Regularities in the Lattice Homology of Seifert Homology Spheres

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Manifolds

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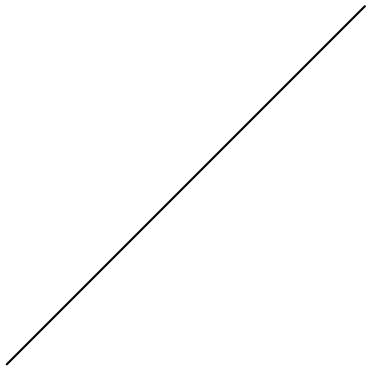

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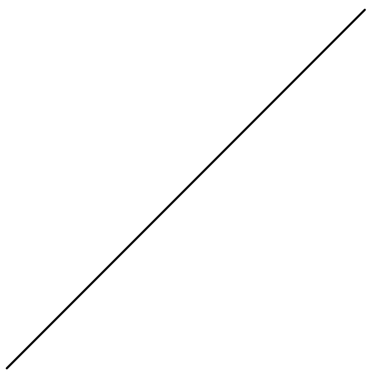

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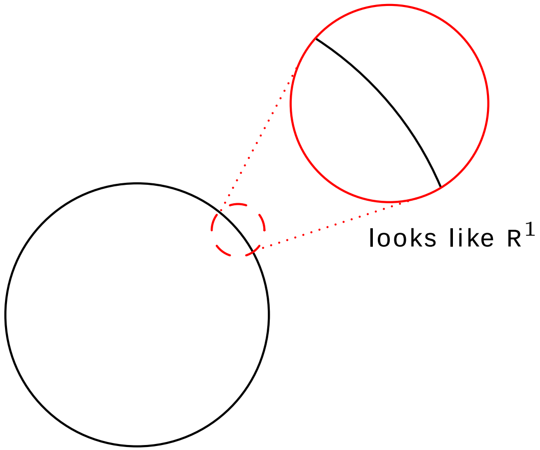

S^1. The 1-dimensional circle. A small ant sitting on a circle looks around and sees a line.

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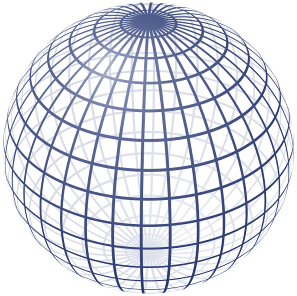

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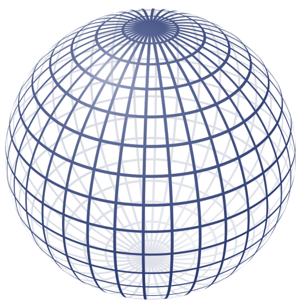

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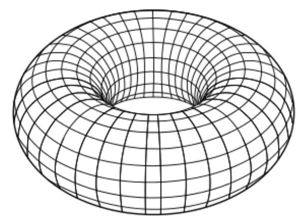

$T^2 = S^1 \quad S^1$. The 2-dimensional surface of a torus. An ant sitting on a torus looks around and sees a plane.

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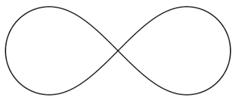

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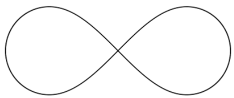

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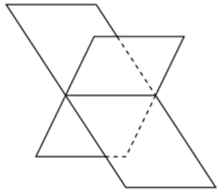

Two intersecting planes do not form a manifold, since an ant sitting on the line of intersection looks around and sees two intersecting planes, not R^2.

Manifolds with Boundary

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A manifold–with–boundary is an extension of the notion of a manifold with a section called a ***boundary***, where each point in the boundary has a small region around it that looks like the half-space $\mathsf{R}^{n\quad 1}\quad \mathsf{R}_{\quad 0}$. The boundary is a manifold (without boundary) of one lower dimension.

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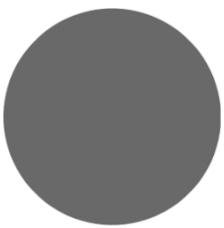

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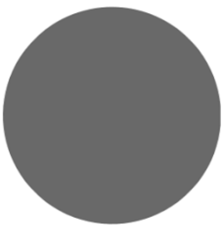

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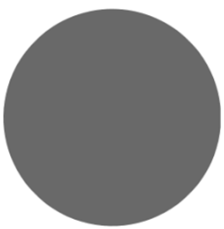

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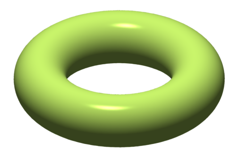

The filled-in torus D^2 S^1 has boundary S^1 S^1.

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Very nontrivial fact: this holds true for $n = 3$ as well.

$n = 4$ is when we get our first example of an n-dimensional manifold that *isn't* the boundary of some $(n+1)$-dimensional manifold, e.g. CP^2.

Cobordisms

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This concept is best illustrated through some examples.

