A Minkowski-type inequality in AdS-Melvin space

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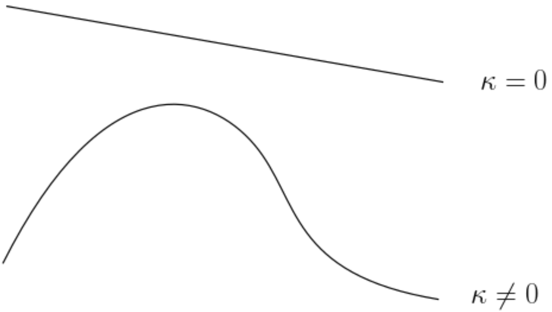
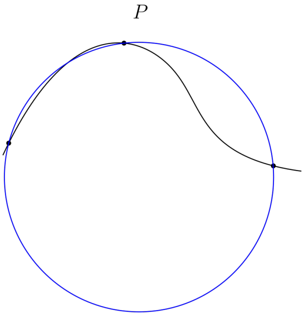
Curvature

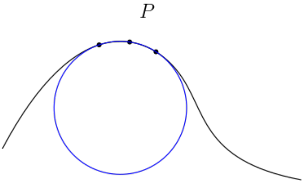
- How can we tell the difference between a straight line and a curve?
- We can define the *curvature* κ, which measures how fast the curve changes direction.



Curvature

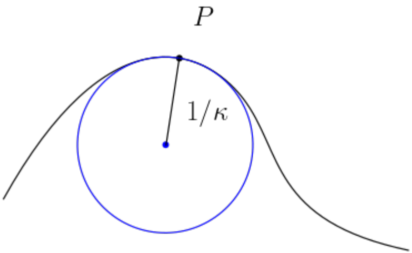
- To define the curvature at a point P, we can try comparing the curve to a circle.
- Circles with larger radii change direction less rapidly, so the radius is a good metric.
- Pick two points close to P, and draw the circle through the three points.





Curvature

- As the two points move closer to P, the circle approaches the *osculating* circle.

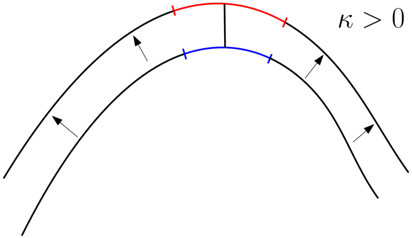


Definition

The *curvature* κ at P is the reciprocal of the radius of the osculating circle.

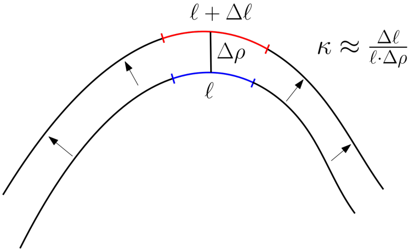
Another Look at Curvature

- Suppose the curvature at P is positive, and consider a small segment containing P.
- We *expand* the curve outwards, mapping the old segment to a new segment.
- The new segment is longer than the old segment when $\kappa > 0$.
- If $\kappa < 0$ the opposite holds.



Another Look at Curvature

- When we *expand* the curve, we move each point a distance $\Delta\rho$ along the outward normal.
- Suppose our old segment has length ℓ, and our new segment has length $\ell + \Delta\ell$.



Definition

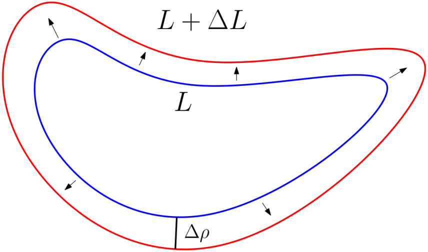
The *curvature* κ is the limit of $\frac{\Delta\ell}{\ell\cdot\Delta\rho}$ as ℓ and $\Delta\rho$ go to zero.

Another Look at Curvature

Corollary

Suppose we expand a curve by a distance $\Delta\rho$, changing its length from L to ΔL. Then

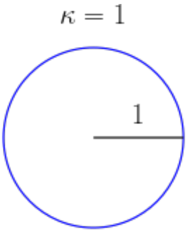
$$\lim_{\Delta\rho\to 0} \frac{\Delta L}{\Delta\rho} = \int \kappa \, ds.$$



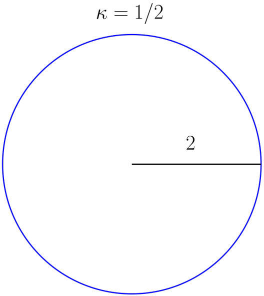
$$\frac{\Delta L}{\Delta\rho} \approx \int \kappa \, ds$$

