

PRIMES General Math Problem Set

PRIMES 2021

Due December 1, 2020

Dear PRIMES applicant:

This is the PRIMES 2021 Math Problem Set. Please send us your solutions as part of your PRIMES application by **December 1, 2020**. For complete rules, see <http://math.mit.edu/research/highschool/primes/apply.php>

- You can type the solutions or write them up by hand and then scan them. Please attach your solutions to the application as a PDF file. The name of the attached file must start with your last name, for example, “etingof-solutions.pdf” or similar. Include your full name in the heading of the file.
- Please write not only answers, but also proofs (and partial solutions/results/ideas if you cannot completely solve the problem). Besides the admission process, your solutions will be used to decide which projects would be most suitable for you if you are accepted to PRIMES.
- Submissions in L^AT_EX are preferred, but handwritten submissions are also accepted.
- You are allowed to use any resources to solve these problems, *except other people’s help*. This means that you can use calculators, computers, books, and the Internet. However, if you consult books or Internet sites, please give us a reference.
- **Note that posting these problems on problem-solving websites before the application deadline is strictly forbidden!** Applicants who do so will be disqualified, and their parents and recommenders will be notified.

Note that some of these problems are tricky. We recommend that you do not leave them for the last day. Instead, think about them, on and off, over some time, perhaps several days. We encourage you to apply if you can solve at least 50% of the problems.

Enjoy!

Why it makes no sense to cheat

PRIMES expects its participants to adhere to MIT rules and standards for honesty and integrity in academic studies. As a result, **any cases of plagiarism, unauthorized collaboration, cheating, or facilitating academic dishonesty during the application process or during the work at PRIMES may result in immediate disqualification from the program, at the sole discretion of PRIMES.** In addition, PRIMES reserves the right to notify a participant’s parents, schools, and/or recommenders in the event it determines that a participant did not adhere to these

expectations. For explanation of these expectations, see *What is Academic Integrity?*, integrity.mit.edu.

Moreover, even if someone gets into PRIMES by cheating, it would immediately become apparent that their background is weaker than expected, and they are not ready for research. This would prompt an additional investigation with serious consequences. By trying to get into PRIMES by cheating, students run very serious risks of exposing their weak background and damaging their college admissions prospects.

General Math Problems

Problem G1. A polynomial $f(x)$ has complex coefficients. It turns out that $f(x) \cdot f'(x)$ is a degree five polynomial whose x^5 , x^4 , x^1 , x^0 coefficients are respectively 3, 10, 25, 12. Determine the polynomial f .

Problem G2. Scientists have found a vaccine that produces undesirable side effects with probability p . Initially, the number p is distributed uniformly across the interval $[0, 0.1]$. To test the vaccine, the scientists test the vaccine on 148374 volunteers and find that no one experiences adverse side effects.

Find the smallest real number λ such that the scientists can assert $p < \lambda$ with probability at least 95%. Round your answer to four significant figures.

Problem G3. Let p be an odd prime number. Calculate the number of triples $(a, b, c) \in \mathbb{F}_p \times \mathbb{F}_p \times \mathbb{F}_p$ for which $a + b + c = a^3 + b^3 + c^3 = 1$.

Problem G4. We roll a fair six-sided die and let s_1 be the result of the roll. Then, we roll s_1 fair six-sided dice and let s_2 be the sum of the rolls. Then, we roll s_2 fair six-sided dice and let s_3 be the sum of the rolls. The process continues to generate an infinite sequence (s_1, s_2, \dots) .

- (a) Find the probability that 3 appears in the sequence.
- (b) Find the expected value of s_n , for each integer n .
- (c) We say the sequence grows exponentially if there exists a constant $c > 1$ such that $s_n > c^n$ for all sufficiently large integers n . Does the sequence grow exponentially almost surely?

Problem G5. For each positive integer $n \geq 4$, find all positive real numbers a_1, a_2, \dots, a_n such that

$$a_i^2 = 19a_{i+1} + 20a_{i+2} + 21a_{i+3}$$

holds for all $i = 1, \dots, n$ with indices taken modulo n .

Problem G6. If s is a finite binary string, then we denote by $f(s)$ the sum of the squares of the lengths of the *consecutive runs* of f . For example, $f(10110001111100) = 1^2 + 1^2 + 2^2 + 3^2 + 5^2 + 2^2 = 44$.

Suppose that a binary string s of length n is specified by letting the i th bit be 1 with probability p_i and 0 with probability $1 - p_i$, all independent. We wish to calculate the expected value of $f(s)$ given the values of n, p_1, p_2, \dots, p_n .

- (a) Exhibit an algorithm with the best runtime you can find, in terms of n .
- (b) Give the best lower bounds you can on the runtime of such an algorithm.