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Let $M = S^1$ and $N = S^1 \sqcup S^1$. Then, the "pair of pants" manifold displays a cobordism between the M and N.

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The study of cobordisms has been of intense interest the last few decades.

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Bringing back the pair of pants analogy:

Note that the top circle S^1 bounds a disc D^2. Since the S^1 on top is cobordant to the $S^1 \sqcup S^1$ on the bottom through the pair of pants, $S^1 \sqcup S^1$ also bounds a 2-dimensional manifold, specifically the pair of pants with the top capped off with a disc.

Cobordism Classes of 3-manifolds

Note that cobordism is an equivalence relation (in particular, if X and Y are cobordant and Y and Z are cobordant, then we can see X and Z are cobordant). Therefore, it makes sense to talk about the cobordism class of a manifold X (it's simply the set of all manifolds cobordant to X).

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Because of this, we will actually study a slight specialization of cobordism called homology cobordism between 3-manifolds, which we will de ne later. In this case, there are in nitely many homology cobordism classes of 3-manifolds, and the classi cation problem is far from solved.

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As we saw earlier, $x^2 + y^2 + z^2 = 1$ in 3-dimensional space is S^2, the surface of a sphere. Generalizing, $w^2 + x^2 + y^2 + z^2 = 1$ in 4-dimensional hyperspace is S^3, the 3-dimensional sphere.

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You can also think of S^3 as a cube except you grab all the points on the faces and fuse them together into a single point. Therefore, S^3 is roughly R^3, just the outside points are wrapped around and fused together.

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3. The resulting manifold is obtained from M via surgery along L.

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Surgery is a very weird process that is practically impossible to visualize, but it is important since basically all 3-manifolds can be obtained via this process:

Theorem (Lickorish and Wallace)

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To represent the process in a better way, we can use *surgery diagrams*: we draw the link in S^3 (which is basically $\mathbb{R}^3$), and then label each with a number representing how we twist each solid torus (from thickening each knot in the link) when we glue it back in.

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Homology

Before, we noted that the problem of cobordisms between 3-manifolds is not very interesting, as they are all cobordant to each other. Therefore, we instead study a variant of cobordism called homology cobordism, which is much more interesting. To introduce this notion, we must first define homology.

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We say that a 4-manifold X is an *homology 4-cylinder* if X has the same homology groups H_i as $S^3 \times [0;1]$ for all $i \geq 0$.

Homology Cobordism

Now, we are interested in this specialization of cobordism:

Definition

Two homology spheres M and N are *homology cobordant* if there exists some homology cylinder W such that the disjoint union of M and N bounds W.

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- The **maximal monotone subroot** is a graded root derived from the lattice homology. Like the *d*-invariant, it is also a homology cobordism invariant.

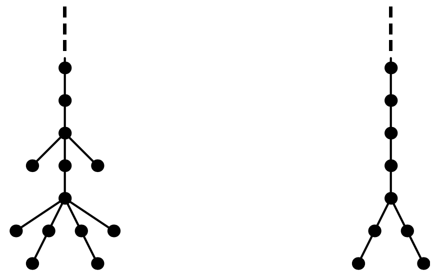
Homology Cobordism Invariants

Visualizing examples of homology spheres and homology cobordisms is very hard, so we turn to invariants to help us understand them.

The ***d*-invariant**, which assigns an even integer to each homology sphere. It is a homology cobordism invariant, which means if two homology spheres are homology cobordant, then they share the same *d*-invariant.

The **lattice homology** is an invariant which assigns a graded root to each homology sphere. A graded root is a symmetric rooted tree with an infinite stem of nodes pointing upwards. It is ***not*** a homology cobordism invariant.

The **maximal monotone subroot** is a graded root derived from the lattice homology. Like the *d*-invariant, it is also a homology cobordism invariant.



Our Results

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Theorem

Let $a_1; a_2; :::; a_n \quad 2$ be pairwise coprime integers, and let $\quad = a_1 a_2 \quad a_{n \quad 1}$. Then,

$$d(\Sigma(a_1; a_2; :::; a_n)) = d(\Sigma(a_1; a_2; :::; a_{n \quad 1}; a_n + \quad)):$$

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$$d(\Sigma(a_1, a_2, \ldots, a_n)) = d(\Sigma(a_1, a_2, \ldots, a_{n-1}, a_n + \delta)).$$

Theorem

Let $a_1, a_2, \ldots, a_n \geq 2$ be pairwise coprime integers, and let $\delta = a_1 a_2 \cdots a_{n-1}$. Then, the maximal monotone subroots of the lattice homologies of $\Sigma(a_1, a_2, \ldots, a_n)$ and $\Sigma(a_1, a_2, \ldots, a_{n-1}, a_n + 2\delta)$ are the same.

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Remark

In general, the maximal monotone subroots of the lattice homologies of $\Sigma(a_1;a_2;:::;a_n)$ *and* $\Sigma(a_1;a_2;:::;a_{n \quad 1};a_n + \quad)$ *are not the same.*

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