Another Look at Curvature

- This motivates us to consider the integral $\int \kappa \, ds$.



$$\int \kappa \, ds = 2\pi$$



$$\int \kappa \, ds = 2\pi$$

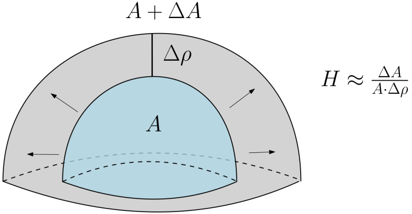
Theorem

For any closed curve, we have

$$\int \kappa \, ds = 2\pi.$$

Mean Curvature

- We can extend this new definition into higher dimensions.
- We *expand* a surface by moving each point a distance $\Delta\rho$ along the unit outward normal.
- This maps a small patch of area A around point P to a new patch of area $A + \Delta A$.

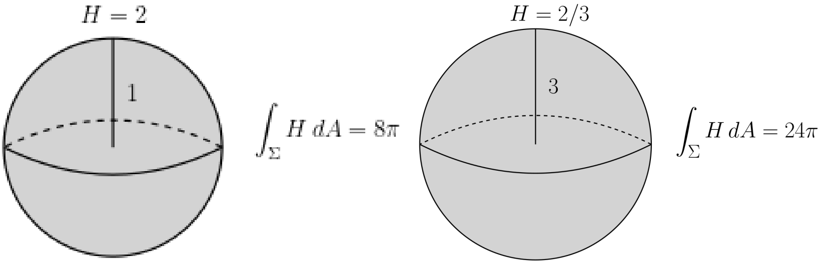


Definition

The *mean curvature* H is the limit of $\frac{\Delta A}{A \cdot \Delta\rho}$ as A and $\Delta\rho$ go to zero.

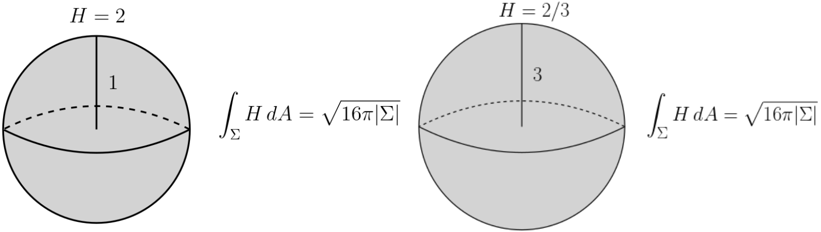
Classical Minkowski Inequality

- Similar to the two-dimensional case, let's try integrating the mean curvature over a surface in $\mathbb{R}^3$.
- Unfortunately, the integral of mean curvature over different spheres is not constant.



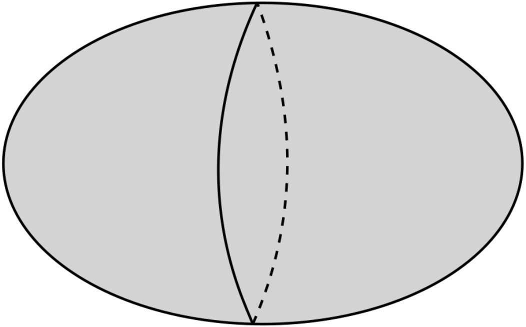
Classical Minkowski Inequality

- We can normalize: for any sphere Σ, the surface integral of H is equal to $\sqrt{16\pi|\Sigma|}$, where $|\Sigma|$ is its surface area.



Classical Minkowski Inequality

- This doesn't hold for other surfaces, however.



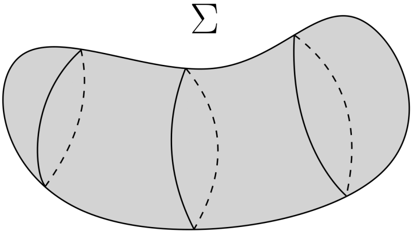
$$\int_{\Sigma} H \, dA > \sqrt{16\pi|\Sigma|}$$

Classical Minkowski Inequality

Theorem

For a surface $\Sigma \subset \mathbb{R}^3$ with area $|\Sigma|$, we have the inequality

$$\int_\Sigma H \,\mathrm{dA} \geq \sqrt{16\pi|\Sigma|}.$$



- This holds for any dimension $\mathbb{R}^n$.

Riemannian Metrics

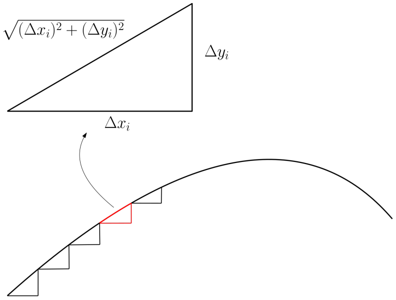
- So far, we've only worked in Euclidean spaces: $\mathbb{R}^2$ and $\mathbb{R}^3$.
- We can work in more general spaces as well, but we need to introduce the concept of a *Riemannian metric*.
- A Riemannian metric is, roughly speaking, a method of assigning lengths to curves.

