A High-order Cumulant-based Sparse Ruler

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The Sparse Ruler
The Sparse Ruler

Definition 1.1. A sparse ruler is a set of integers \( S = \{s_1, s_2, \ldots, s_n\} \). We say that \( S \) generates the set of lags \( \Phi(S) \) if for any integer \( \phi \in \Phi(S) \), there are \( i, j \) such that \( s_i - s_j = \phi \).

Problem: given a fixed number \( (n) \) of marks, how do we construct the ruler \( (S) \) to maximize the number of consecutive lags \( (\Phi) \)?
Motivation

- NP-Completeness
- Information Theory
- Error-Correcting Code
- Signal Processing
Nested Ruler

\[ S = S_1 \cup S_2 \]
\[ S_1 = \{ n_1 N_2 \mid n_1 = 1, 2, \ldots, N_1 \} \]
\[ S_2 = \{ n_2 \mid n_2 = 1, 2, \ldots, N_2 \} \]

Then, \( \Phi(S) = \{ \mu \mid -N_1 N_2 + 1 \leq \mu \leq N_2 N_2 - 1 \} \)

Example: take

1, 2, 3, \ldots (10)
10, 20, 30, \ldots 100
Nested Ruler

\[ S = S_1 \cup S_2 \]
\[ S_1 = \{ n_1 N_2 \mid n_1 = 1, 2, \ldots, N_1 \} \]
\[ S_2 = \{ n_2 \mid n_2 = 1, 2, \ldots, N_2 \} \]
Then, \( \Phi(S) = \{ \mu \mid -N_1 N_2 + 1 \leq \mu \leq N_2 N_2 - 1 \} \)

Variations:
Coprime Ruler

General form: For integers $P, Q$ where $\gcd(P, Q) = 1$

$$S = S_1 \cup S_2$$

$$S_1 = \{ n_1 \cdot P \mid n_1 = 0, 1, 2, \ldots, Q1 \}$$

$$S_2 = \{ n_2 \cdot Q \mid n_2 = 0, 1, 2, \ldots, 2P - 1 \}$$

Then, $\Phi(S) = \{ \mu \mid -PQ - P + 1 \leq \mu \leq PQ + P - 1 \}$
Definition 2.1. Consider a ruler $S$. The set of $2q$-th order lags

$$\Phi^{2q}(S) = \left\{ \sum_{i=1}^{q} p_{ni} - \sum_{i=q+1}^{2q} p_{ni} \mid n_i \in [1, N] \right\}$$

We denote $\Phi^2$ as $\Phi$ for short.

Benefit: Increased lag generation from $O(N^2)$ to $O(N^{2q})$
Cumulants and High-Order Rulers

Definition 2.1. Consider a ruler $\mathcal{S}$. The set of $2q$-th order lags

$$\Phi^{2q}(\mathcal{S}) = \left\{ \sum_{i=1}^{q} p_{n_i} - \sum_{i=q+1}^{2q} p_{n_i} \mid n_i \in [1, N] \right\}$$

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But... Not trivial:

- 4th-Order: $s_a, s_b, s_c, s_d$ has $\binom{4}{2} = 6$ sign permutations
- $2q$th-Order has $\binom{2q}{q}$
4th-Order: General Form

\[ S = P_1 \cup P_2 \cup Q_1 \cup Q_2 \]

Examine: \[ \pm (p_1 + p_2) - (q_1 + q_2) \]
\[ \pm (p_1 - p_2) - (q_1 - q_2) \]
\[ \pm (p_1 + p_2) + (q_1 + q_2) \]

- Nested
- Nested
- Coprime
4th-Order: General Form

\[ S = P_1 \cup P_2 \cup Q_1 \cup Q_2 \]

\[ P_1 = \{(n_1 N_2 + \left\lfloor \frac{q}{2} \right\rfloor)p \mid n_1 = 0, 1, 2, \ldots, N_1\} \]

\[ P_2 = \{(n_2 + q)p \mid n_2 = 0, 1, 2, \ldots, N_2\} \]

\[ Q_1 = \{(n_3 N_4 - \left\lfloor \frac{p}{2} \right\rfloor)q \mid n_4 = 0, 1, 2, \ldots, N_4\} \]

\[ Q_2 = \{(n_4 - \left\lfloor \frac{p}{2} \right\rfloor)q \mid n_5 = 0, 1, 2, \ldots, N_5\} \]
4th-Order: General Form

\[ S = P_1 \cup P_2 \cup Q_1 \cup Q_2 \]

\[ P_1 = \{(n_1 N_2 + \left\lfloor \frac{q}{2} \right\rfloor)p \mid n_1 = 0, 1, 2, \ldots, N_1\} \]

\[ P_2 = \{(n_2 + q)p \mid n_2 = 0, 1, 2, \ldots, N_2\} \]

