Square Tilings of Translation Surfaces

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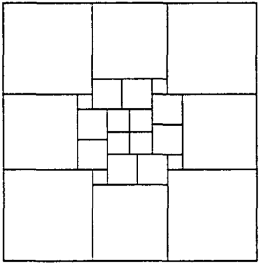
An Introduction: Square Tilings

Definition

A *square tiling* of P is a set of non-overlapping squares which cover P and which are all contained by P.

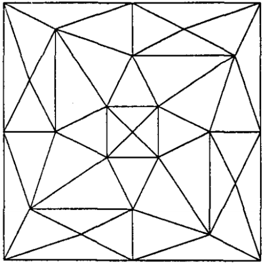
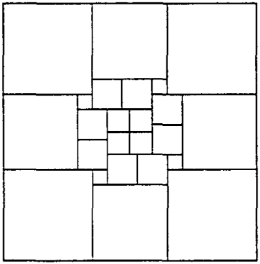


An example of a square tiling

How to Analyze Tilings? The Contacts Graph

Definition (Contacts Graph)

- Graph G
- Vertices S correspond to squares Z_S
- Edge between U and V if and only if squares Z_U and Z_V share a side

A square tiling (right) with its contacts graph (left)

Contacts Graphs and Triangulations, Part I

Definition

A *triangulation* of P is a covering of P by non-overlapping triangles

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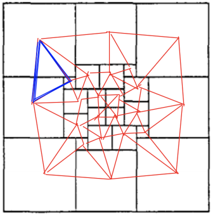
Lemma

The contacts graph is always a triangulation.



Contacts Graphs and Triangulations, Part II

Main Questions

- Given a triangulation, is there always a tiling with it as contacts graph?
- Is it unique?
- Can we construct such a tiling?

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Answer [Schramm 1993]

Yes!

Main Proof Idea:

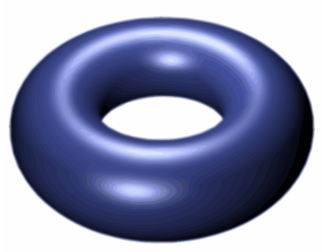
- Extremal Metric!

From Squares to Translation Surfaces: Step I, the Torus!

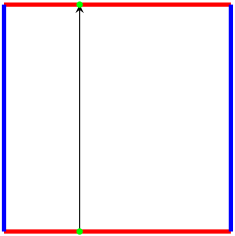
Definition

The *torus*:

- donut
- square with opposite sides identified as the same

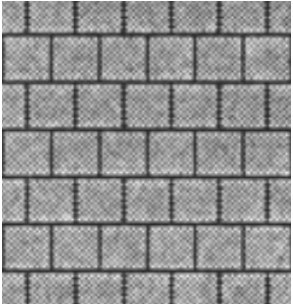


Torus



Torus as a square with identified edges.

Example Tiling of a Torus



Doubly Periodic / Torus Tiling

Theorems on the Torus

Theorem (Schramm 1996, C. 2020)

- *For any triangulation, a square tiling with it as contacts graph always exists! [Schramm]*
- *It is unique up to horizontally translating each square, or vertically translating each square. We can construct it if we know the square sizes. [Our result!]*

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Proof Sketch

- Existence: proved by Oded Schramm via conformal geometry methods
- Uniqueness and Construction Method: obtained by **adapting the extremal metric**

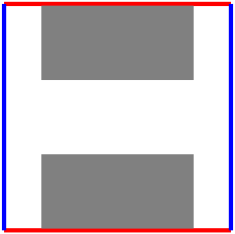
Translation Surfaces (Finally!)

Definition (Translation Surface)

- Take polygon with pairs of parallel sides
- Identify the opposite sides

Example

The torus!



A square on a torus.

The Generalized Problems: Tiling on Translation Surfaces!

Questions

- When can we tile a translation surface with squares?
- Does every triangulation correspond to a tiling?
- Is the tiling unique?
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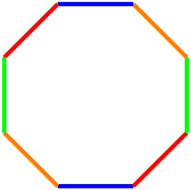
Our general problem is very difficult, so let's look at a particular case!

Octagonal Translation Surfaces

- An octagonal translation surface is formed by identifying opposite edges of an octagon with four pairs of parallel sides.
- The surface encloses a region of genus 2.



Square Tiling Octagonal Translation Surfaces, Part I

Theorem (C. 2020)

- *We cannot square tile the translation surface generated by the regular octagon.*
- *There exists a vertical stretch of the regular octagon, R, such that the translation surface generated by R is square tileable.*

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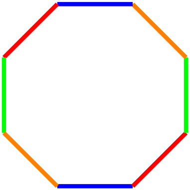
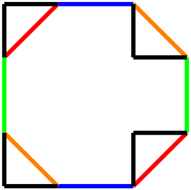
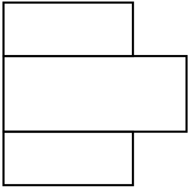


Regular Octagon Translation Surface



Moving Triangles



Rectangle Tiling

Square Tiling Octagonal Translation Surfaces, Part II

Proof Ideas:

- Define a notion of special side length and area
- Length is an additive function
- Area is product of width and height
- Choose function so the total area is negative

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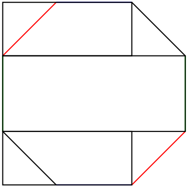
Proof Ideas:

- Define a notion of special side length and area
- Length is an additive function
- Area is product of width and height
- Choose function so the total area is negative
- Tile the octagonal surface with rectangles and then apply a vertical stretch



Three rectangles tiling the regular octagonal translation surface. Applying a vertical stretch with a factor of $1 + \sqrt{2}$ makes them squares.

Summary and Future Work

Our work:

- Proof that the square tiling corresponding to any triangulation
- Algorithm to construct such tilings
- Octagonal translation surface can be tiled in certain cases

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Natural continuations:

- Triangulations/contacts graphs
- Other translation surfaces
- Adapt metric idea

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