Riemannian Metrics

Example (metric of $\mathbb{R}^2$)

- The metric of $\mathbb{R}^2$ is $dx^2 + dy^2$.
- If we cut a curve into differential pieces, each piece has a length $\sqrt{dx^2 + dy^2}$.
- So a curve defined by $(x(t), y(t))$ has length

$$\int \sqrt{(x'(t))^2 + (y'(t))^2}\, dt.$$

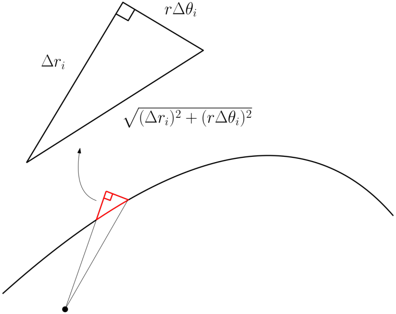


Riemannian Metrics

Example (polar $\mathbb{R}^2$)

- The metric of $\mathbb{R}^2$ is $dr^2 + r^2 d\theta^2$.
- If we cut a curve into differential pieces, each piece has a length $\sqrt{dr^2 + r^2 d\theta^2}$.
- So a curve defined by $(r(t), \theta(t))$ has length

$$\int \sqrt{(r'(t))^2 + (r\theta'(t))^2}\, dt.$$

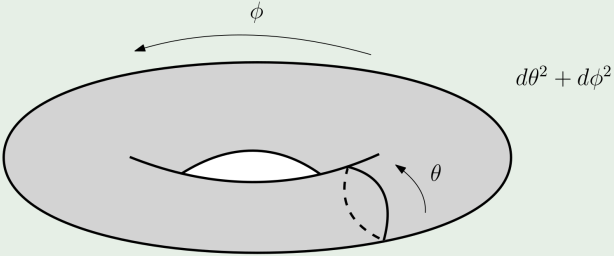


Riemannian Metrics

Example (torus)

- We can equip a torus with the metric

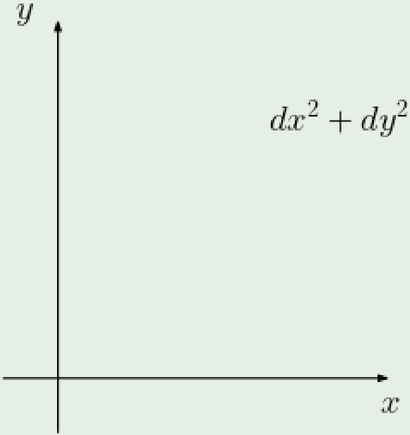
$$d\theta^2 + d\phi^2.$$



Riemannian Metrics

Example (torus)

- Same metric as the Euclidean space:

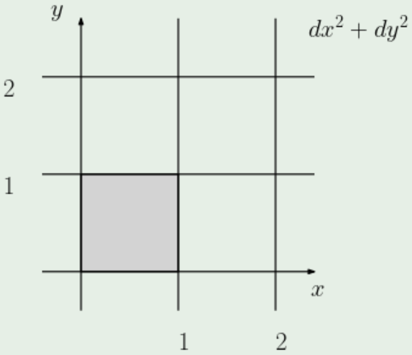


$$dx^2 + dy^2$$

Riemannian Metrics

Example (torus)

- How can we generate a torus from $\mathbb{R}^2$?

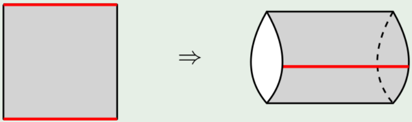


Riemannian Metrics

Example (torus)

- *Identify* the top and bottom edges:

$$dx^2 + dy^2$$

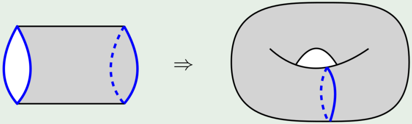


Riemannian Metrics

Example (torus)

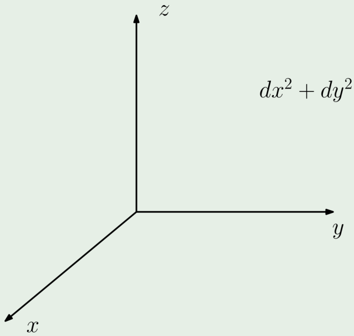
- *Identify* the left and right edges:

$$dx^2 + dy^2$$



Riemannian Metrics

Example ($\mathbb{R}^3$)