- Nested Ruler
- Common multiple of \( p \)
- Shifted by a factor
  - Lemma: Shifting adds up
4th-Order: Integration

\[ S = \mathbb{P}_1 \cup \mathbb{P}_2 \cup \mathbb{Q}_1 \cup \mathbb{Q}_2 \]

\[ \mathbb{P}_1 = \{(n_1 N_2 + \left\lfloor \frac{q}{2} \right\rfloor)p \mid n_1 = 0, 1, 2, \ldots, N_1\} \]

\[ \mathbb{P}_2 = \{(n_2 + q)p \mid n_2 = 0, 1, 2, \ldots, N_2\} \]

\[ \mathbb{Q}_1 = \{(n_3 N_4 - \left\lfloor \frac{p}{2} \right\rfloor)q \mid n_4 = 0, 1, 2, \ldots, N_4\} \]

\[ \mathbb{Q}_2 = \{(n_4 - \left\lfloor \frac{p}{2} \right\rfloor)q \mid n_5 = 0, 1, 2, \ldots, N_5\} \]

- \( \mathbb{P} \) and \( \mathbb{Q} \): larger coprime structure

- Orienting \((p_1 + p_2) - (q_1 + q_2), (p_1 - p_2) - (q_1 + q_2), (p_1 + p_2) + (q_1 + q_2)\)
4th-Order: Integration + Result

- Orienting \((p_1 + p_2) - (q_1 + q_2), (p_1 - p_2) - (q_1 + q_2), (p_1 + p_2) + (q_1 + q_2)\)

\[
M_{max}^6 = \begin{cases} 
\left\lfloor \frac{5}{2}pq \right\rfloor & \text{when } q \text{ is even} \\
\left\lfloor \frac{5}{2}pq \right\rfloor - q & \text{when } q \text{ is odd}
\end{cases}
\leq \left\lfloor \frac{5}{2}N_1N_2N_3N_4 \right\rfloor
\]
6th-Order: General Form

\[ S = P_1 \cup P_2 \cup P_3 \cup Q_1 \cup Q_2 \cup Q_3 \]

Examine 9 Combinations of:

\(-P_1 + P_2 + P_3\) and \(-Q_1 + Q_2 + Q_3\)
\(+P_1 - P_2 + P_3\) and \(+Q_1 - Q_2 + Q_3\)
\(P_1 + P_2 - P_3\)
\(+Q_1 + Q_2 - Q_3\)

Nested

Nested

Coprime
6th-Order: General Form

\[ S = P_1 \cup P_2 \cup P_3 \cup Q_1 \cup Q_2 \cup Q_3 \]

\[ P_1 = \{ (n_1 N_2 N_3)p \mid n_1 = 0, 1, 2, \ldots, N_1 \} \]

\[ P_2 = \{ (n_2 N_3 + q)p \mid n_2 = 0, 1, 2, \ldots, N_2 \} \]

\[ P_3 = \{ (n_3 + \left\lfloor \frac{3q}{2} \right\rfloor)p \mid n_3 = 0, 1, 2, \ldots, N_3 \} \]

\[ Q_1 = \{ (n_4 N_5 N_6 - \left\lfloor \frac{5p}{2} \right\rfloor)q \mid n_4 = 0, 1, 2, \ldots, N_4 \} \]

\[ Q_2 = \{ (n_5 N_6 - \left\lfloor \frac{7p}{2} \right\rfloor)q \mid n_5 = 0, 1, 2, \ldots, N_5 \} \]

\[ Q_3 = \{ (n_6 - 5p)q \mid n_6 = 0, 1, 2, \ldots, N_6 \} \]
### 6th-Order: Integration

