Topological Entropy of Simple Braids

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What is a braid?

We can think of a braid as formed by $n$ strands (think of pieces of string) that can cross over and under one another.

Braids are related to other topological objects, including knots and links.

[images from https://arxiv.org/abs/1103.5628]
What is a braid?

The braids on $n$ strands form a group $B_n$. For example, the product of

![Braid diagrams](https://users.math.msu.edu/users/wengdap1/filling_to_cluster.html)
What is a braid?

The multiplication operation in $B_n$ is not commutative in general.

The group $B_n$ is generated by $n-1$ elements $\sigma_1, \ldots, \sigma_{n-1}$.

$$\sigma_i = \begin{bmatrix} \vdots & \vdots \\ \cdots & \times & \cdots \\ \cdots & \vdots & \cdots \\ i & i+1 \\ \vdots & \vdots \\ \end{bmatrix}$$

They satisfy the relations $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i-j| \geq 2$; they also satisfy $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ for $1 \leq i \leq n-2$.

[images from https://arxiv.org/abs/1103.5628]
What is a simple braid?

There is a natural map from $B_n$ to $S_n$ (the group of permutations of \{1, \ldots, n\}) where $\sigma_i$ maps to the transposition swapping $i$ and $i + 1$.

The *simple braids* are natural preimages of the $n!$ elements of $S_n$.

[Image made using https://users.math.msu.edu/users/wengdap1/filling_to_cluster.html]
What is a simple braid?

One notable simple braid is the *half twist*, which corresponds to the permutation $i \mapsto n + 1 - i$. For $n \geq 3$, the square of the half twist generates the center of $B_n$.

[Image from https://arxiv.org/abs/1302.6536]
The Nielsen–Thurston classification

Braids can be classified as
- periodic,
- reducible and not periodic, or
- pseudo-Anosov.
The Nielsen–Thurston classification

A braid is periodic if it can be raised to some power to equal some power of the full twist. For example, cubing this braid gives the full twist.

[images made using https://users.math.msu.edu/users/wengdap1/filling_to_cluster.html]
The Nielsen–Thurston classification

A braid is reducible if it is possible to draw some loops to get something like the image below.

\[ C_{1,1} \quad C_{1,2} \quad C_{1,3} \quad C_{2,1} \quad C_{2,2} \quad C_{3,1} \]

[Image from Juan González-Meneses, “The nth root of a braid is unique up to conjugacy”]
The Nielsen–Thurston classification

A braid that is not periodic or reducible is pseudo-Anosov.

[Image made using https://users.math.msu.edu/users/wengdap1/filling_to_cluster.html]
Topological entropy

If a braid is periodic or is reducible with all components periodic, it’s “orderly” and has topological entropy zero. Otherwise (if it is pseudo-Anosov or is reducible with at least one pseudo-Anosov component), it’s “chaotic” and has positive topological entropy.
Topological entropy

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[image made using https://users.math.msu.edu/users/wengdap1/filling_to_cluster.html]
Topological entropy

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[Image from Benson Farb and Dan Margalit, A Primer on Mapping Class Groups]
Topological entropy

Topological entropy of braids has applications in real life to the mixing of fluids.

The property of having topological entropy zero is preserved under raising to a power.
The Burau representation

There is a useful homomorphism from $B_n$ to the group of invertible $(n-1) \times (n-1)$ matrices whose entries are polynomials with integer coefficients in $t$ and $t^{-1}$.

$\sigma_1 \mapsto \begin{bmatrix} -t & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \sigma_2 \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ t & -t & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$

$\sigma_3 \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & t & -t & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \sigma_4 \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & t & -t \end{bmatrix}$

Kolev found a relationship between the topological entropy of a braid and the eigenvalues of its image under the Burau representation, for $t$ on the unit circle in the complex numbers.
Theorem (R.–Trinh)

The images of simple braids obey certain patterns, as the example below illustrates.

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
4 & 7 & 2 & 5 & 8 & 6 & 1 & 3 \\
\end{array}
\]
Simple braids and the Burau representation

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\]
**Main theorem**

**Theorem (R.–Trinh)**

The proportion of simple braids in $B_n$ that have positive topological entropy goes to 1 as $n$ goes to infinity.
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