On the Wasserstein Distance Between $k$-Step Probability Measures on Finite Graphs

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Guavas?
Guava Juice!
We must transport guava juice stored in warehouses from the first distribution to the second distribution via roads.
Transporting 1 gallon of guava juice along 1 road costs $1. Let’s try transporting the juice and see how much it costs!
Transporting Guava Juice

cost = 2 \cdot 2
TRANSPORTING GUAVA JUICE

![Graph Diagram]

The diagram illustrates the transportation of guava juice between different locations, with nodes representing different points and edges indicating routes. Numbers on the nodes and edges depict quantities or distances. The green arrow indicates a change or movement.
TRANSPORTING GUAVA JUICE

\[\text{cost} = 2 \cdot 2 + 1 \cdot 1 = 5\]
Here’s a more cost-effective way of transporting the guava juice:

\[
\text{cost} = 1.5 \cdot 1 + 0.5 \cdot 2 + 1 \cdot 1 + 0.5 \cdot 1 = 4
\]
A natural question:
What is the most cost-effective way of transporting the juice?

Wasserstein distance = minimum cost of transportation.
We study the Wasserstein distance between the $k$-step probability distributions of random walks with laziness on a finite graph.
We study the Wasserstein distance between the \textit{k-step probability distributions} of random walks with laziness on a finite graph.

At each vertex, the proportion $\alpha$ of the mass stays while the rest of the mass splits evenly among its neighbors.

$\alpha = 0.2$
Defining the Guvab

The following definition captures our object of study.

**Definition**

We define a *Guvab* to be a tuple \((G, u, v, \alpha, \beta)\) where \(G\) is a finite simple connected graph, \(u, v \in V(G)\), and \(\alpha, \beta \in [0, 1]\) with \(\alpha \leq \beta\).

Given a Guvab and a nonnegative integer \(k\), consider the \(k\)-step probability distributions of the two random walks with starting vertices \(u, v\) and lazinesses \(\alpha, \beta\), respectively. We denote by \(W_k\) the Wasserstein distance between these two \(k\)-step probability distributions.
Motivation:

- $W_1$ is used to determine Lin-Lu-Yau-Ollivier-Ricci curvature ([LLY11])
- Applications in drug design, cancer networks, and economic risk ([SGR^{+}15], [SGT16], [WX21])

Our Question:

- What about $W_k$ as $k$ gets larger and larger?
- Does it converge? When? To what? How fast?
When \( \lim_{k \to \infty} W_k \) is well-defined, call it \( W \).

**Theorem (Classifying End Behavior)**

All Guvabs fit into one of four categories, and we know when they fit into each category:

1. \( W = 1 \) and \( \alpha, \beta < 1 \)
   - \( G \) bipartite, \( \alpha = \beta = 0 \), \( d(u, v) \) is odd

2. \( W = \frac{1}{2} \) and \( \alpha, \beta < 1 \)
   - \( G \) bipartite, \( \alpha = 0 < \beta < 1 \)

3. \( W = 0 \) and \( \alpha, \beta < 1 \)
   - all other Guvabs with \( \alpha, \beta < 1 \)

4. \( \beta = 1 \)
Main Result #2: Exponential Convergence

For any Guvab, \( \lim_{k \to \infty} W_{2k} \) and \( \lim_{k \to \infty} W_{2k+1} \) are well-defined (due to Main Result 1).

Theorem (Exponential Convergence of W-Dist)

For any Guvab, we have that:

- either \( \{ W_{2k} \} \) is eventually constant, or there exists a constant \( \lambda_{\text{even}} \in (-1, 1) \) and a positive constant \( c_{\text{even}} > 0 \) such that
  \[
  |W_{2k} - \lim_{k \to \infty} W_{2k}| \sim c_{\text{even}} \cdot |\lambda_{\text{even}}|^{2k}
  \]

- either \( \{ W_{2k+1} \} \) is eventually constant, or there exists a constant \( \lambda_{\text{odd}} \in (-1, 1) \) and a positive constant \( c_{\text{odd}} > 0 \) such that
  \[
  |W_{2k+1} - \lim_{k \to \infty} W_{2k+1}| \sim c_{\text{odd}} \cdot |\lambda_{\text{odd}}|^{2k+1}
  \]
Theorem (Characterization of Constancy)

When $\alpha, \beta < 1$, we have that $\{W_k\}$ is eventually constant if and only if one of the following holds:

1. $\alpha = \beta = 0$, $G$ is bipartite, and $d(u, v)$ is odd (here $W = 1$),
2. $\alpha = 0$, $\beta = \frac{1}{2}$, and $G$ is bipartite (here $W = \frac{1}{2}$),
3. $\alpha = \beta = 0$ and $N(u) = N(v)$ (here $W = 0$),
4. $\alpha = \beta = \frac{1}{\deg u + 1}$, the edge $(u, v) \in E(G)$, and if the edge $(u, v)$ were removed from $E(G)$ then $u, v$ would have $N(u) = N(v)$ (here $W = 0$),
5. $\alpha = \beta$ and $u = v$ (here $W = 0$).
Main Result #3: Characterization of Constancy

1. \( \alpha = \beta = 0 \), \( G \) is bipartite, and \( d(u, v) \) is odd (here \( W = 1 \)),
2. \( \alpha = 0, \beta = \frac{1}{2} \), and \( G \) is bipartite (here \( W = \frac{1}{2} \)),
3. \( \alpha = \beta = 0 \) and \( N(u) = N(v) \) (here \( W = 0 \)),

4. \( \alpha = \beta = \frac{1}{\deg u + 1} \), the edge \( (u, v) \in E(G) \), and if the edge \( (u, v) \) were removed from \( E(G) \) then \( u, v \) would have \( N(u) = N(v) \) (here \( W = 0 \)),

5. \( \alpha = \beta \) and \( u = v \) (here \( W = 0 \)).
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Thanks for listening! Any questions?
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