A (simple) graph has a vertex set $V$ and an edge set $E$, where an edge is an unordered pair of distinct vertices of $V$. 

![Graph diagram]

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On the Distance Spectra of Extended Double Stars
Definition (Adjacency Matrix)

The adjacency matrix of a graph with vertices $x_1, x_2, \ldots x_n$ is the $n$ by $n$ matrix $A$ where $A_{ij}$ is equal to 0 if $x_i$ and $x_j$ don’t have an edge connecting them, and equal to 1 if they do.

$$
\begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 \\
0 & 1 & 1 & 0
\end{bmatrix}
$$
Definition (Distance Matrix)

The distance matrix of a graph with vertices \( x_1, x_2, \ldots, x_n \) is the \( n \) by \( n \) matrix \( D \) where \( D_{ij} \) is equal to the smallest number of edges that need to be traversed to get from \( x_i \) to \( x_j \).

\[
\begin{bmatrix}
0 & 1 & 2 & 2 \\
1 & 0 & 1 & 1 \\
2 & 1 & 0 & 1 \\
2 & 1 & 1 & 0
\end{bmatrix}
\]
Matrix Multiplication

General Formula for Multiplying a Matrix by a Vector

\[
\begin{pmatrix}
    a_{11} & a_{12} & \ldots & a_{1n} \\
    a_{21} & a_{22} & \ldots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{m1} & a_{m2} & \ldots & a_{mn}
\end{pmatrix}
\begin{pmatrix}
    y_1 \\
    y_2 \\
    \vdots \\
    y_n
\end{pmatrix}
= 
\begin{pmatrix}
    a_{11}y_1 + a_{12}y_2 + \ldots + a_{1n}y_n \\
    a_{21}y_1 + a_{22}y_2 + \ldots + a_{2n}y_n \\
    \vdots \\
    a_{m1}y_1 + a_{m2}y_2 + \ldots + a_{mn}y_n
\end{pmatrix}
\]

Example

\[
\begin{pmatrix}
    1 & 2 & 3 \\
    4 & 5 & 6 \\
    7 & 8 & 9
\end{pmatrix}
\begin{pmatrix}
    10 \\
    11 \\
    12
\end{pmatrix}
= 
\begin{pmatrix}
    1 \cdot 10 + 2 \cdot 11 + 3 \cdot 12 \\
    4 \cdot 10 + 5 \cdot 11 + 6 \cdot 12 \\
    7 \cdot 10 + 8 \cdot 11 + 9 \cdot 12
\end{pmatrix}
= 
\begin{pmatrix}
    68 \\
    167 \\
    276
\end{pmatrix}
\]
Definition (Eigenvalues)
An eigenvalue of an $n \times n$ matrix $A$ is a scalar $\lambda$ in which there exists some non-zero $n$ by 1 vector $v$ such that $Av = \lambda v$.

Definition (Spectrum)
The spectrum of a matrix is the “set” of its eigenvalues.

Example
\[
\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\]
Thus, we see that both 1 and 2 are eigenvalues of $\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$. 
Definition (Cospectral)

Two graphs are (distance) cospectral if they have the same (distance) spectrum. We refer to a graph as determined by its (distance) spectrum if it is not (distance) cospectral to any non-isomorphic graph.
Definition (Diameter of a Graph)

The diameter of a graph is the maximum distance between any pair of vertices of the graph.

Existing Results

Existing results have focused on the adjacency spectrum. Diameter 2 graphs follow the pattern $D = 2J - 2I - A$, and most results on the distance spectra focus on low diameter.

\[ x_4 \quad x_3 \]
\[ x_1 \qquad x_2 \]
**Definition (Stars)**

A *star* is a complete bipartite graph denoted by $K_{1,a}$, the graph formed by a single central vertex with $a$ leaves connected to it.

**Figure: $K_{1,3}$**
Definition (Double Stars)

An *double star*, denoted by $S(a, b)$, is the graph formed by joining the centers of the stars $K_{1,a}$ and $K_{1,b}$.

Figure: $S(3, 4)$
An extended double star, denoted by $T(a, b)$, is the graph formed by joining the centers of the stars $K_{1,a}$ and $K_{1,b}$ to a common vertex.
Motivation

Triple Stars

We started with some diameter 4 trees, such as triple stars.
Definition (Induced subgraph)
For $S \subseteq V(G)$, an induced subgraph of $G$, denoted by $G[S]$, is the subgraph of $G$ whose vertex set is $S$ and whose edge set consists of all edges of $G$ which have both ends in $S$.

Definition (Principal submatrix)
A principal submatrix is obtained by removing certain row indices and the same column indices.
Theorem (Interlacing)

Let $G$ be a graph with $n$ vertices and distance matrix $D(G)$. Denote its eigenvalues as

$$
\lambda_1(D(G)) \geq \lambda_2(D(G)) \geq \ldots \geq \lambda_n(D(G)).
$$

Let $H$ be an induced subgraph of $G$ with $m$ vertices and distance spectrum

$$
\mu_1(D(H)) \geq \mu_2(D(H)) \ldots \geq \mu_m(D(H)).
$$

If $D(H)$ is a principal submatrix of $D(G)$, then

$$
\lambda_{n-m+i}(D(G)) \leq \mu_i(D(H)) \leq \lambda_i(D(G))
$$

for $i = 1, 2, \ldots, m$.

Definition (Forbidden Subgraph)

A graph $H$ is a forbidden subgraph of a graph $G$ if the set of induced subgraphs of $G$ does not include a graph isomorphic to $H$. 
Our Results

Theorem

*Extended double stars are determined by their distance spectrum.*

Notation

The graph $G$ is cospectral to an extended double star.
Theorem (2013)

The complete graph $K_n$ is determined by its distance spectrum.

Corollary (2016)

If $G$ is a graph with order $n$ and diameter 2, then $|E(G)| < |E(T)|$.

Theorem

If the diameter of $G$ is greater than 4, then $G$ and $T$ are not cospectral.
Identifying a Path

There exist two vertices $x_1, x_5 \in V(G)$ such that $d_{x_1x_5} = 4$. Denote $X = \{x_1, x_2, x_3, x_4, x_5\}$ the vertex set of a path of length 4.

Definition (Vertex sets $V_i$)

Denote by $V_i (i = 0, 1, 2, 3, 4, 5)$ the vertex subset of $V\setminus X$ consisting of vertices adjacent to $i$ vertices of $X$. 
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Example

Here $x_6 \in V_5$ and $x_7 \in V_2$. 
Forbidden Graphs

To prove $V_5$ empty, just show this subgraph is forbidden!
Proof Outline

- Prove $V_4$, $V_3$, $V_1$, $V_0$ are empty
- Prove $V_2$ is empty
Definition

In $V_2$, interlacing eliminates all subgraphs except the one shown below. We call such a vertex in $V_2$ adjacent to $x_2$ and $x_3$ a \textit{hat}.
Theorem

There can not be more than 5 hats in $G$.

Theorem

There can not be 0 hats in $G$. 
Theorem

There can not be more than 5 hats in $G$.

Theorem

There can not be 0 hats in $G$.

Theorem

$V_2$ is empty.
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