since the values can be determined by $T(t) \cap B$, and $T(t)$ only has 2^4 subsets, we can check that the inequality holds for each of the 2^4 configurations of $T(t) \cap B$, and conclude that it holds for all t. Now, we have

$$\sum_{t \in S_{i+1}(B)} C_{t,\mathrm{WRG},i-1}(B) + C_{t,\mathrm{WCG},i-1}(B) - 2C_{t,\mathrm{ZG},i-1}(B) + 2C_{t,\mathrm{SG},i-1}(B) + C_{t,i+1}(B) \geq 0$$

which is equivalent to

$$h_{\mathrm{WRG},i-1}(B) + h_{\mathrm{WCG},i-1}(B) - 2h_{\mathrm{ZG},i-1}(B) + 2h_{\mathrm{SG},i-1}(B) + \frac{1}{4}(2i+2)ih_{i+1}(B) \geq 0.$$

Next, for $t \in S_{i+2}(B)$, we have the inequality

$$C_{t,\mathrm{ZG},i}(B) - C_{t,\mathrm{SG},i}(B) - C_{t,\mathrm{WRG},i}(B) - C_{t,\mathrm{WCG},i}(B) + \frac{1}{2}C_{t,i+1}(B) + \frac{1}{2}C_{t,i+2}(B) \geq 0.$$

Again, the inequality holds for each of the 2^4 configurations of $T(t) \cap B$, so this inequality holds for all t. Now, we have

$$\sum_{t \in S_{i+2}(B)} C_{t,\mathrm{ZG},i}(B) - C_{t,\mathrm{SG},i}(B) - C_{t,\mathrm{WRG},i}(B) - C_{t,\mathrm{WCG},i}(B) + \frac{1}{2}C_{t,i+1}(B) + \frac{1}{2}C_{t,i+2}(B) \geq 0$$

which is equivalent to

$$\begin{aligned} h_{\mathrm{ZG},i}(B) - h_{\mathrm{SG},i}(B) - h_{\mathrm{WRG},i}(B) - h_{\mathrm{WCG},i}(B) \\ + \frac{1}{4}(2i+2)(n-i-1)h_{i+1}(B) + \frac{1}{4}(i+1)(i+2)h_{i+2}(B) \geq 0. \end{aligned}$$

□

Now we are ready to complete the proof of Theorem 4.12.

Proof of Theorem 4.12. Coupling inequality (9) and inequality (10) with the two inequalities $\frac{1}{4}(n-i)(3n+i-3)h_i(B) \geq 0$ and $h_{\mathrm{ZG},i-2}(B) - h_{\mathrm{SG},i-2}(B) \geq 0$, when we add all 4 of these inequalities, we get

$$\begin{aligned} C_i(B) = &h_{\mathrm{ZG},i-2}(B) - h_{\mathrm{SG},i-2}(B) + \frac{1}{4}(n-i)(3n+i-3)h_i(B) + h_{\mathrm{WRG},i-1}(B) + h_{\mathrm{WCG},i-1}(B) \\ &-2h_{\mathrm{ZG},i-1}(B) + 2h_{\mathrm{SG},i-1}(B) + h_{\mathrm{ZG},i}(B) - h_{\mathrm{SG},i}(B) - h_{\mathrm{WRG},i}(B) - h_{\mathrm{WCG},i}(B) \\ &+\frac{1}{4}(2i+2)(n-i-1)h_{i+1}(B) + \frac{1}{4}(i+1)(i+2)h_{i+2}(B) \geq 0. \end{aligned}$$

□

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Appendix A Bipartite graphs with enough connected components

Below, we provide a proof for Lemma 3.7.

Lemma A.1 (Lemma 3.7)**.** *If $G = G(A)$ for some A in an orbit of size $(q-1)^{k+1}$, then G is isomorphic to one of* $\mathrm{ZG} \bigsqcup e^{k-2}, \mathrm{SG} \bigsqcup e^{k-2}, \mathrm{WCG} \bigsqcup e^{k-1}$, *or* $\mathrm{WRG} \bigsqcup e^{k-1}$.

