Products of reflections in smooth Bruhat intervals
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Not every set and operation make a group. A group’s set and operation must obey some special rules. These are called \textbf{group axioms}. We won’t have time to go over these.
Introduction

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  *Generators* are elements such that we can write any other element as products of generators. Every element of the group can be written as a word or string in terms of generators.
Definition (Group Example)

The **symmetric group** $S_n$ is the group of permutations of $\{1, \ldots, n\}$. 

Any permutation is the composition of adjacent transpositions, i.e. adjacent swaps. Hence the group's generators are $s_i = (i \ i + 1)$ for each $i = 1, \ldots, n - 1$. 

The group's identity $e$ is the identity permutation. 

Example: The permutation $24135$ in $S_5$ can be obtained by the following:

1. $e = 12345$
2. $s_1 \rightarrow 21345$
3. $s_3 \rightarrow 21435$
4. $s_2 \rightarrow 24135$

So we write $24135 = s_1 s_3 s_2$. (In particular, $s_1 s_3 s_2$ is a word in terms of the letters $s_1, s_2, s_3, s_4$.)

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Coxeter Groups

Definition

Given some $n^2$ integers $m_{i,j}$, with

- $m_{i,i} = 1$ for all $i$, and
- $m_{i,j} \geq 2$ for all $i \neq j$,

the Coxeter Group for this $(m_{ij})$ is a group with generators $s_1, \ldots, s_n$ given by

$$(s_is_j)^{m_{i,j}} = e$$

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Definition

The Coxeter graph for a Coxeter Group $G$ has $n$ vertices $s_1, \ldots, s_n$, and has an edge from $i$ to $j$ if and only if $m_{i,j} \geq 3$. 
$S_n$ is a Coxeter Group

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- $s_i^2 = e$ for any $i$, since swapping $i$ and $i + 1$ twice does nothing.
- $(s_i s_j)^2 = e$ for $j - i \geq 2$. Indeed, “non-adjacent” $s_i$ and $s_j$ commute, i.e. $s_i s_j = s_j s_i$, since these two swaps do not interfere with each other. So $(s_i s_j)^2 = e$. 

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- $(s_is_j)^2 = e$ for $j - i \geq 2$. Indeed, “non-adjacent” $s_i$ and $s_j$ commute, i.e. $s_is_j = s_j s_i$, since these two swaps do not interfere with each other. So $(s_is_j)^2 = e$.
- $(s_is_{i+1})^3 = e$ for $i = 1, \ldots, n - 1$. First swap $(i, i + 1)$, then swap $(i + 1, i + 2)$:

\[
\begin{align*}
    i &\rightarrow i \rightarrow i + 1, \\
    i + 1 &\rightarrow i + 2 \rightarrow i + 2, \\
    i + 2 &\rightarrow i + 1 \rightarrow i,
\end{align*}
\]

which is the 3-cycle $(i, i + 1, i + 2)$, which cycles back to itself upon cubing.
These match up with the Coxeter Group rules! **Hence $S_n$ is a Coxeter Group**, given by the following $m$-values:

- $m_{i,i} = 1$ for all $i$,
- $m_{i,j} = 2$ for all $j - i \geq 2$,
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$S_n$’s Coxeter graph is therefore a line graph:

![Coxeter Graph](image)

where there are $n - 1$ dots, each representing $s_1, \ldots, s_{n-1}$. This is called a Coxeter group of type $A_{n-1}$ or type $S_n$. 
The **Bruhat order** is a **partial order**, or **poset**, on elements of a Coxeter Group.
Brief Introduction to Bruhat Order

The **Bruhat order** is a **partial order**, or **poset**, on elements of a Coxeter Group. For the symmetric group, levels of the poset are given by the number of inversions in the permutation. Covering relations are swaps that increase the number of inversions by exactly 1.
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![Bruhat Order Diagram](image)
A reflection in $S_n$ is an element that swaps two positions.
Smoothness and Background Theorem

- A **reflection** in $S_n$ is an element that swaps two positions.
- Some elements of a Coxeter Group are called **smooth**.

The following theorem was background for our research:

**Theorem (Gilboa and Lapid, 2020)**

For any smooth $w \in S_n$, let \{t_1, \ldots, t_k\} be the set of reflections less than or equal to $w$ in Bruhat order. There exists a (compatible) order $t_1 \prec t_2 \prec \cdots \prec t_k$ for which $t_1 t_2 \cdots t_k = w$.

**Example**

In $S_3$, everything is smooth. The set of reflections less than or equal to $w = s_1 s_2 s_1$ in Bruhat order is \{s_1 s_2 s_1, s_1, s_2\}. The claimed ordering exists: $s_2 \cdot (s_1 s_2 s_1) \cdot s_1 = s_2 s_1 s_2 = s_1 s_2 s_1$. 

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Some elements of a Coxeter Group are called smooth. There is an algebraic geometry viewpoint to smoothness. But in $S_n$, smooth elements can concretely be characterized as all permutations avoiding the patterns 3412 and 4231.
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Our research generalizes Gilboa and Lapid's theorem by proving further structure exists for these reflections in the Bruhat order:

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$e \to t_1 \to t_1 t_2 \to \cdots \to t_1 \cdots t_k = w$.

$e \to t_k \to t_k t_{k-1} \to \cdots \to t_k \cdots t_1$ is a saturated chain in Bruhat order.

Our next goal is to generalize the above further to any compatible order, a kind of order used in the combinatorial constructions for these products of reflections.
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- $e \rightarrow t_1 \rightarrow t_1 t_2 \rightarrow \cdots \rightarrow t_1 \cdots t_k$ is a saturated chain in Bruhat order.
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Our next goal is to generalize the above further to any *compatible order*, a kind of order used in the combinatorial constructions for these products of reflections.
In $S_3$, consider $w = s_1 s_2 s_1 \in S_3$. We use the order $s_2 \prec s_1 s_2 s_1 \prec s_1$:

(Note $s_1 s_2 s_1 = s_2 s_1 s_2$.) Above on the left, the reflections $s_2$, then $s_1 s_2 s_1$, then $s_1$ are what we multiply in covering relations to make a saturated Bruhat chain. The right is a different chain, in the reverse (suffix products) order.
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