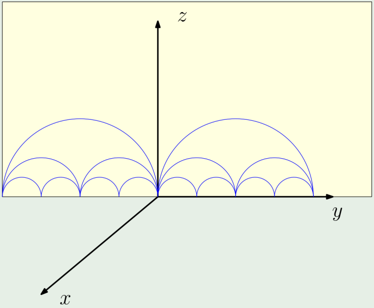
$$dx^2 + dy^2 + dz^2$$

Riemannian Metrics

Example (hyperbolic space)

- The *hyperbolic* space is given by the metric

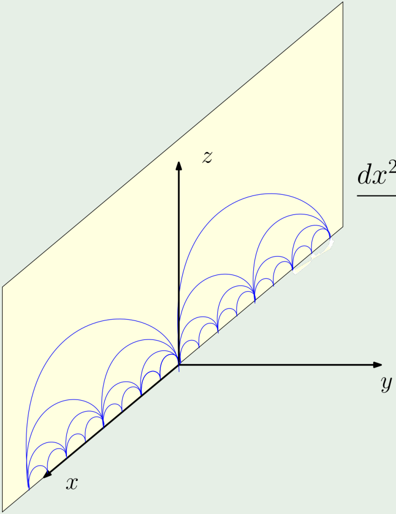
$$\frac{dx^2 + dy^2 + dz^2}{z^2}.$$



$$\frac{dx^2 + dy^2 + dz^2}{z^2}$$

Riemannian Metrics

Example (hyperbolic space)



$$\frac{dx^2 + dy^2 + dz^2}{z^2}$$

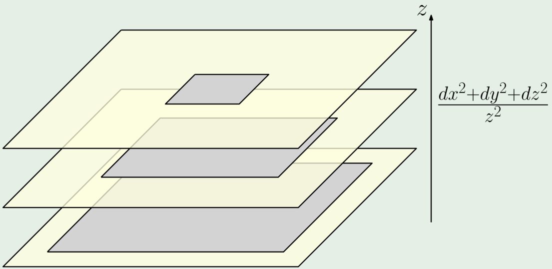
Riemannian Metrics

Example (hyperbolic space)

- Constant z cross-sections have the metric

$$\frac{dx^2 + dy^2}{z^2}.$$

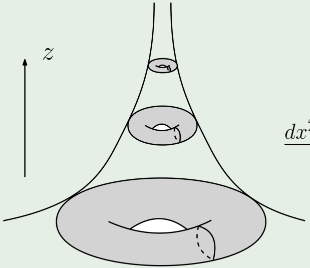
- These cross-sections get smaller for larger z.



Riemannian Metrics

Example (hyperbolic space)

- The space is *toroidal*: cross-sections are toruses.



$$\frac{dx^2+dy^2+dz^2}{z^2}$$

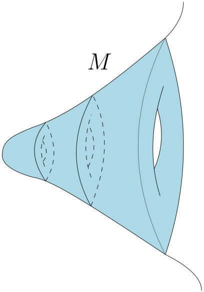
AdS-Melvin space

Definition

The *AdS-Melvin space* is given by the metric

$$z^{-2}F(z)^{-1}\mathrm{d}z^2 + z^{-2}\,\mathrm{d}x^2 + z^{-2}F(z)\,\mathrm{d}y^2$$

where $F(r) = 1 - z^3 - bz^4$ for some $b > 0$.



Properties:

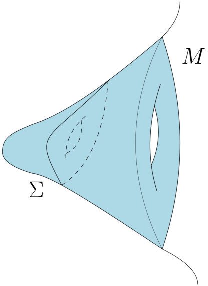
- toroidal
- negative mass

Main Question

Conjecture

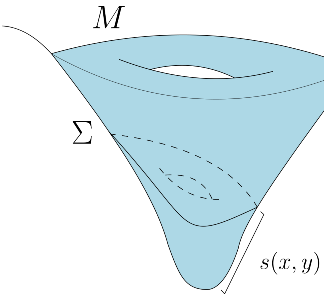
For a surface Σ embedded in the AdS-Melvin space enclosing a region Ω, we have the inequality

$$\underbrace{\int_\Sigma Hz^{-1}\,dA - 6\int_\Omega z^{-1}\,dvol}_{Q(\Sigma)} + \text{mass} \geq 0.$$



Main Question

- We assume that Σ is a *graph*; that is, it is defined by $z^{-1} = s(x, y)$ for some function s.



Theorem

If Σ is a graph embedded in the AdS-Melvin space, then we have the inequality

$$Q(\Sigma) + \text{mass} \geq 0.$$

Acknowledgements

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