| 12 | 17 | 22 | 27 | 32 | 37 | 42 | 47 | 52 | 57 | 62 | 67 | 72 | 77 | 82 | 87 | 92 | 97 | 102 | 107 | 112 | 117 | 122 | 127 | 132 | 137 | 142 | 147 | 152 | 157 | 162 | 167 | 172 | 177 | 182 | 187 |
|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 6  | 11 | 16 | 21 | 26 | 31 | 36 | 41 | 46 | 51 | 56 | 61 | 66 | 71 | 76 | 81 | 86 | 91 | 96 | 101 | 106 | 111 | 116 | 121 | 126 | 131 | 136 | 141 | 146 | 151 | 156 | 161 | 166 | 171 | 176 | 181 |
| 0  | 5  | 10 | 15 | 20 | 25 | 30 | 35 | 40 | 45 | 50 | 55 | 60 | 65 | 70 | 75 | 80 | 85 | 90 | 95 | 100 | 105 | 110 | 115 | 120 | 125 | 130 | 135 | 140 | 145 | 150 | 155 | 160 | 165 | 170 | 175 |
| 6  | 1  | 4  | 9  | 14 | 19 | 24 | 29 | 34 | 39 | 44 | 49 | 54 | 59 | 64 | 69 | 74 | 79 | 84 | 89 | 94 | 99 | 104 | 109 | 114 | 119 | 124 | 129 | 134 | 139 | 144 | 149 | 154 | 159 | 164 | 169 |
| 12 | 7  | 2  | 3  | 8  | 13 | 18 | 23 | 28 | 33 | 38 | 43 | 48 | 53 | 58 | 63 | 68 | 73 | 78 | 83 | 88 | 93 | 98 | 103 | 108 | 113 | 118 | 123 | 128 | 133 | 138 | 143 | 148 | 153 | 158 | 163 |

$$M_{max}^6 = \left[ \frac{17}{2} pq \right] \leq \left[ \frac{17}{2} N_1 N_2 N_3 N_4 N_5 N_6 \right]$$
2q-th Order: Layering

- 6-6-6-6...6
- 6-6-6-6...4
- 6-6-6-6...2
2q-th Order: Layering

\[ S_1 = \{ \alpha_1, \alpha_2, \ldots, \alpha_{N_1} \} \quad \text{with} \quad \Phi^{2q_1}(S_1) = \{ -\mu_1 \leq \mu \leq \mu_1 \} \]

\[ S_2 = \{ \beta_1, \beta_2, \ldots, \beta_{N_2} \} \quad \text{with} \quad \Phi^{2q_2}(S_2) = \{ -\mu_2 \leq \mu \leq \mu_2 \} \]

Take a new \( 2(q_1 + q_2) \)-th order ruler:

\[ S_1 \cup S_2' \]

\[ S_2' = \{ 2\beta_1 \mu_1, 2\beta_2 \mu_1, \ldots, 2\beta_{N_2} \mu_1 \} \]

This generates:

\[ \Phi^{2(q_1+q_2)}(S_1 \cup S_2') = \{ -2\mu_1 \mu_2 - \mu_1 \leq \mu \leq 2\mu_1 \mu_2 + \mu_1 \} \]
2q-th Order: Result

- 6-6-6-6...6 \( O\left(17^{\frac{q}{3}} N^{2q}\right) \)
- 6-6-6-6...4 \( O\left(2 \cdot 17^{\frac{q-1}{3}} N^{2q}\right) \)
- 6-6-6-6...2 \( O\left(5 \cdot 17^{\frac{q-2}{3}} N^{2q}\right) \)
<table>
<thead>
<tr>
<th>Ruler Structures</th>
<th>Lag Generation</th>
</tr>
</thead>
<tbody>
<tr>
<td>2qL-NA</td>
<td>$O(2N^{2q})$</td>
</tr>
<tr>
<td>SE-2qL-NA</td>
<td>$O(2^q N^{2q})$</td>
</tr>
<tr>
<td>(proposed)</td>
<td>$O(17^{\frac{q}{3}} N^{2q})$</td>
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</tbody>
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References


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Questions?
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THANK YOU!