Combinatorial Aspects of the Card Game War

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- Game ends when a player has no cards left

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- We represent a *state* of the game as

$$a_1 a_2 \ldots a_i | a_{i+1} a_{i+2} \ldots a_n,$$

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- E.g. 2|134 with WL-putback:

$$\begin{aligned} & 2 \mid 134 \\ \Longrightarrow\ & 21 \mid 34 \\ \Longrightarrow\ & 1 \mid 432 \\ \Longrightarrow\ & \mid 3241. \end{aligned}$$

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- Bob has won on his first passthrough
- Alice lost on her second passthrough
- **We only consider games where Bob wins within his first passthrough.**

Visualizing Passthroughs

(Source: commons.wikimedia.org/wiki/File:Customer_divider_bar_1.jpg)

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- **Each block acts as its own mini-game of War.**

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Note that in these single-use states, the order Bob puts his cards back doesn't matter because they don't show up in the game again. We can regard cards Bob wins as discarded.

Blocks in Single-Use States

- Consider a state $a_1 a_2 \ldots a_m | a_{m+1} \ldots a_n$, where $n \geq m \cdot 2^k$. If this is a level-k single-use state, we have the following:

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- We call a block $a_1 | a_2 \ldots a_{2^k}$ a level-k block if it is a level-k single-use state when played as a game of War.
- From now on we only discuss blocks where Alice has a single card.

Probability that a State is Level-k Single-Use

Theorem

The chance a random permutation of the 2^k cards in a state $a_1|a_2 \dots a_{2^k}$ is a level-k block is P_k, where $P_1 = \frac{1}{2}$ and recursively

$$P_{k+1} = \frac{1}{2} + \frac{1}{2} P_k^2.$$

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The first few terms in this sequence are $P_2 = \frac{5}{8}$, $P_3 = \frac{89}{128}$, $P_4 = \frac{24305}{32768}$.

Win-Loss Sequences

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Example:

- Take 2|134 with WL-putback.
- $2|134 \implies 21|34 \implies 1|432 \implies |3241$
- So win-loss sequence is W/LL.

More on Win-Loss Sequences

- Before each subsequent passthrough, the number of cards Alice has is twice the number of wins she had in the previous passthrough, because each win yields two cards back to Alice's stack

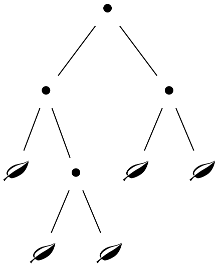
Theorem

Win-loss sequences are in bijection with full binary trees.

Full Binary Trees

Definition (Full Binary Tree)

A full binary tree (FBT) is a binary tree in which every node has either 2 children (left child and right child) or 0 children. A node with 0 children is called a **leaf**, and a node with 2 children is called a **non-leaf**.



Expressing Win-Loss Sequences as Full Binary Trees

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- Each loss yields zero cards in the next passthrough.
- **non-leaves = W's, leaves = L's, levels of FBT = passthroughs**

From Win-Loss Sequence to Full Binary Tree

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Finding FBT for W/WW/LWLL/LL

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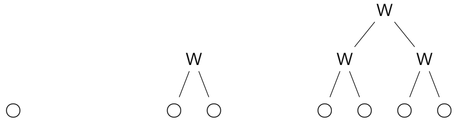

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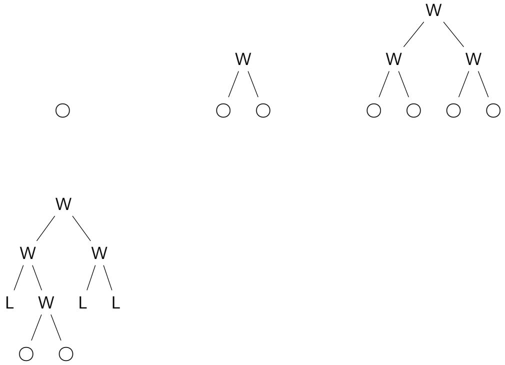