Proof. Let T be the set of vertices in G that are incident to at least one edge. Observe that for each vertex in T, it is in a connected component of size at least 2. This means

$$m+n-k-1 = C(G) = C(T) + m + n - \#T \leq m + n - \frac{\#T}{2}.$$

Therefore $\#T \leq 2(k+1)$. Now, observe that there are at least k rows that have degree at least 1. If not, suppose there is a matrix A of rank k such that $G(\mathrm{supp}(A)) = G$. Then, A has rank less than k. Similarly, we see there are at least k columns of degree at least 1. Each of these rows and columns correspond to a vertex in T. Let $\#x, \#y$ be the number of rows and columns in T. The possible $(\#x, \#y)$ pairs are confined to the following possibilities:

$$(k,k),\ (k,k+1),\ (k,k+2),\ (k+1,k+1),\ (k+1,k),\ (k+2,k).$$

Consider the case where $(\#x, \#y) = (k, k+2)$. There exists at least one row with two non-zero elements, otherwise there are at most k columns with nonzero elements. This row connects to 2 columns, so it is in a connected component of at least size 3. This means:

$$m+n-k-1 = C(G) = C(T) + m + n - \#T \leq m + n - \#T + 1 + \frac{\#T-3}{2}.$$

This implies $\frac{\#T+1}{2} \leq k+1$. However, $\#T + 1 = 2k+3$, so this is a contradiction. We get a similar result if $(\#x, \#y) = (k+2, k)$.

Next, if $(\#x, \#y) = (k+1, k+1)$, by the same reasoning as above, all vertices in T must be in connected components of size 2. This means that there is an injective mapping between rows and columns from the graph G, so each row and column contain exactly one nonzero element. However, there are $k+1$ such rows and columns, which means the rank of A is exactly $k+1$, so no such matrices A exists.

If $(\#x, \#y) = (k, k+1)$, then the number of connected components in T must be exactly k, because

$$C(T) = \#T - k - 1 = k.$$

This means each row must be in distinct connected components. Each row must also connect to at least one column. Since there are exactly $k+1$ columns to choose from, there exists one row that connects to two columns, and the rest connect to exactly one. This is the $\mathrm{WCG} \bigsqcup e^{k-1}$ graph, where k is the number of rows. Similarly, for $(\#x, \#y) = (k+1, k)$, we get the $\mathrm{WRG} \bigsqcup e^{k-1}$ graph.

Next, we resolve the case where $(\#x, \#y) = (k, k)$. We have $C(T) = \#T - k - 1 = k - 1$. This means there is exactly one pair of rows, (a, b), that are in the same connected component. There are 2 cases.

Case 1: There is exactly one column c where $(a, c), (b, c) \in E(G)$.

Then, if c is the only column incident to either a or b, we can conclude there exists a different pair of columns in the same connected component. If $G(A)$ is isomorphic to such a graph, then A is equivalent to (up to permutation)

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & \\ \hline & 0 & & T \end{array}\right].$$

The 3 by 3 square matrix has rank at most 2. Since there are $k - 3$ more non-attacking rooks placed on the board, this means the entire matrix has rank at most $k - 1$. Therefore this case produces 0 valid matrices.

If there exists another column d incident to either a or c, without loss of generality, let it be incident to a. Then, c and d are in the same connected component, so all other rows/columns form an injective mapping. This graph is $\mathrm{ZG} \bigsqcup e^{k-2}$, where k is the number of rows/columns.

Case 2: There are two columns c, d where $(a, c), (a, d), (b, c)$, and (b, d) are incident.

Then, every row besides these two must form an injective mapping. This is the $\mathrm{SG} \bigsqcup e^{k-2}$ graph, where k is the number of rows/columns. □

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