Square Tilings of Translation Surfaces

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An Introduction: Square Tilings

Definition

A *square tiling* of $\mathcal{P}$ is a set of non-overlapping squares which cover $\mathcal{P}$ and which are all contained by $\mathcal{P}$.

An example of a square tiling
How to Analyze Tilings? The Contacts Graph

Definition (Contacts Graph)

- Graph $G$
- Vertices $S$ correspond to squares $Z_S$
- Edge between $U$ and $V$ if and only if squares $Z_U$ and $Z_V$ share a side

A square tiling (right) with its contacts graph (left)
Definition

A *triangulation* of $\mathcal{P}$ is a covering of $\mathcal{P}$ by non-overlapping triangles.
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Lemma

The contacts graph is always a triangulation.
Main Questions

- Given a triangulation, is there always a tiling with it as contacts graph?
- Is it unique?
- Can we construct such a tiling?
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Answer [Schramm 1993]
Yes!

Main Proof Idea:
- Extremal Metric!
Definition

The *torus*:
- donut
- square with opposite sides identified as the same
Example Tiling of a Torus

Doubly Periodic / Torus Tiling
Theorem (Schramm 1996, C. 2020)

- For any triangulation, a square tiling with it as contacts graph always exists! [Schramm]
- It is unique up to horizontally translating each square, or vertically translating each square. We can construct it if we know the square sizes. [Our result!]
Theorem (Schramm 1996, C. 2020)

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Proof Sketch

- Existence: proved by Oded Schramm via conformal geometry methods
- Uniqueness and Construction Method: obtained by adapting the extremal metric
Translation Surfaces (Finally!)

Definition (Translation Surface)
- Take polygon with pairs of parallel sides
- Identify the opposite sides

Example
The torus!

A square on a torus.
Questions

- When can we tile a translation surface with squares?
- Does every triangulation correspond to a tiling?
- Is the tiling unique?
- How can we construct the tiling?
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Our general problem is very difficult, so let’s look at a particular case!
An octagonal translation surface is formed by identifying opposite edges of an octagon with four pairs of parallel sides.

The surface encloses a region of genus 2.
Theorem (C. 2020)

- We cannot square tile the translation surface generated by the regular octagon.
- There exists a vertical stretch of the regular octagon, $R$, such that the translation surface generated by $R$ is square tileable.
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- *We cannot square tile the translation surface generated by the regular octagon.*
- *There exists a vertical stretch of the regular octagon, \( R \), such that the translation surface generated by \( R \) is square tileable.*
Proof Ideas:

- Define a notion of special side length and area
- Length is an additive function
- Area is product of width and height
- Choose function so the total area is negative
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- Length is an additive function
- Area is product of width and height
- Choose function so the total area is negative
- Tile the octagonal surface with rectangles and then apply a vertical stretch

Three rectangles tiling the regular octagonal translation surface. Applying a vertical stretch with a factor of $1 + \sqrt{2}$ makes them squares.
Summary and Future Work

Our work:

- Proof that the square tiling corresponding to any triangulation
- Algorithm to construct such tilings
- Octagonal translation surface can be tiled in certain cases
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Natural continuations:
- Triangulations/contacts graphs
- Other translation surfaces
- Adapt metric idea
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