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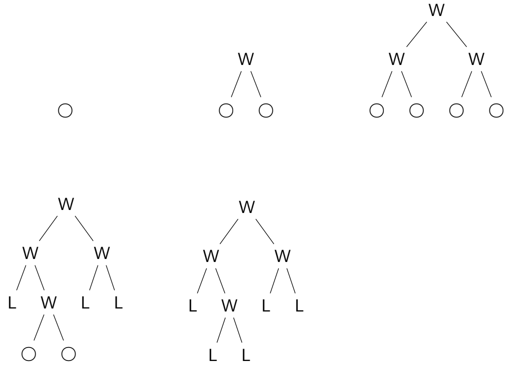

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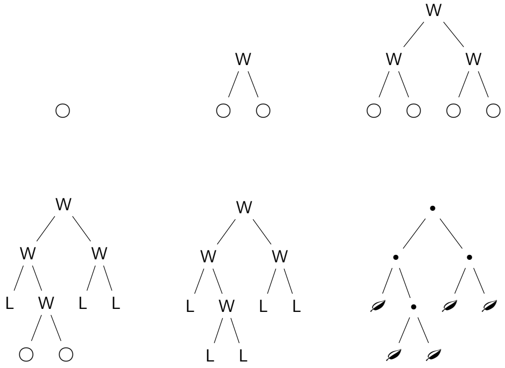

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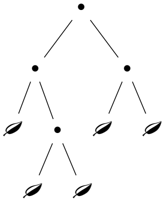
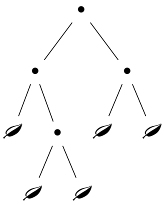


From Full Binary Tree Back To Win-Loss Sequence

- write W in all non-leaves and L in all leaves



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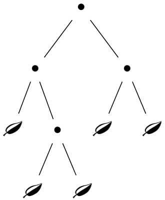
- write W in all non-leaves and L in all leaves
- read left-to-right, top-to-bottom



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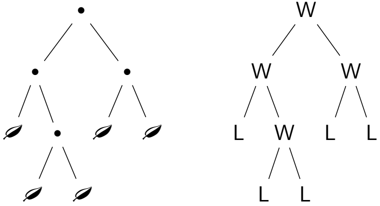

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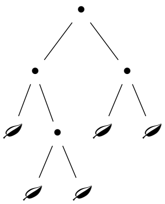

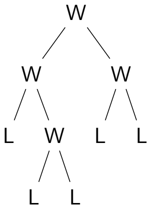

Reading left-to-right, top-to-bottom, we get $W/WW/LWLL/LL$.

A Couple Consequences of the Bijection

Recall levels of a full binary tree are passthroughs of a win-loss sequence.

Proposition

The number of win-loss sequences A_k that end within k passthroughs for Alice satisfies $A_1 = 1$ and $A_{k+1} = A_k^2 + 1$.

Proposition

The number of win-loss sequences B_k that end in exactly $2k+1$ rounds is C_k, where C_k is the k'th Catalan number $\frac{1}{k+1}\binom{2k}{k}$.

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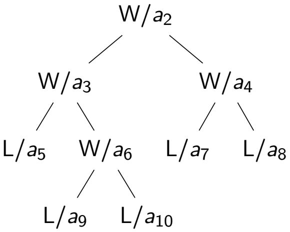

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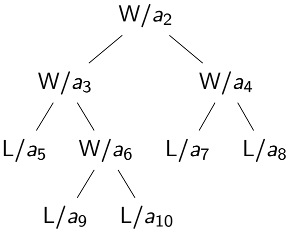

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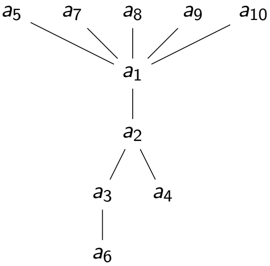

Poset for Random Putback

Continued Research

- Counting states that end in a certain number of rounds with WL-putback

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- Dr. Slava Gerovitch, Prof. Pavel Etingof, and the entire PRIMES program

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